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On special Cesàro–Denjoy–Stieltjes integral


In this paper, we have defined special Cesàro–Denjoy–Stieltjes integral [or, (CDS)-integral] with respect to ω(x) in a way analogous to that of Saks [13] using (ω) C-derivative {Definition 2.2} and ACG* – ω (C-sense) functions. Then we have shown that the (CPS)-integral and the (CDS)-integral are equivalent.

2. Preliminaries. Let ω(x) be a non-decreasing function defined on the closed interval [a, b]. Outside the interval it is defined by ω(x) = ω(a) for x < a and ω(x) = ω(b) for x > b. Let S denote the set of points of continuity of ω(x), D = [a, b] – S and let S₀ denote the union of the pairwise disjoint open intervals (aᵢ,bᵢ) on each of which ω(x) is constant. Let S₁ = {a₁,b₁,a₂,b₂,...}, S₂ = SS₁,S₃ = [a, b] S – (S₀+S₂). Further let S²⁻ and S²⁺ denote the set of those points of S₂ which are, respectively, the set of the left and the right end-points of the intervals of S₀.

Jeffery [10] has defined the class $\mathcal{U}$ of functions $F(x)$ in the following way: $F(x)$ is defined on $[a, b]S$ such that $F(x)$ is continuous on $[a, b]S$ with respect to the set S. If $x₀ \in D$, then $F(x)$ tends to limits as x tends to $x₀+$ and to $x₀-$ over the points of S. For $x < a$, $F(x) = F(a+)$ and for $x > b$, $F(x) = F(b-)$. $F(x)$ may or may not be defined at the points of D. Suppose $\mathcal{U}_0 \subset \mathcal{U}$ contains those functions $F(x)$ in $\mathcal{U}$ such that for every $x₀ \in D$ both $F(x₀+)$ and $F(x₀-)$ are finite.

Notations. $[c, d]$ denotes the closed interval $c \leq x \leq d$ and $(c, d)$ denotes the open interval $c < x < d$. $\bar{A}$ denotes the closure of a set.


A. $F_\omega'(x)$ denotes the $\omega$-derivative [10] of the function $F(x)$ at the point $x$. $D^+ F_\omega(x)$ and $D^- F_\omega(x)$ denote the right-hand upper and lower $\omega$-derivatives [10] and $D^+ F_\omega(x)$ and $D^- F_\omega(x)$ denote the left-hand upper and lower $\omega$-derivatives. $\bar{\omega}(x)$ will denote the function which is defined as follows: $\bar{\omega}(x) = \omega(x)$ for $x \in S$ and $\bar{\omega}(x) = \frac{1}{2}(\omega(x+) + \omega(x-))$ for $x \in D$.

**Definition of (PS)-integral** [9]. Let $f(x)$ be defined on $[a, b]$. A function $M(x) \in \mathcal{U}_0$ will be a (PS)-major function of $f(x)$ on $[a, b]$ if (i) $M(a-)=0$; (ii) $M(x)$ is non-decreasing in each of the open intervals $(a, b) \subset S_0$; (iii) $D_+ M_\omega(x) > -\infty$ for $x \in S_3 + S_2^+$, $D_- M_\omega(x) > -\infty$ for $x \in S_3 + S_2^-$; (iv) $D_+ M_\omega(x) \geq f(x)$ for $x \in S_3 + S_2^+$, $D_- M_\omega(x) \geq f(x)$ for $x \in S_3 + S_2^-$ and $M'_\omega(x) \geq f(x)$ for $x \in D$. Analogously a (PS)-minor function is defined.

$f(x)$ will be said to be Perron–Stieltjes integrable [or, (PS)-integrable] on $[a, b]$ if (a) it has at least one (PS)-major function $M(x)$ and at least one (PS)-minor function $m(x)$, and (b) $\inf \{M(b+)\} = \sup \{m(b+)\}$. If $f(x)$ is (PS)-integrable on $[a, b]$ the common value $\inf \{M(b+)\} = \sup \{m(b+)\}$ is called the (PS)-integral of $f(x)$ on $[a, b]$ and is denoted by $F(x) \omega x a b$. Analogously, (PS)-integrals are defined on the open intervals $[a, b)$ and $(a, b]$. We need the following results [9] of the (PS)-integrals in the sequel.

(i) The indefinite Perron–Stieltjes integral belongs to class $\mathcal{U}_0$.

(ii) If $F(x)$ is the indefinite (PS)-integral of the function $f(x)$ on $[a, b]$, then $F'_\omega(x) = f(x)$ almost everywhere in $[a, b]$ [i.e. except for a set of points in $[a, b]$ having measure zero].

In [9], the class $\mathcal{U}_1$ of functions $F(x)$ possessing the following properties has been defined: (i) $F(x)$ is defined finitely on $[a, b]$ such that $F(x)$ is (PS)-integrable on $[a, b]$; (ii) at each point $x_0$ of $D$, $F(x)$ tends to a finite limit as $x$ tends to $x_0$ over the points of $S$; (iii) at a point $x_0 \in D$, $F(x)$ has the value $\frac{1}{2} \{F(x_0+) + F(x_0-)\}$; (iv) $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$.

We require the following known definitions and results:

**Definition 2.1** [9]. Let a real function $F(x)$ be defined finitely on $[a, b]$ and let it be (PS)-integrable on $[a, b]$. Write

$$ (\omega) C(F; a, b) = \frac{1}{\omega(b+) - \omega(a-)} (PS) \int_a^b F(x) d\omega. $$

$F(x)$ is said to be Cesàro-continuous relative to $\omega$ or $(\omega) C$-continuous at $x_0$ if

$$ \lim_{h \to 0} (\omega) C(F; x_0, x_0 + h) = F(x_0), \quad x_0 + h \in S. $$
where

\[
(\omega) C(F; x_0, x_0+h) = \begin{cases} 
\frac{1}{\omega(x_0+h)-\omega(x_0-)} (PS) \int_{x_0}^{x_0+h} F(t) d\omega, & h > 0, \\
\frac{1}{\omega(x_0+h)-\omega(x_0+)} (PS) \int_{x_0}^{x_0+h} F(t) d\omega, & h < 0, \\
F(x_0+h), & \omega(x_0+h)-\omega(x_0+) \neq 0, \\
\omega(x_0+h)-\omega(x_0-) = 0.
\end{cases}
\]

It is easily seen [9] that \( F(x) \) is \( (\omega) C \)-continuous at \( x \in D \).

Definition 2.2 [9]. Let \( F(x) \in \mathcal{U}_1 \). For a point \( x \in S \) and for \( h \neq 0 \) with \( x+h \in S \), the function \( \Phi(x, h) \) is defined by

\[
\Phi(x, h) = \begin{cases} 
(\omega) C(F; x, x+h) - F(x), & \omega(x+h)-\omega(x) \neq 0, \\
\frac{1}{2} [\omega(x+h)-\omega(x)], & \omega(x+h)-\omega(x) = 0.
\end{cases}
\]

The upper and lower limits of \( \Phi(x, h) \) as \( h \to 0^+ (x+h \in S) \) are called respectively the upper and lower Cesàro-derivates with respect to \( \omega \) [or, upper and lower \( (\omega) C \)-derivates] of \( F(x) \) at \( x \) on the right and are denoted by \( CD^+ F_\omega(x) \) and \( CD^- F_\omega(x) \) respectively. If \( CD^+ F_\omega(x) = CD^- F_\omega(x) \), the common value is called the \( (\omega) C \)-derivative of \( F(x) \) at \( x \) on the right and is denoted by \( CDF_\omega(x) \). Similarly the \( (\omega) C \)-derivates \( CD^- F_\omega(x), CD_+ F_\omega(x) \) and the left \( (\omega) C \)-derivative \( \overline{CDF}_\omega(x) \) of \( F(x) \) at \( x \) are defined. If \( CD^+ F_\omega(x) = CD^- F_\omega(x) \), the common value is called the \( (\omega) C \)-derivative of \( F(x) \) at \( x \) and is denoted by \( CD\overline{F}_\omega(x) \).

Definition 2.3 [3]. A function \( F(x) \in \mathcal{U}_1 \) is said to be \( AC^*-\omega \) (Cesàro-sense), or briefly, \( AC^*-\omega \) (C-sense) over a set \( E \subset [a, b] \) if for every positive number \( \varepsilon \) there exists a positive number \( \delta \) such that for any set of non-overlapping open intervals \( \{(c_r, d_r)\} \) having end-points in \( E \) with

\[
\sum_r \{\omega(d_r+)-\omega(c_r-)] < \delta
\]

the relations

\[
\sum_r \text{bound}_{c_r < x < d_r} |(\omega) C(F; c_r, x) - F(c_r)| < \varepsilon
\]

and

\[
\sum_r \text{bound}_{c_r < x < d_r} |(\omega) C(F; d_r, x) - F(d_r)| < \varepsilon
\]

hold.

Definition 2.4 [3]. A function \( F(x) \in \mathcal{U}_1 \) is said to be \( ACG^*-\omega \) (Cesàro-sense), or briefly, \( ACG^*-\omega \) (C-sense) on \( [a, b] \), if it is \( (\omega) C \)-con-
continuous on \([a, b]\) and if the interval \([a, b]\) can be expressed as the sum of a countable number of closed sets on each of which \(F(x)\) is \(AC^* - \omega\) (C-sense).

**Definition 2.5** [3]. A function \(F(x) \in \mathcal{U}_1\) is said to be \(AC - \omega\) on a set \(E \subseteq [a, b]\) if for every \(\varepsilon > 0\) there exists a positive number \(\delta\) such that for any set of non-overlapping open intervals \(\{(c_r, d_r)\}\) having endpoints on \(E\) for which

\[
\sum_r \{\omega(d_r +) - \omega(c_r -)\} < \delta
\]

we have

\[
\sum_r |F(d_r) - F(c_r)| < \varepsilon.
\]

The \(\omega\)-derivative [10] and approximate \(\omega\)-derivative [7] are originally defined for functions \(\in \mathcal{U}\). Here we modify the concepts of \(\omega\)-derivative and approximate \(\omega\)-derivative to be applicable at the points of \([a, b] \cap S\) for any function \(g(x)\) defined on \([a, b]\) in the following way:

**Definition 2.6.** For any \(x \in S\) and a point \(\xi (\neq x)\) in \(S\) we define

\(\chi(x, \xi)\) as follows:

\[
\chi(x, \xi) = \begin{cases} 
\frac{g(\xi) - g(x)}{\omega(\xi) - \omega(x)}, & \omega(\xi) - \omega(x) \neq 0, \\
0, & \omega(\xi) - \omega(x) = 0.
\end{cases}
\]

If \(\chi(x, \xi)\) tends to a limit as \(\xi\) tends to \(x\) over the points of \(S\), then this limit is called the \(\omega\)-derivative of \(g(x)\) at \(x\) and is denoted by \(g'_\omega(x)\) and if \(\chi(x, \xi)\) tends to a limit as \(\xi\) tends to \(x\) over the points of \(S\) except for a subset of \(S\) of \(\omega\)-density [6] zero at \(x\), then this limit is called the approximate \(\omega\)-derivative of \(g(x)\) at \(x\) and is denoted by \((ap)g'_\omega(x)\).

**Theorem 2.1** [9]. If \(F(x)\) is in class \(\mathcal{U}_1\), then the four (\(\omega\))C-derivates of \(F(x)\) are \(\omega\)-measurable [10] on \([a, b] \cap S\).

**Theorem 2.2** [3]. Let \(Q \subseteq [a, b]\) be a closed set having end-points \(c, d\) and complementary intervals \(\{(c_n, d_n)\}\). The sufficient conditions for a function \(F(x) \in \mathcal{U}_1\) to be \(AC^* - \omega\) (C-sense) on \(Q\) are that (i) \(F(x)\) is \(AC - \omega\) on \(Q\),

\[
\sum \text{bound}_{n \in \mathbb{N}} |(\omega)C(F; c_n, x) - F(c_n)| < \infty,
\]

(ii)

\[
\sum \text{bound}_{n \in \mathbb{N}} |(\omega)C(F; d_n, x) - F(d_n)| < \infty,
\]

and (iii) if \(\omega(\beta +) - \omega(\alpha -) = 0 (\alpha, \beta \in \mathbb{Q})\), then \(F(x)\) is constant on \([\alpha, \beta]\).

If \(F(x)\) is \((\omega)C\)-continuous on \([c, d]\), then conditions (i), (ii) and (iii) are also necessary for \(F(x)\) to be \(AC^* - \omega\) (C-sense) on \(Q\).

**Theorem 2.3** [3]. If a function \(F(x) \in \mathcal{U}_1\) is \(ACG^* - \omega\) (C-sense) on \([a, b]\),

then \(CDF_\omega(x)\) exists finitely \(\omega\)-almost everywhere in \([a, b] \cap S\). Also \(CDF_\omega(x)\) is equal to \((ap)F'_\omega(x)\) \(\omega\)-almost everywhere in \([a, b] \cap S\).
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Theorem 2.4 [3]. If a function $F(x) \in \mathcal{U}_1$ is $ACG^{*\omega}$ (C-sense) on $[a, b]$ and $CDF_\omega(x) = 0$ $\omega$-almost everywhere in $[a, b]$ $S$ and if $F(x^+) = F(x^-)$ for $x \in D$, then $F(x)$ is constant on $[a, b]$.

3. The (CPS)-integral. In this article we present the definition of the (CPS)-integral [9] and some of its properties which we shall require in the sequel.

Definition 3.1 [9]. Let a function $f(x)$ be defined [not necessarily finite] on $[a, b]$. A function $M(x) \in \mathcal{U}_1$ is said to be a (CPS)-major function of $f(x)$ on $[a, b]$ if
(a) $M(x)$ is $(\omega)C$-continuous on $[a, b] - D$,
(b) $M(a) = 0$,
(c) $M(x)$ is non-decreasing on each $(a_j, b_j) \subset S_0$,
(d) $CD_- M_\omega(x) > -\infty$ for $x \in S_3 + S_2^-$, $CD_+ M_\omega(x) > -\infty$ for $x \in S_3 + S_2^+$,
(e) $CD_- M_\omega(x) \geq f(x)$ for $x \in S_3 + S_2^-$, $CD_+ M_\omega(x) \geq f(x)$ for $x \in S_3 + S_2^+$,
(f) $M(x^+) - M(x^-) \geq f(x) [\omega(x^+) - \omega(x^-)]$ for $x \in D$.

Analogously a (CPS)-minor function is defined.

Definition 3.2. A function $f(x)$ defined on $[a, b]$ is said to be integrable in the Cesàro-Stieltjes sense relative to $\omega$ [or, to be (CPS)-integrable] on $[a, b]$ if (i) it has at least one (CPS)-major function and at least one (CPS)-minor function, and (ii) inf $\{M(b)\} = sup \{m(b)\}$. If $f(x)$ is (CPS)-integrable on $[a, b]$, the common value inf $\{M(b)\} - sup \{m(b)\}$ is called the Cesàro-Perron-Stieltjes integral [or, (CPS)-integral] of the function $f(x)$ on $[a, b]$ and is denoted by $\int_a^b f(x) d\omega$.

Theorem 3.1 [9]. The indefinite (CPS)-integral of $f(x)$ is $(\omega)C$-continuous.

Theorem 3.2 [9]. If $f(x)$ is (CPS)-integrable on $[a, b]$ and $F(x)$ be its indefinite (CPS)-integral and $M(x)$, $m(x)$ are a (CPS)-major function and a (CPS)-minor function for $f(x)$, then each of the differences $M(x) - F(x)$ and $F(x) - m(x)$ is non-decreasing on $[a, b]$.

Theorem 3.3 [9]. If $F(x)$ is the indefinite (CPS)-integral of the function $f(x)$ defined on $[a, b]$, then $CDF_\omega(x) = f(x)$ $\omega$-almost everywhere in $[a, b]$ $S$. Further for every $x \in D, F(x^+) - F(x^-) = f(x) [\omega(x^+) - \omega(x^-)]$.

Theorem 3.4 [4]. Let a function $f(x)$ defined on $[a, b]$ be summable (LS) ([6], [10]) over a closed set $Q \subset [a, b]$ with end-points $c, d$ and complementary intervals $\{(c_n, d_n)\}$ and let $f(x)$ be (CPS)-integrable on each $[c_n, d_n]$. If
$$\sum_n \text{bound} |(\omega)C(F_n; c_n, x)| < \infty,$$
and
$$\sum_n \text{bound} |(\omega)C(F_n; d_n, x) - F_n(d_n)| < \infty,$$
where
\[ F_n(x) = \begin{cases} 0 & \text{for } x = c_n, \\ \int_{c_n}^{x} f(t) \, dt & \text{for } c_n < x \leq d_n, \end{cases} \]
then \( f(x) \) is (CPS)-integrable on the whole interval \([c, d]\) and
\[
(CPS) \int_{c}^{d} f(x) \, dx = (LS) \int_{c}^{d} f(x) \, dx + \sum_{n} (CPS) \int_{c_n}^{d_n} f(x) \, dx - \frac{1}{2} W_n - \frac{1}{2} \{ f(c)[\omega(c+) - \omega(c-)] + f(d)[\omega(d+) - \omega(d-)] \},
\]
where
\[ W_n = f(c_n)[\omega(c_n+) - \omega(c_n-)] + f(d_n)[\omega(d_n+) - \omega(d_n-)]. \]

Theorem 3.5 [4]. Suppose the function \( f(x) \) defined on \([a, b]\) is (CPS)-integrable on every segment \([c, \beta]\), where \( a \leq c < \beta < d \leq b \) having (CPS)-integral \( F(x) \) which is also (PS)-integrable on \([c, d]\). If \( f(d) \) is finite when \( d \in D \) and if the limits
\[
J_1 = \lim_{\beta \to d^-} \lim_{\beta \to c^+} (C(F; d, \beta) \text{ if } d \in S, \\
J_2 = \lim_{\beta \to d^-} F(\beta) \text{ if } d \in D
\]
exist and are finite, then \( f(x) \) will be (CPS)-integrable on \([c, d]\) and
\[
(CPS) \int_{c}^{d} f(x) \, dx = J_1 \text{ if } d \in S
\]
and
\[
(CPS) \int_{c}^{d} f(x) \, dx = J_2 + \frac{1}{2} f(d)[\omega(d+) - \omega(d-)] \text{ if } d \in D.
\]

Using similar arguments, the following theorem can be proved:

Theorem 3.6. Suppose the function \( f(x) \) defined on \([a, b]\) is (CPS)-integrable on every segment \([\alpha, d]\), where \( a \leq c < \alpha < d \leq b \) having (CPS)-integral \( F(x) \) which is also (PS)-integrable on \([c, d]\). If \( f(c) \) is finite when \( c \in D \) and if the limits
\[
K_1 = \lim_{\alpha \to c^+} \lim_{\alpha \to e^-} (C(F; c, \alpha) \text{ if } c \in S, \\
K_2 = \lim_{\alpha \to c^+} F(\alpha) \text{ if } c \in D
\]
exist and are finite, then \( f(x) \) will be (CPS)-integrable on \([c, d]\) and
\[
(CPS) \int_{c}^{d} f(x) \, dx = K_1 \text{ if } c \in S.
\]
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\[ (CPS) \int_c^d f(x) \, d\omega = K_2 + \frac{1}{2}[\omega(c+) - \omega(c-)] \quad \text{if } c \in D. \]

4. The \((CDS)\)-integral. Here we shall introduce the definition of \((CDS)\)-integral and shall prove a few important properties.

Definition 4.1. Let \(f(x)\) be a function defined on \([a,b]\). If there exists a function \(F(x) \in \mathcal{W}_1\) which is \(ACG^* - \omega\) (C-sense) on \([a,b]\) and which is such that \(CDF_\omega(x) = f(x)\) \(\omega\)-almost everywhere on \([a,b]S\) and \(F(x+) - F(x-) = f(x) \ [\omega(x+) - \omega(x-)]\) for \(x \in D\), then \(f(x)\) is said to be special Cesàro–Denjoy–Stieltjes integrable \([\text{or, \(CDS\)-integrable}]\) on \([a,b]\) and the function \(F(x)\) is called indefinite \((CDS)\)-integral of \(f(x)\) on \([a,b]\); the difference \(F(b) - F(a)\) is termed definite \((CDS)\)-integral of \(f(x)\) over \([a,b]\) and is denoted by \((CDS) \int_a^b f(x) \, d\omega\).

It follows by Theorem 2.4 that if \(F(x)\) and \(G(x)\) are any two indefinite \((CDS)\)-integrals of \(f(x)\) on \([a,b]\), then \(F(x) - G(x)\) is constant on \([a,b]\). The definite \((CDS)\)-integral of a function \(f(x)\), \((CDS)\)-integrable on \([a,b]\) is therefore unique.

Theorem 4.1. A function \(f(x)\) which is \((CDS)\)-integrable on \([a,b]\) is \(\omega\)-measurable on \([a,b]\).

**Proof.** Let \(F(x)\) be an indefinite \((CDS)\)-integral of \(f(x)\). Then \(CDF_\omega(x) = f(x)\) \(\omega\)-almost everywhere in \([a,b]S\). So by Theorem 2.1 \(f(x)\) is \(\omega\)-measurable on \([a,b]S\). Since the set \(D\) is at most denumerable, \(f(x)\) is \(\omega\)-measurable on \([a,b]\).

Theorem 4.2. A function \(f(x)\) which is \((CDS)\)-integrable on \([a,b]\) is finite \(\omega\)-almost everywhere.

**Proof.** The proof follows from Theorem 2.3 and Definition 4.1.

5. The \((CDS)\)-integral includes the \((CPS)\)-integral.

Preliminary lemmas. Let \(F(x) \in \mathcal{W}_1\) be \((\omega)C\)-continuous on \([a,b]\) and non-decreasing on each of the open intervals \((a_i, b_i) \subset S_0\) and let for every natural number \(n\), \(E_n\) denote the set of points \(x\) of \([a,b]\) such that for \(x + h \in S\) with \(|h| < 1/n\) we have

1. \((\omega)C(F; x, x+h) - F(x) \geq -\frac{1}{n} [\omega(x+h) - \omega(x-)], \quad h > 0;\)
2. \(F(x) - (\omega)C(F; x, x+h) \geq -\frac{1}{n} [\omega(x+) - \omega(x+h)], \quad h < 0.\)

Let

\[ G(x) = \begin{cases} 
0, & x = a, \\
(PS) \int_a^x F(t) \, d\omega, & a < x \leq b.
\end{cases} \]
Lemma 5.1. If \( \{x_k\} \) is a convergent sequence of points of \( E_n \) and if the limit of the sequence belongs to \( S \), then

\[
\lim_{x_k \to x} F(x_k) = F(x).
\]

Proof. Choose \( h > 0 \) with \( h < 1/n \) such that \( x + h \in S \). We consider those \( x_k \) for which \( x_k + h_k = x + h, 0 < h_k < 1/n \). From (1) we get

\[
(\omega) C(F; x_k, x + h) - F(x_k) \geq -\frac{1}{n} [\omega(x + h) - \omega(x_k)].
\]

Case (a). Let \( \omega(x + h) - \omega(x) \neq 0 \). Letting \( x_k \to x \) in (3) we get

\[
(\omega) C(F; x, x + h) \geq \lim_{x_k \to x} F(x_k) - \frac{1}{n} [\omega(x + h) - \omega(x)].
\]

Since \( F(x) \) is \((\omega) C\)-continuous, taking limit as \( h \to 0 \) we get

\[
F(x) \geq \lim_{x_k \to x} F(x_k).
\]

Case (b). Let \( \omega(x + h) - \omega(x) = 0 \). Firstly, let \( x_k \to x \) from the right. Then since on the right of \( x \), \( F(x) \) is continuous we have

\[
F(x) = \lim_{x_k \to x} F(x_k)
\]

and the lemma is proved. Next let \( x_k \to x \) from the left. Then taking limit as \( x_k \to x \) in (3), we get

\[
F(x) = \lim_{x_k \to x} F(x_k).
\]

Therefore in any case we have

\[
F(x) \geq \lim_{x_k \to x} F(x_k).
\]

Similarly choosing \( h' < 0 \) with \( |h'| < 1/n \) and using relation (2) we get

\[
F(x) = \lim_{x_k \to x} F(x_k).
\]

From (4) and (5)

\[
F(x) = \lim_{x_k \to x} F(x_k).
\]

This completes the proof of the lemma.

Lemma 5.2. If \( x \in S \) is a limit point of \( E_n \), then \( x \in E_n \) and if \( x \in D \) is a limit point of \( E_n \) on the right, then relation (1) holds for \( x = x \) and relation (2) with \( F(x) \) replaced by \( F(x+) \) holds for \( x = x \). Further if \( x \in D \) is a limit point of \( E_n \) on the left, then relation (2) holds for \( x = x \) and relation (1) with \( F(x) \) replaced by \( F(x-) \) holds for \( x = x \).
Proof. Case (i). Let the limit point $\alpha$ of $E_n$ belong to $S$. Suppose $\{\alpha_k\}$ is a convergent sequence of points of $E_n$ of which $\alpha$ is the limit. Choosing $h > 0$ with $h < 1/n$ we get as in Lemma 5.1

\begin{equation}
(\omega) C(F; \alpha_k, \alpha + h) - F(\alpha_k) \geq -\frac{1}{n} \left( \omega(\alpha + h) - \omega(\alpha_k) \right).
\end{equation}

We can suppose that $\omega(\alpha + h) - \omega(\alpha) \neq 0$. Otherwise it is clear that $F(\alpha + h) - F(\alpha) \geq 0$ and so

\begin{equation}
(\omega) C(F; \alpha, \alpha + h) - F(\alpha) \geq -\frac{1}{n} \left( \omega(\alpha + h) - \omega(\alpha) \right).
\end{equation}

Now as $\alpha_k \to \alpha$ we get from (6) using Lemma 5.1

\begin{equation}
(\omega) C(F; \alpha, \alpha + h) - F(\alpha) \geq -\frac{1}{n} \left( \omega(\alpha + h) - \omega(\alpha) \right).
\end{equation}

Thus $\alpha$ satisfies relation (1). Similarly we can show that $\alpha$ satisfies (2). So $\alpha \in E_n$.

Case (ii). Next, let $\alpha \in D$ be a limit point of $E_n$ on the right. Suppose $\{\alpha_k\}$ is a sequence of points of $E_n$ converging from right to $\alpha$. Choose $h' < 0$ with $|h'| < 1/n$ such that $\alpha + h' \in S$. In this case we can choose $h'_k$ with $|h'_k| < 1/n$ corresponding to each $\alpha_k$ for sufficiently large $k$ such that $\alpha_k + h'_k = \alpha + h'$. We have

\begin{equation}
F(\alpha_k) - (\omega) C(F; \alpha_k, \alpha + h') \geq -\frac{1}{n} \left( \omega(\alpha_k) - \omega(\alpha + h') \right).
\end{equation}

Letting $k$ tend to infinity we get

\begin{equation}
F(\alpha) - (\omega) C(F; \alpha, \alpha + h') \geq -\frac{1}{n} \left( \omega(\alpha) - \omega(\alpha + h') \right),
\end{equation}

from which we get as $h' \to 0$

\begin{equation}
F(\alpha) - F(\alpha) \geq -\frac{1}{n} \left( \omega(\alpha) - \omega(\alpha -) \right).
\end{equation}

Now choose $h > 0$ with $0 < h < 1/n$. Then for $\alpha + h \in S$ we get as above

\begin{equation}
(\omega) C(F; \alpha_k, \alpha + h) - F(\alpha_k) \geq -\frac{1}{n} \left( \omega(\alpha + h) - \omega(\alpha_k) \right)
\end{equation}

and so for sufficiently large $k$ for which $\omega(\alpha + h) - \omega(\alpha_k) \neq 0$ we have

\begin{equation}
G(\alpha + h) - G(\alpha_k) - F(\alpha_k) [\omega(\alpha + h) - \omega(\alpha_k)] \geq -\frac{1}{n} \left[ \omega(\alpha + h) - \omega(\alpha_k) \right]^2.
\end{equation}

Letting $k$ tend to infinity we have

\begin{equation}
G(\alpha + h) - G(\alpha) - F(\alpha) [\omega(\alpha + h) - \omega(\alpha)] \geq -\frac{1}{n} \left[ \omega(\alpha + h) - \omega(\alpha) \right]^2.
\end{equation}

From (8) and (9) and the relation

\begin{equation}
G(\alpha) - G(\alpha) = F(\alpha) [\omega(\alpha) - \omega(\alpha)] \quad \text{[by result (ii) of (PS)-integral]},
\end{equation}

we get

\begin{equation}
(\omega) C(F; \alpha, \alpha + h) - F(\alpha) \geq -\frac{1}{n} \left[ \omega(\alpha + h) - \omega(\alpha) \right]
\end{equation}

\begin{equation}
> -\frac{1}{n} \left[ \omega(\alpha + h) - \omega(\alpha -) \right].
\end{equation}
If for all \( k, \omega(\alpha + h) - \omega(\alpha) = 0 \), then \( \omega(\alpha + h) - \omega(\alpha) = 0 \) and so
\[
\omega C(F; \alpha, \alpha + h) = F(\alpha),
\]
and again we get relation (10) which together with relation (7) prove the relevant assertions made in the lemma.

**Case (iii).** The case when \( \alpha \in D \) is a limit point of \( E_n \) on the left, can be treated as in case (ii). This completes the proof of the lemma.

**Theorem 5.1.** A function \( f(x) \) which is (CPS)-integrable on \([a, b]\) is (CDS)-integrable on \([a, b]\) and
\[
(CDS) \int_a^b f(x) \, d\omega = (CPS) \int_a^b f(x) \, d\omega.
\]

**Proof.** Let \( F(x) \) be the indefinite (CPS)-integral of \( f(x) \) on \([a, b]\). Let \( \varepsilon > 0 \) be chosen arbitrarily. Then \( f(x) \) has a (CPS)-major function \( U(x) \) and a (CPS)-minor function \( V(x) \) such that \( U(b) - F(b) < \varepsilon/3 \) and \( F(b) - V(b) < \varepsilon/3 \). Let for every natural number \( m \), \( A_m \) denote the set of points \( x \) of \([a, b]\) such that for \( x + h \in S \) with \( |h| < 1/m \) we have
\[
\begin{align*}
(11) & \quad (\omega) C(U; x, x + h) - U(x) \geq -\frac{1}{2} m [\omega(x + h) - \omega(x)], \quad h > 0, \\
(12) & \quad U(x) - (\omega) C(U; x, x + h) \geq -\frac{1}{2} m [\omega(x + h) - \omega(x)], \quad h < 0;
\end{align*}
\]
and let for every natural number \( n \), \( B_n \) denote the set of points \( x \) of \([a, b]\) such that for \( x + b \in S \) with \( |b| < 1/n \) we have
\[
\begin{align*}
(13) & \quad (\omega) C(V; x, x + h) - V(x) \leq \frac{1}{2} n [\omega(x + h) - \omega(x)], \quad h > 0, \\
(14) & \quad V(x) - (\omega) C(V; x, x + h) \leq \frac{1}{2} n [\omega(x + h) - \omega(x)], \quad h < 0.
\end{align*}
\]
Let \( E_{mn} = A_m B_n \), \( p = \max(m, n) \) and \( E_{mnj} \) denote the common part of \( E_{mn} \) and the closed interval \([j/p+1, j+1/p+1]\). Then
\[
[a, b] = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} E_{mnj}.
\]
Now we shall show that \( F(x) \) is \( AC^* - \omega \) (C-sense) on \( E_{mnj} \). Let \( \{(c_r, d_r)\} \) be any set of non-overlapping intervals having end-points in \( E_{mnj} \).

**Case (a).** Let \( c_r \) be a point of \( E_{mnj} \) or a limit point of \( E_{mnj} \) in case \( c_r \in S \) or else a limit point of \( E_{mnj} \) on the right when \( c_r \in D \). Then for \( c_r < x \leq d_r \) with \( \omega(x) - \omega(c_r) \neq 0 \)
\[
\begin{align*}
(15) & \quad (\omega) C(F; c_r, x) - F(c_r) \\
& \quad = (\omega) C(U; c_r, x) - U(c_r) - \\
& \quad \quad \frac{1}{\omega(x) - \omega(c_r)} \quad (PS) \int_{c_r}^{x} [U(t) - F(t)] \, d\omega + U(c_r) - F(c_r) \\
& \quad \geq (\omega) C(U; c_r, x) - U(c_r) - [U(d_r) - F(d_r)] + [U(c_r) - F(c_r)].
\end{align*}
\]
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\[ \begin{align*}
&\geq -\frac{1}{m} \left[ \omega(x+) - \omega(c_r-) \right] - \left[ U(d_r) - F(d_r) \right] + \left[ U(c_r) - F(c_r) \right] \\
&\text{[by Theorem 3.2]} \\
&\text{[by (11) and Lemma 5.2].}
\end{align*} \]

Relation (15) is obviously satisfied when \( \omega(x+) - \omega(c_r-) = 0 \). Hence

\[ \begin{align*}
\text{bound } \left[ (\omega) C(F; c_r, x) - F(c_r) \right] &\geq -\frac{1}{m} \left[ \omega(d_r+) - \omega(c_r-) \right] - \left[ U(d_r) - F(d_r) \right] + \left[ U(c_r) - F(c_r) \right].
\end{align*} \]

Case (b). If \( c_r \in D \) be a limit point of \( E_{mnj} \) on the left, we can show in a similar way

\[ \begin{align*}
\text{bound } \left[ (\omega) C(F; c_r, x) - F(c_r) \right] &\geq -\frac{1}{m} \left[ \omega(d_r+) - \omega(c_r-) \right] - \left[ U(d_r) - F(d_r) \right] + \left[ U(c_r) - F(c_r) \right].
\end{align*} \]

Therefore

\[ \begin{align*}
(16) \quad &\sum_{c_r < x \leq d_r} \text{bound } \left[ (\omega) C(F; c_r, x) - F(c_r) \right] + \sum_{c_r < x \leq d_r} \text{bound } \left[ (\omega) C(F; c_r, x) - F(c_r) \right] \\
\geq &-\frac{1}{m} \sum_r \left[ \omega(d_r+) - \omega(c_r-) \right] - 2 \left[ \{ U(b) - F(b) \} - \{ U(a) - F(a) \} \right] \\
> &-\frac{1}{m} \sum_r \left[ \omega(d_r+) - \omega(c_r-) \right] - 2\varepsilon/3 > -\varepsilon
\end{align*} \]

provided

\[ \sum_r \left[ \omega(d_r+) - \omega(c_r-) \right] < 2\varepsilon/3m, \]

where \( \sum \) and \( \sum \) denote the summations over \( r \) for cases (a) and (b) respectively.

Similarly, using relation (13) and a result analogous to Lemma 5.2 corresponding to the set \( B_n \), which obviously holds, we get

\[ \begin{align*}
(17) \quad &\sum_{c_r < x \leq d_r} \text{bound } \left[ (\omega) C(F; c_r, x) - F(c_r) \right] + \\
&+ \sum_{c_r < x \leq d_r} \text{bound } \left[ (\omega) C(F; c_r, x) - F(c_r) \right] < \varepsilon
\end{align*} \]

provided

\[ \sum_r \left[ \omega(d_r+) - \omega(c_r-) \right] < 2\varepsilon/3n. \]

Combining (16) and (17) we get

\[ \begin{align*}
(18) \quad &\sum_{c_r < x \leq d_r} \text{bound } \left| (\omega) C(F; c_r, x) - F(c_r) \right| + \\
&+ \sum_{c_r < x \leq d_r} \text{bound } \left| (\omega) C(F; c_r, x) - F(c_r) \right| < \varepsilon
\end{align*} \]
provided
\[ \sum_r \text{bound}_{c_r < x < d_r} |(\omega) C(F; c_r, x) - F(c_r)| < \epsilon \]
where
\[ \delta = \min (2\epsilon/3m, 2\epsilon/3n). \]
From (18) we get
\[ (19) \sum^{(2)} |F(c_r) - F(c_r^-)| \leq \epsilon. \]
So from (18) and (19) we get
\[ \sum^{(2)} \text{bound}_{c_r < x < d_r} |(\omega) C(F; c_r, x) - F(c_r)| < 2\epsilon \]
provided
\[ \sum_r [\omega(d_r^+) - \omega(c_r^-)] < \delta. \]
Similarly using relations (12) and (14) we get
\[ \sum^{(2)} \text{bound}_{c_r < x < d_r} |(\omega) C(F; d_r, x) - F(d_r)| < 2\epsilon \]
provided
\[ \sum_r [\omega(d_r^+) - \omega(c_r^-)] < \delta. \]
It follows that \( F(x) \) is \( AC^* - \omega \) (C-sense) on \( \bar{E}_{mnj} \). Since each \( \bar{E}_{mnj} \) is closed and since (by Theorem 3.1), \( F(x) \) is \( (\omega) C \)-continuous on \([a, b] \), \( F(x) \) is \( ACG^* - \omega \) (C-sense) on \([a, b] \). Again by Theorem 3.3, \( CDF_{\omega}(x) = f(x) \omega \)-almost everywhere in \([a, b] \) and \( F(x^+) - F(x^-) = f(x) [\omega(x^+) - \omega(x^-)] \) for \( x \in D \), and so \( f(x) \) is (CDS)-integrable on \([a, b] \) and
\[ (CDS) \int_a^b f(x) d\omega = F(b) - F(a) = (CPS) \int_a^b f(x) d\omega. \]
This completes the proof of the theorem.

6. The (CPS)-integral includes the (CDS)-integral.

Lemma 6.1. If \( F(x) \in \mathcal{U}_1 \) is \( AC - \omega \) on a closed set \( Q \), then it is \( BV \) on \( Q \).

The proof can be completed by proceeding as in the proof of Theorem 5 [1].

Lemma 6.2. If a function \( F(x) \) is \( BV \) on \([a, b] \), then \( F'(x) \) exists finitely \( \omega \)-almost everywhere on \([a, b] \) and is summable (LS) on \([a, b] \).

The proof follows by usual arguments (cf. [11], Theorem 5.14 and [6], Theorem 6.3).
Theorem 6.1. If a function \( f(x) \) is (CDS)-integrable on \([a, b]\), then it is (CPS)-integrable on \([a, b]\).

Proof. Let \( F(x) \) be an indefinite (CDS)-integral of \( f(x) \) on \([a, b]\). Let \( K \) be the set of points \( x \) of \([a, b]\) throughout no closed neighbourhood of which \( f(x) \) is (CPS)-integrable. Then it is easily seen that \( K \) is a closed set. We now show that \( K \) is a null set. To prove this we assume that \( K \) is not null. Let \((\alpha_r, \beta_r)\) be any complementary interval of \( K \) and let \( p_r, q_r \) be two points of \( S \) such that \( \alpha_r < p_r < q_r < \beta_r \). Then \( f(x) \) is (CPS)-integrable on \([p_r, q_r]\) and by Theorem 5.1

\[
(CPS) \int_{p_r}^{q_r} f(t) \, d\omega = F(q_r) - F(p_r).
\]

Since \( F(x) \in \mathcal{W}_1, F(q_r) \) and \( F(p_r) \) tend to finite limits as \( q_r, p_r \) tend to \( \beta_r, \alpha_r \) respectively when \( \beta_r, \alpha_r \in D \) and since \( f(x) \) is \((\omega)C\)-continuous

\[
\lim_{x \to \beta_r} (\omega)C(F; \beta_r, x) = F(\beta_r)
\]

and

\[
\lim_{x \to \alpha_r} (\omega)C(F; \alpha_r, x) = F(\alpha_r)
\]

and hence by Theorems 3.5 and 3.6, \( f(x) \) is (CPS)-integrable on \([\alpha_r, \beta_r]\). Therefore \( K \) has no isolated points. Since \( f(x) \) is \( ACG^* - \omega \) (C-sense) on \([a, b]\), there exist a countable number of closed sets \( E_n \) such that \([a, b] = \sum E_n \) and \( F(x) \) is \( AC^* - \omega \) (C-sense) on each \( E_n \). Since \( K = \sum KE_n \), there exists, by Baire's theorem, a closed interval \([l, m]\) and a positive integer \( n \) such that \( K(l, m) \) is not null and \( K[l, m] = KE_n[l, m] = Q \) (say). Thus \( F(x) \) is \( AC^* - \omega \) (C-sense) on \( Q \). Let \([c, d]\) be the smallest closed interval containing \( Q \). Denote the component intervals of \([c, d] - Q\) by \( \{(c_n, d_n)\} \). We now define the function \( G(x) \) as follows:

\[
G(x) = \begin{cases} 
F(x) & \text{for } x \in Q, \\
\bar{F}(c_n) + \frac{\bar{\omega}(x) - \omega(c_n^+)}{\omega(d_n^-) - \omega(c_n^+)} \{\bar{F}(d_n) - \bar{F}(c_n)\} & \text{for } c_n < x < d_n, \omega(d_n^-) \neq \omega(c_n^+), \\
\bar{F}(c_n) = \bar{F}(d_n) & \text{for } c_n < x < d_n, \omega(d_n^-) = \omega(c_n^+), \\
F(c) & \text{for } x < c, \\
F(d) & \text{for } x > d;
\end{cases}
\]

where

\[
\bar{F}(c_n) = \begin{cases} 
F(c_n) & \text{when } c_n \in S, \\
F(c_n^+) & \text{when } c_n \in D;
\end{cases}
\]
and
\[ F(d_n) = \begin{cases} F(d_n) & \text{when } d_n \in S, \\ F(d_n-) & \text{when } d_n \in D. \end{cases} \]

Since \( F(x) \) is \( AC^* - \omega \) (C-sense) on \( Q \), it is \( \overline{AC} - \omega \) on \( Q \) and so by Lemma 6.1, it is \( BV \) on \( Q \). Therefore (cf. [7], Theorem 3.1) \( G(x) \) is \( BV \) on \([c, d]\) and so by Lemma 6.2 \( G'(x) \) exists finitely \( \omega \)-almost everywhere in \([c, d]\) \( S \). Now \( G'(x) = (ap) F'(x) \) \( \omega \)-almost everywhere in \( QS \). Therefore by Theorem 2.3 \( G'(x) = CDF'(x) = f(x) \) \( \omega \)-almost everywhere in \( QS \). Therefore by Lemma 6.2, \( f(x) \) is summable \( (LS) \) on \( QS \). Again
\[ \frac{G(x+) - G(x-)}{\omega(x+) - \omega(x-)} = \frac{F(x+) - F(x-)}{\omega(x+) - \omega(x-)} = f(x) \]
for \( x \in QD \). Therefore \( f(x) \) is summable \( (LS) \) on \( QD \). It follows that \( f(x) \) is summable \( (LS) \) on \( Q \). Since \( F(x) \) is \( AC^* - \omega \) (C-sense) on \( Q \), by Theorem 2.2,
\[ \sum_{n} \text{bound } |(\omega) C(F; c_n, x) - F(c_n)| < \infty \]
and
\[ \sum_{n} \text{bound } |(\omega) C(F; d_n, x) - F(d_n)| < \infty \]
and so
\[ \sum_{n} \text{bound } |(\omega) C(F_n; c_n, x)| < \infty \]
and
\[ \sum_{n} \text{bound } |(\omega) C(F_n; d_n, x) - F_n(d_n)| < \infty, \]
where \( F_n(x) = F(x) - F(c_n) \). Therefore by Theorem 3.4, \( f(x) \) is \( (CPS) \)-integrable in \([c, d]\). This is clearly impossible, since \( c \) and \( d \) are end-points of a closed subset of \( K \). The set \( K \) must therefore be null. This completes the proof of the theorem.

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References


