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On special Cesàro–Denjoy–Stieltjes integral

1. Introduction. Burkill [5] has introduced the definition of Cesàro–Perron integral which generalizes Perron integral. Then Dutta [9] has introduced the definition of Cesàro–Perron–Stieltjes integral which generalizes Cesàro–Perron integral [or, (CPS)-integral]. In [4] the author has established the Cauchy and Harnack properties [12] for the (CPS)-integral and in [3] the definition of $ACG^*-\omega$ (C-sense) function has been introduced.

In this paper, we have defined special Cesàro–Denjoy–Stieltjes integral [or, (CDS)-integral] with respect to $\omega(x)$ in a way analogous to that of Saks [13] using $(\omega)\,C$-derivative {Definition 2.2} and $ACG^*-\omega$ (C-sense) functions. Then we have shown that the (CPS)-integral and the (CDS)-integral are equivalent.

2. Preliminaries. Let $\omega(x)$ be a non-decreasing function defined on the closed interval $[a,b]$. Outside the interval it is defined by $\omega(x) = \omega(a)$ for $x < a$ and $\omega(x) = \omega(b)$ for $x > b$. Let S denote the set of points of continuity of $\omega(x)$, $D = [a,b]-S$ and let S_0 denote the union of the pairwise disjoint open intervals (a_i, b_i) on each of which $\omega(x)$ is constant. Let $S_1 = \{a_1, b_1, a_2, b_2, \ldots\}$, $S_2 = SS_1$, $S_3 = [a,b]S-(S_0+S_2)$. Further let S_2^- and S_2^+ denote the set of those points of S_2 which are, respectively, the set of the left and the right end-points of the intervals of S_0.

Jeffery [10] has defined the class $\mathscr{U}$ of functions $F(x)$ in the following way: $F(x)$ is defined on $[a,b]S$ such that $F(x)$ is continuous on $[a,b]S$ with respect to the set S. If $x_0 \in D$, then $F(x)$ tends to limits as x tends to x_0+ and to x_0- over the points of S. For $x < a$, $F(x) = F(a+)$ and for $x > b$, $F(x) = F(b-)$. $F(x)$ may or may not be defined at the points of D. Suppose $\mathscr{U}_0 \subset \mathscr{U}$ contains those functions $F(x)$ in $\mathscr{U}$ such that for every $x_0 \in D$ both $F(x_0+)$ and $F(x_0-)$ are finite.

Notations. $[c,d]$ denotes the closed interval $c \leqslant x \leqslant d$ and (c,d) denotes the open interval $c < x < d$. $\bar{A}$ denotes the closure of a set

A. $F'_\omega(x)$ denotes the ω-derivative [10] of the function $F(x)$ at the point x. $D^+ F_\omega(x)$ and $D_+ F_\omega(x)$ denote the right-hand upper and lower ω-derivates [10] and $D^- F_\omega(x)$ and $D_- F_\omega(x)$ denote the left-hand upper and lower ω-derivates. $\bar{\omega}(x)$ will denote the function which is defined as follows: $\bar{\omega}(x) = \omega(x)$ for $x \in S$ and $\bar{\omega}(x) = \frac{1}{2}\{\omega(x+)+\omega(x-)\}$ for $x \in D$.

Definition of (*PS*)-integral [9]. Let $f(x)$ be defined on $[a, b]$. A function $M(x) \in \mathscr{U}_0$ will be a (*PS*)-major function of $f(x)$ on $[a, b]$ if (i) $M(a-) = 0$; (ii) $M(x)$ is non-decreasing in each of the open intervals $(a_i, b_i) \subset S_0$; (iii) $D_+ M_\omega(x) > -\infty$ for $x \in S_3+S_2^+$, $D_- M_\omega(x) > -\infty$ for $x \in S_3+S_2^-$; (iv) $D_+ M_\omega(x) \geqslant f(x)$ for $x \in S_3+S_2^+$, $D_- M_\omega(x) \geqslant f(x)$ for $x \in S_3+S_2^-$ and $M'_\omega(x) \geqslant f(x)$ for $x \in D$. Analogously a (*PS*)-minor function is defined. $f(x)$ will be said to be *Perron–Stieltjes integrable* [or, *(PS)-integrable*] on $[a, b]$ if (a) it has at least one (*PS*)-major function $M(x)$ and at least one (*PS*)-minor function $m(x)$, and (b) $\inf\{M(b+)\} = \sup\{m(b+)\}$. If $f(x)$ is (*PS*)-integrable on $[a, b]$ the common value $\inf\{M(b+)\} = \sup\{m(b+)\}$ is called the *(PS)-integral* of $f(x)$ on $[a, b]$ and is denoted by $(PS)\int_a^b f(x)\,d\omega$.

We need the following results [9] of the (*PS*)-integrals in the sequel.

(i) The indefinite Perron–Stieltjes integral belongs to class $\mathscr{U}_0$.

(ii) If $F(x)$ is the indefinite (*PS*)-integral of the function $f(x)$ on $[a, b]$, then $F'_\omega(x) = f(x)$ ω-almost everywhere in $[a, b]$ [i.e. except for a set of points in $[a, b]$ having ω-measure zero].

In [9], the class $\mathscr{U}_1$ of functions $F(x)$ possessing the following properties has been defined: (i) $F(x)$ is defined finitely on $[a, b]$ such that $F(x)$ is (*PS*)-integrable on $[a, b]$; (ii) at each point x_0 of D, $F(x)$ tends to a finite limit as x tends to x_0+ or x_0- over the points of S; (iii) at a point $x_0 \in D$, $F(x)$ has the value $\frac{1}{2}\{F(x_0+)+F(x_0-)\}$; (iv) $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$.

We require the following known definitions and results:

Definition 2.1 [9]. Let a real function $F(x)$ be defined finitely on $[a, b]$ and let it be (*PS*)-integrable on $[a, b]$. Write

$$(\omega)C(F;\, a, b) = \frac{1}{\omega(b+)-\omega(a-)}(PS)\int_a^b F(x)\,d\omega.$$

$F(x)$ is said to be *Cesàro-continuous relative to* ω or (ω)*C-continuous* at x_0 if

$$\lim_{\substack{h\to 0\\ x_0+h\in S}} (\omega)C(F;\, x_0, x_0+h) = F(x_0),$$

where

$$(\omega) C(F;\ x_0, x_0+h) = \begin{cases} \dfrac{1}{\omega(x_0+h)-\omega(x_0-)} (PS) \displaystyle\int_{x_0}^{x_0+h} F(t)\,d\omega, & h > 0, \\ & \omega(x_0+h)-\omega(x_0-) \neq 0, \\ \dfrac{1}{\omega(x_0+h)-\omega(x_0+)} (PS) \displaystyle\int_{x_0}^{x_0+h} F(t)\,d\omega, & h < 0, \\ & \omega(x_0+h)-\omega(x_0+) \neq 0, \\ F(x_0+h), & \omega(x_0+h)-\omega(x_0\pm) = 0. \end{cases}$$

It is easily seen [9] that $F(x)$ is $(\omega)C$-continuous at $x \in D$.

DEFINITION 2.2 [9]. Let $F(x) \in \mathscr{U}_1$. For a point $x \in S$ and for $h \neq 0$ with $x+h \in S$, the function $\Phi(x, h)$ is defined by

$$\Phi(x, h) = \begin{cases} \dfrac{(\omega) C(F;\ x, x+h)-F(x)}{\frac{1}{2}[\omega(x+h)-\omega(x)]}, & \omega(x+h)-\omega(x) \neq 0, \\ 0, & \omega(x+h)-\omega(x) = 0. \end{cases}$$

The upper and lower limits of $\Phi(x, h)$ as $h \to 0+$ $(x+h \in S)$ are called respectively the *upper* and *lower Cesàro-derivates* with respect to ω [or, *upper* and *lower* (ω)*C-derivates*] of $F(x)$ at x on the right and are denoted by $CD^+ F_\omega(x)$ and $CD_+ F_\omega(x)$ respectively. If $CD^+ F_\omega(x) = CD_+ F_\omega(x)$, the common value is called the (ω)*C-derivative* of $F(x)$ at x on the right and is denoted by $CDF_{+\omega}(x)$. Similarly the $(\omega)C$-derivates $CD^- F_\omega(x)$, $CD_- F_\omega(x)$ and the left $(\omega)C$-derivative $CDF_{-\omega}(x)$ of $F(x)$ at x are defined. If $CDF_{+\omega}(x) = CDF_{-\omega}(x)$, the common value is called the (ω)*C-derivative* of $F(x)$ at x and is denoted by $CDF_\omega(x)$.

DEFINITION 2.3 [3]. A function $F(x) \in \mathscr{U}_1$ is said to be $AC^*-\omega$ (*Cesàro-sense*), or briefly, $AC^*-\omega$ (*C-sense*) over a set $E \subset [a, b]$ if for every positive number ε there exists a positive number δ such that for any set of non-overlapping open intervals $\{(c_r, d_r)\}$ having end-points in E with

$$\sum_r \{\omega(d_r+)-\omega(c_r-)\} < \delta$$

the relations

$$\sum_r \overline{\text{bound}}_{c_r < x \leq d_r} |(\omega) C(F;\ c_r, x)-F(c_r)| < \varepsilon$$

and

$$\sum_r \overline{\text{bound}}_{c_r \leq x < d_r} |(\omega) C(F;\ d_r, x)-F(d_r)| < \varepsilon$$

hold.

DEFINITION 2.4 [3]. A function $F(x) \in \mathscr{U}_1$ is said to be $ACG^*-\omega$ (*Cesàro-sense*), or briefly, $ACG^*-\omega$ (*C-sense*) on $[a, b]$, if it is $(\omega)C$-continuous on $[a, b]$ and if the interval $[a, b]$ can be expressed as the sum of a countable number of closed sets on each of which $F(x)$ is $AC^*-\omega$ (C-sense).

DEFINITION 2.5 [3]. A function $F(x) \in \mathscr{U}_1$ is said to be $\overline{AC}-\omega$ on a set $E \subset [a, b]$ if for every $\varepsilon > 0$ there exists a positive number δ such that for any set of non-overlapping open intervals $\{(c_r, d_r)\}$ having end-points on E for which

$$\sum_r \{\omega(d_r+)-\omega(c_r-)\} < \delta$$

we have

$$\sum_r |F(d_r)-F(c_r)| < \varepsilon .$$

The ω-derivative [10] and approximate ω-derivative [7] are originally defined for functions $\in \mathscr{U}$. Here we modify the concepts of ω-derivative and approximate ω-derivative to be applicable at the points of $[a, b]S$ for any function $g(x)$ defined on $[a, b]$ in the following way:

DEFINITION 2.6. For any $x \in S$ and a point ξ $(\neq x)$ in S we define $\chi(x, \xi)$ as follows:

$$\chi(x, \xi) = \begin{cases} \dfrac{g(\xi)-g(x)}{\omega(\xi)-\omega(x)}, & \omega(\xi)-\omega(x) \neq 0, \\ 0, & \omega(\xi)-\omega(x) = 0. \end{cases}$$

If $\chi(x, \xi)$ tends to a limit as ξ tends to x over the points of S, then this limit is called the *ω-derivative* of $g(x)$ at x and is denoted by $g'_\omega(x)$ and if $\chi(x, \xi)$ tends to a limit as ξ tends to x over the points of S except for a subset of S of ω-density [6] zero at x, then this limit is called the *approximate ω-derivative* of $g(x)$ at x and is denoted by $(ap)g'_\omega(x)$.

THEOREM 2.1 [9]. *If $F(x)$ is in class $\mathscr{U}_1$, then the four $(\omega)C$-derivates of $F(x)$ are ω-measurable* [10] *on $[a, b]S$.*

THEOREM 2.2 [3]. *Let $Q \subset [a, b]$ be a closed set having end-points c, d and complementary intervals $\{(c_n, d_n)\}$. The sufficient conditions for a function $F(x) \in \mathscr{U}_1$ to be $AC^*-\omega$ (C-sense) on Q are that* (i) *$F(x)$ is $\overline{AC}-\omega$ on Q,*

(ii)

$$\sum_n \overline{\underset{c_n<x\leqslant d_n}{\text{bound}}} |(\omega)C(F;\ c_n, x)-F(c_n)| < \infty ,$$

$$\sum_n \overline{\underset{c_n\leqslant x<d_n}{\text{bound}}} |(\omega)C(F;\ d_n, x)-F(d_n)| < \infty ,$$

and (iii) *if $\omega(\beta+)-\omega(\alpha-) = 0$ $(\alpha, \beta \in Q)$, then $F(x)$ is constant on $[\alpha, \beta]$. If $F(x)$ is $(\omega)C$-continuous on $[c, d]$, then conditions* (i), (ii) *and* (iii) *are also necessary for $F(x)$ to be $AC^*-\omega$ (C-sense) on Q.*

THEOREM 2.3 [3]. *If a function $F(x) \in \mathscr{U}_1$ is $ACG^*-\omega$ (C-sense) on $[a, b]$, then $CDF_\omega(x)$ exists finitely ω-almost everywhere in $[a, b]S$. Also $CDF_\omega(x)$ is equal to $(ap)F'_\omega(x)$ ω-almost everywhere in $[a, b]S$.*

THEOREM 2.4 [3]. *If a function* $F(x) \in \mathscr{U}_1$ *is* $ACG^* - \omega$ *(C-sense) on* $[a, b]$ *and* $CDF_\omega(x) = 0$ *ω-almost everywhere in* $[a, b] S$ *and if* $F(x+) = F(x-)$ *for* $x \in D$, *then* $F(x)$ *is constant on* $[a, b]$.

3. The (*CPS*)**-integral.** In this article we present the definition of the (*CPS*)-integral [9] and some of its properties which we shall require in the sequel.

DEFINITION 3.1 [9]. Let a function $f(x)$ be defined [not necessarily finite] on $[a, b]$. A function $M(x) \in \mathscr{U}_1$ is said to be a *(CPS)-major function* of $f(x)$ on $[a, b]$ if

(a) $M(x)$ is (ω) C-continuous on $[a, b] - D$,

(b) $M(a) = 0$,

(c) $M(x)$ is non-decreasing on each $(a_j, b_i) \subset S_0$,

(d) $CD_- M_\omega(x) > -\infty$ for $x \in S_3 + S_2^-$, $CD_+ M_\omega(x) > -\infty$ for $x \in S_3 + S_2^+$,

(e) $CD_- M_\omega(x) \geqslant f(x)$ for $x \in S_3 + S_2^-$, $CD_+ M_\omega(x) \geqslant f(x)$ for $x \in S_3 + S_2^+$,

(f) $M(x+) - M(x-) \geqslant f(x)[\omega(x+) - \omega(x-)]$ for $x \in D$.

Analogously a (*CPS*)-minor function is defined.

DEFINITION 3.2. A function $f(x)$ defined on $[a, b]$ is said to be *integrable in the Cesàro–Stieltjes sense* relative to ω [or, to be *(CPS)-integrable*] on $[a, b]$ if (i) it has at least one (*CPS*)-major function and at least one (*CPS*)-minor function, and (ii) $\inf\{M(b)\} = \sup\{m(b)\}$. If $f(x)$ is (*CPS*)-integrable on $[a, b]$, the common value $\inf\{M(b)\} = \sup\{m(b)\}$ is called the *Cesàro–Perron–Stieltjes integral* [or, *(CPS)-integral*] of the function $f(x)$ on $[a, b]$ and is denoted by $(CPS)\int_a^b f(x)\,d\omega$.

THEOREM 3.1 [9]. *The indefinite (CPS)-integral of* $f(x)$ *is* (ω) C*-continuous.*

THEOREM 3.2 [9]. *If* $f(x)$ *is (CPS)-integrable on* $[a, b]$ *and* $F(x)$ *be its indefinite (CPS)-integral and* $M(x)$, $m(x)$ *are a (CPS)-major function and a (CPS)-minor function for* $f(x)$, *then each of the differences* $M(x) - F(x)$ *and* $F(x) - m(x)$ *is non-decreasing on* $[a, b]$.

THEOREM 3.3 [9]. *If* $F(x)$ *is the indefinite (CPS)-integral of the function* $f(x)$ *defined on* $[a, b]$, *then* $CDF_\omega(x) = f(x)$ *ω-almost everywhere in* $[a, b] S$. *Further for every* $x \in D$, $F(x+) - F(x-) = f(x)[\omega(x+) - \omega(x-)]$.

THEOREM 3.4 [4]. *Let a function* $f(x)$ *defined on* $[a, b]$ *be summable* (*LS*) ([6], [10]) *over a closed set* $Q \subset [a, b]$ *with end-points* c, d *and complementary intervals* $\{(c_n, d_n)\}$ *and let* $f(x)$ *be (CPS)-integrable on each* $[c_n, d_n]$. *If*

$$\sum_n \overline{\underset{c_n < x \leqslant d_n}{\text{bound}}}\, |(\omega) C(F_n;\ c_n, x)| < \infty,$$

and

$$\sum_n \overline{\underset{c_n \leqslant x < d_n}{\text{bound}}}\, |(\omega) C(F_n;\ d_n, x) - F_n(d_n)| < \infty,$$

where

$$F_n(x) = \begin{cases} 0 & \text{for } x = c_n, \\ (CPS)\int\limits_{c_n}^{x} f(t)\,d\omega & \text{for } c_n < x \leqslant d_n, \end{cases}$$

then $f(x)$ *is (CPS)-integrable on the whole interval* $[c, d]$ *and*

$$(CPS)\int\limits_{c}^{d} f(x)\,d\omega = (LS)\int\limits_{Q} f(x)\,d\omega + \sum_{n} (CPS)\int\limits_{c_n}^{d_n} f(x)\,d\omega -$$
$$- \sum_{n} \tfrac{1}{2}W_n - \tfrac{1}{2}\{f(c)[\omega(c+)-\omega(c-)] + f(d)[\omega(d+)-\omega(d-)]\},$$

where

$$W_n = f(c_n)[\omega(c_n+)-\omega(c_n-)] + f(d_n)[\omega(d_n+)-\omega(d_n-)].$$

THEOREM 3.5 [4]. *Suppose the function* $f(x)$ *defined on* $[a, b]$ *is (CPS)-integrable on every segment* $[c, \beta]$, *where* $a \leqslant c < \beta < d \leqslant b$ *having (CPS)-integral* $F(x)$ *which is also (PS)-integrable on* $[c, d]$. *If* $f(d)$ *is finite when* $d \in D$ *and if the limits*

$$J_1 = \lim_{\substack{\beta \to d- \\ \beta \in S}} (\omega)\,C(F;\ d, \beta) \qquad \text{if } d \in S,$$

$$J_2 = \lim_{\substack{\beta \to d- \\ \beta \in S}} F(\beta) \qquad \text{if } d \in D$$

exist and are finite, then $f(x)$ *will be (CPS)-integrable on* $[c, d]$ *and*

$$(CPS)\int\limits_{c}^{d} f(x)\,d\omega = J_1 \qquad \text{if } d \in S$$

and

$$(CPS)\int\limits_{c}^{d} f(x)\,d\omega = J_2 + \tfrac{1}{2}f(d)[\omega(d+)-\omega(d-)] \qquad \text{if } d \in D.$$

Using similar arguments, the following theorem can be proved:

THEOREM 3.6. *Suppose the function* $f(x)$ *defined on* $[a, b]$ *is (CPS)-integrable on every segment* $[\alpha, d]$, *where* $a \leqslant c < \alpha < d \leqslant b$ *having (CPS)-integral* $F(x)$ *which is also (PS)-integrable on* $[c, d]$. *If* $f(c)$ *is finite when* $c \in D$ *and if the limits*

$$K_1 = \lim_{\substack{\alpha \to c+ \\ \alpha \in S}} (\omega)\,C(F;\ c, \alpha) \qquad \text{if } c \in S,$$

$$K_2 = \lim_{\substack{\alpha \to c+ \\ \alpha \in S}} F(\alpha) \qquad \text{if } c \in D$$

exist and are finite, then $f(x)$ *will be (CPS)-integrable on* $[c, d]$ *and*

$$(CPS)\int\limits_{c}^{d} f(x)\,d\omega = K_1 \qquad \text{if } c \in S$$

and

$$(CPS)\int_c^d f(x)\,d\omega = K_2+\tfrac{1}{2}[\omega(c+)-\omega(c-)] \quad \textit{if } c\in D.$$

4. The (*CDS*)-integral. Here we shall introduce the definition of (*CDS*)-integral and shall prove a few important properties.

DEFINITION 4.1. Let $f(x)$ be a function defined on $[a, b]$. If there exists a function $F(x)\in\mathscr{U}_1$ which is $ACG^*-\omega$ (*C*-sense) on $[a, b]$ and which is such that $CDF_\omega(x) = f(x)$ ω-almost everywhere on $[a, b]S$ and $F(x+)-F(x-) = f(x)\,[\omega(x+)-\omega(x-)]$ for $x\in D$, then $f(x)$ is said to be *special Cesàro–Denjoy–Stieltjes integrable* [or, (*CDS*)-*integrable*] on $[a, b]$ and the function $F(x)$ is called *indefinite* (*CDS*)-*integral* of $f(x)$ on $[a, b]$; the difference $F(b)-F(a)$ is termed definite (*CDS*)-integral of $f(x)$ over $[a, b]$ and is denoted by $(CDS)\int_a^b f(x)\,d\omega$.

It follows by Theorem 2.4 that if $F(x)$ and $G(x)$ are any two indefinite (*CDS*)-integrals of $f(x)$ on $[a, b]$, then $F(x)-G(x)$ is constant on $[a, b]$. The definite (*CDS*)-integral of a function $f(x)$, (*CDS*)-integrable on $[a, b]$ is therefore unique.

THEOREM 4.1. *A function $f(x)$ which is (CDS)-integrable on $[a, b]$ is ω-measurable on $[a, b]$.*

Proof. Let $F(x)$ be an indefinite (*CDS*)-integral of $f(x)$. Then $CDF_\omega(x) = f(x)$ ω-almost everywhere in $[a, b]S$. So by Theorem 2.1 $f(x)$ is ω-measurable on $[a, b]S$. Since the set D is at most denumerable, $f(x)$ is ω-measurable on $[a, b]$.

THEOREM 4.2. *A function $f(x)$ which is (CDS)-integrable on $[a, b]$ is finite ω-almost everywhere.*

Proof. The proof follows from Theorem 2.3 and Definition 4.1.

5. The (*CDS*)-integral includes the (*CPS*)-integral.

Preliminary lemmas. Let $F(x)\in\mathscr{U}_1$ be $(\omega)C$-continuous on $[a, b]$ and non-decreasing on each of the open intervals $(a_i, b_i)\subset S_0$ and let for every natural number n, E_n denote the set of points x of $[a, b]$ such that for $x+h\in S$ with $|h| < 1/n$ we have

$$(1)\qquad (\omega)C(F;\ x, x+h)-F(x) \geqslant -\tfrac{1}{2}n[\omega(x+h)-\omega(x-)], \qquad h > 0;$$

$$(2)\qquad F(x)-(\omega)C(F;\ x, x+h) \geqslant -\tfrac{1}{2}n[\omega(x+)-\omega(x+h)], \qquad h < 0.$$

Let

$$G(x) = \begin{cases} 0, & x = a,\\ (PS)\int_a^x F(t)\,d\omega, & a < x \leqslant b.\end{cases}$$

LEMMA 5.1. *If* $\{\alpha_k\}$ *is a convergent sequence of points of* E_n *and if the limit of the sequence belongs to* S, *then*

$$\lim_{\alpha_k\to\alpha} F(\alpha_k) = F(\alpha).$$

Proof. Choose $h > 0$ with $h < 1/n$ such that $\alpha+h \in S$. We consider those α_k for which $\alpha_k+h_k = \alpha+h$, $0 < h_k < 1/n$. From (1) we get

$$(\omega)C(F;\ \alpha_k, \alpha+h) - F(\alpha_k) \geqslant -\tfrac{1}{2}n[\omega(\alpha+h)-\omega(\alpha_k-)]. \tag{3}$$

Case (a). Let $\omega(\alpha+h)-\omega(\alpha) \neq 0$. Letting $\alpha_k \to \alpha$ in (3) we get

$$(\omega)C(F;\ \alpha, \alpha+h) \geqslant \overline{\lim_{\alpha_k\to\alpha}}\, F(\alpha_k) - \tfrac{1}{2}n[\omega(\alpha+h)-\omega(\alpha)].$$

Since $F(x)$ is $(\omega)C$-continuous, taking limit as $h\to 0$ we get

$$F(\alpha) \geqslant \overline{\lim_{\alpha_k\to\alpha}}\, F(\alpha_k).$$

Case (b). Let $\omega(\alpha+h)-\omega(\alpha) = 0$. Firstly, let $\alpha_k \to \alpha$ from the right. Then since on the right of α, $F(x)$ is continuous we have

$$F(\alpha) = \lim_{\alpha_k\to\alpha} F(\alpha_k)$$

and the lemma is proved. Next let $\alpha_k \to \alpha$ from the left. Then taking limit as $\alpha_k \to \alpha$ in (3), we get

$$F(\alpha) \geqslant \overline{\lim_{\alpha_k\to\alpha}}\, F(\alpha_k).$$

Therefore in any case we have

$$F(\alpha) \geqslant \overline{\lim_{\alpha_k\to\alpha}}\, F(\alpha_k). \tag{4}$$

Similarly choosing $h' < 0$ with $|h'| < 1/n$ and using relation (2) we get

$$F(\alpha) = \underline{\lim_{\alpha_k\to\alpha}}\, F(\alpha_k). \tag{5}$$

From (4) and (5)

$$F(\alpha) = \lim_{\alpha_k\to\alpha} F(\alpha_k).$$

This completes the proof of the lemma.

LEMMA 5.2. *If* $\alpha \in S$ *is a limit point of* E_n, *then* $\alpha \in E_n$ *and if* $\alpha \in D$ *is a limit point of* E_n *on the right, then relation* (1) *holds for* $x = \alpha$ *and relation* (2) *with* $F(x)$ *replaced by* $F(x+)$ *holds for* $x = \alpha$. *Further if* $\alpha \in D$ *is a limit point of* E_n *on the left, then relation* (2) *holds for* $x = \alpha$ *and relation* (1) *with* $F(x)$ *replaced by* $F(x-)$ *holds for* $x = \alpha$.

Proof. *Case* (i). Let the limit point α of E_n belong to S. Suppose $\{\alpha_k\}$ is a convergent sequence of points of E_n of which α is the limit. Choosing $h > 0$ with $h < 1/n$ we get as in Lemma 5.1

$$(6)\qquad (\omega)C(F;\ \alpha_k, \alpha+h) - F(\alpha_k) \geqslant -\tfrac{1}{2}n[\omega(\alpha+h)-\omega(\alpha_k-)].$$

We can suppose that $\omega(\alpha+h)-\omega(\alpha) \neq 0$. Otherwise it is clear that $F(\alpha+h)-F(\alpha) \geqslant 0$ and so

$$(\omega)C(F;\ \alpha, \alpha+h)-F(\alpha) \geqslant -\tfrac{1}{2}n[\omega(\alpha+h)-\omega(\alpha)].$$

Now as $\alpha_k \to \alpha$ we get from (6) using Lemma 5.1

$$(\omega)C(F;\ \alpha, \alpha+h)-F(\alpha) \geqslant -\tfrac{1}{2}n[\omega(\alpha+h)-\omega(\alpha)].$$

Thus α satisfies relation (1). Similarly we can show that α satisfies (2). So $\alpha \in E_n$.

Case (ii). Next, let $\alpha \in D$ be a limit point of E_n on the right. Suppose $\{\alpha_k\}$ is a sequence of points of E_n converging from right to α. Choose $h' < 0$ with $|h'| < 1/n$ such that $\alpha+h' \in S$. In this case we can choose h'_k with $|h'_k| < 1/n$ corresponding to each α_k for sufficiently large k such that $\alpha_k+h'_k = \alpha+h'$. We have

$$F(\alpha_k)-(\omega)C(F;\ \alpha_k, \alpha+h') \geqslant -\tfrac{1}{2}n[\omega(\alpha_k+)-\omega(\alpha+h')].$$

Letting k tend to infinity we get

$$(7)\qquad F(\alpha+)-(\omega)C(F;\ \alpha, \alpha+h') \geqslant -\tfrac{1}{2}n[\omega(\alpha+)-\omega(\alpha+h')],$$

from which we get as $h' \to 0$

$$(8)\qquad F(\alpha+)-F(\alpha) \geqslant -\tfrac{1}{2}n[\omega(\alpha+)-\omega(\alpha-)].$$

Now choose $h > 0$ with $0 < h < 1/n$. Then for $\alpha+h \in S$ we get as above

$$(\omega)C(F;\ \alpha_k, \alpha+h)-F(\alpha_k) \geqslant -\tfrac{1}{2}n[\omega(\alpha+h)-\omega(\alpha_k-)]$$

and so for sufficiently large k for which $\omega(\alpha+h)-\omega(\alpha_k-) \neq 0$ we have

$$G(\alpha+h)-G(\alpha_k-)-F(\alpha_k)[\omega(\alpha+h)-\omega(\alpha_k-)] \geqslant -\tfrac{1}{2}n[\omega(\alpha+h)-\omega(\alpha_k-)]^2.$$

Letting k tend to infinity we have

$$(9)\quad G(\alpha+h)-G(\alpha+)-F(\alpha+)[\omega(\alpha+h)-\omega(\alpha+)] \geqslant -\tfrac{1}{2}n[\omega(\alpha+h)-\omega(\alpha+)]^2.$$

From (8) and (9) and the relation

$G(\alpha+)-G(\alpha-) = F(\alpha)[\omega(\alpha+)-\omega(\alpha-)]$ [by result (ii) of (PS)-integral],

we get

$$(10)\qquad (\omega)C(F;\ \alpha, \alpha+h)-F(\alpha) \geqslant -\tfrac{1}{2}n[\omega(\alpha+h)-\omega(\alpha+)] > -\tfrac{1}{2}n[\omega(\alpha+h)-\omega(\alpha-)].$$

If for all $k, \omega(\alpha+h)-\omega(\alpha_k-) = 0$, then $\omega(\alpha+h)-\omega(\alpha+) = 0$ and so

$$(\omega)\, C(F;\ \alpha, \alpha+h) = F(\alpha),$$

and again we get relation (10) which together with relation (7) prove the relevant assertions made in the lemma.

Case (iii). The case when $\alpha \in D$ is a limit point of E_n on the left, can be treated as in case (ii). This completes the proof of the lemma.

THEOREM 5.1. *A function $f(x)$ which is (CPS)-integrable on $[a, b]$ is (CDS)-integrable on $[a, b]$ and*

$$(CDS)\int_a^b f(x)\, d\omega = (CPS)\int_a^b f(x)\, d\omega.$$

Proof. Let $F(x)$ be the indefinite (CPS)-integral of $f(x)$ on $[a, b]$. Let $\varepsilon > 0$ be chosen arbitrarily. Then $f(x)$ has a (CPS)-major function $U(x)$ and a (CPS)-minor function $V(x)$ such that $U(b)-F(b) < \varepsilon/3$ and $F(b)-V(b) < \varepsilon/3$. Let for every natural number m, A_m denote the set of points x of $[a, b]$ such that for $x+h \in S$ with $|h| < 1/m$ we have

$$(11)\quad (\omega)\, C(U; x, x+h) - U(x) \geqslant -\tfrac{1}{2} m [\omega(x+h)-\omega(x-)], \quad h > 0,$$

$$(12)\quad U(x) - (\omega)\, C(U; x, x+h) \geqslant -\tfrac{1}{2} m [\omega(x+)-\omega(x+h)], \quad h < 0;$$

and let for every natural number n, B_n denote the set of points x of $[a, b]$ such that for $x+h \in S$ with $|h| < 1/n$ we have

$$(13)\quad (\omega)\, C(V;\ x, x+h) - V(x) \leqslant \tfrac{1}{2} n [\omega(x+h)-\omega(x-)], \quad h > 0,$$

$$(14)\quad V(x) - (\omega)\, C(V;\ x, x+h) \leqslant \tfrac{1}{2} n [\omega(x+)-\omega(x+h)], \quad h < 0.$$

Let $E_{mn} = A_m B_n$, $p = \max(m, n)$ and E_{mnj} denote the common part of E_{mn} and the closed interval $[j/p+1, j+1/p+1]$. Then

$$[a, b] = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} \bar{E}_{mnj}.$$

Now we shall show that $F(x)$ is $AC^*-\omega$ (C-sense) on $\bar{E}_{mnj}$. Let $\{(c_r, d_r)\}$ be any set of non-overlapping intervals having end-points in $\bar{E}_{mnj}$.

Case (a). Let c_r be a point of E_{mnj} or a limit point of E_{mnj} in case $c_r \in S$ or else a limit point of E_{mnj} on the right when $c_r \in D$. Then for $c_r < x \leqslant d_r$ with $\omega(x+)-\omega(c_r-) \neq 0$

$$(15)\quad (\omega)\, C(F;\ c_r, x) - F(c_r)$$
$$= (\omega)\, C(U;\ c_r, x) - U(c_r) -$$
$$- \frac{1}{\omega(x+)-\omega(c_r-)} (PS)\int_{c_r}^{x} [U(t)-F(t)]\, d\omega + U(c_r) - F(c_r)$$
$$\geqslant (\omega)\, C(U;\ c_r, x) - U(c_r) - [U(d_r)-F(d_r)] + [U(c_r)-F(c_r)]$$

[by Theorem 3.2]

$$\geqslant -\tfrac{1}{2}m[\omega(x+)-\omega(c_r-)]-[U(d_r)-F(d_r)]+[U(c_r)-F(c_r)]$$

[by (11) and Lemma 5.2].

Relation (15) is obviously satisfied when $\omega(x+)-\omega(c_r-) = 0$. Hence

$$\underline{\text{bound}}_{c_r<x\leqslant d_r}[(\omega)C(F;\ c_r, x)-F(c_r)]$$
$$\geqslant -\tfrac{1}{2}m[\omega(d_r+)-\omega(c_r-)]-[U(d_r)-F(d_r)]+[U(c_r)-F(c_r)].$$

Case (b). If $c_r \in D$ be a limit point of E_{mnj} on the left, we can show in a similar way

$$\underline{\text{bound}}_{c_r<x\leqslant d_r}[(\omega)C(F;\ c_r, x)-F(c_r-)]$$
$$\geqslant -\tfrac{1}{2}m[\omega(d_r+)-\omega(c_r-)]-[U(d_r)-F(d_r)]+[U(c_r-)-F(c_r-)].$$

Therefore

$$\text{(16)}\quad \sum^{(1)}\underline{\text{bound}}_{c_r<x\leqslant d_r}[(\omega)C(F;\ c_r, x)-F(c_r)]+\sum^{(2)}\underline{\text{bound}}_{c_r<x\leqslant d_r}[(\omega)C(F;\ c_r, x)-F(c_r-)]$$
$$\geqslant -\tfrac{1}{2}m\sum_r[\omega(d_r+)-\omega(c_r-)]-2[\{U(b)-F(b)\}-\{U(a)-F(a)\}]$$
$$> -\tfrac{1}{2}m\sum_r[\omega(d_r+)-\omega(c_r-)]-2\varepsilon/3 > -\varepsilon$$

provided

$$\sum_r[\omega(d_r+)-\omega(c_r-)] < 2\varepsilon/3m,$$

where $\sum^{(1)}$ and $\sum^{(2)}$ denote the summations over r for cases (a) and (b) respectively.

Similarly, using relation (13) and a result analogous to Lemma 5.2 corresponding to the set B_n, which obviously holds, we get

$$\text{(17)}\quad \sum^{(1)}\overline{\text{bound}}_{c_r<x\leqslant d_r}[(\omega)C(F;\ c_r, x)-F(c_r)]+$$
$$+\sum^{(2)}\overline{\text{bound}}_{c_r<x\leqslant d_r}[(\omega)C(F;\ c_r, x)-F(c_r-)] < \varepsilon$$

provided

$$\sum_r[\omega(d_r+)-\omega(c_r-)] < 2\varepsilon/3n.$$

Combining (16) and (17) we get

$$\text{(18)}\quad \sum^{(1)}\overline{\text{bound}}_{c_r<x\leqslant d_r}|(\omega)C(F;\ c_r, x)-F(c_r)|+$$

$$+\sum^{(2)} \overline{\text{bound}}_{c_r < x \leq d_r} |(\omega) C(F;\ c_r, x) - F(c_r -)| < \varepsilon$$

provided

$$\sum_r [\omega(d_r +) - \omega(c_r -)] < \delta,$$

where

$$\delta = \min(2\varepsilon/3m, 2\varepsilon/3n).$$

From (18) we get

$$\sum^{(2)} |F(c_r) - F(c_r -)| \leq \varepsilon. \tag{19}$$

So from (18) and (19) we get

$$\sum_r \overline{\text{bound}}_{c_r < x \leq d_r} |(\omega) C(F;\ c_r, x) - F(c_r)| < 2\varepsilon$$

provided

$$\sum_r [\omega(d_r +) - \omega(c_r -)] < \delta.$$

Similarly using relations (12) and (14) we get

$$\sum_r \overline{\text{bound}}_{c_r \leq x < d_r} |(\omega) C(F;\ d_r, x) - F(d_r)| < 2\varepsilon$$

provided

$$\sum_r [\omega(d_r +) - \omega(c_r -)] < \delta.$$

It follows that $F(x)$ is $AC^* - \omega$ (C-sense) on $\bar{E}_{mnj}$. Since each $\bar{E}_{mnj}$ is closed and since (by Theorem 3.1), $F(x)$ is $(\omega)C$-continuous on $[a, b]$, $F(x)$ is $ACG^* - \omega$ (C-sense) on $[a, b]$. Again by Theorem 3.3, $CDF_\omega(x) = f(x)$ ω-almost everywhere in $[a, b]S$ and $F(x+) - F(x-) = f(x)[\omega(x+) - \omega(x-)]$ for $x \in D$, and so $f(x)$ is (CDS)-integrable on $[a, b]$ and

$$(CDS)\int_a^b f(x)\, d\omega = F(b) - F(a) = (CPS)\int_a^b f(x)\, d\omega.$$

This completes the proof of the theorem.

6. The (CPS)-integral includes the (CDS)-integral.

LEMMA 6.1. *If $F(x) \in \mathscr{U}_1$ is $\overline{AC} - \omega$ on a closed set Q, then it is BV on Q.*

The proof can be completed by proceeding as in the proof of Theorem 5 [1].

LEMMA 6.2. *If a function $F(x)$ is BV on $[a, b]$, then $F'_\omega(x)$ exists finitely ω-almost everywhere on $[a, b]S$ and is summable (LS) on $[a, b]S$.*

The proof follows by usual arguments (cf. [11], Theorem 5.14 and [6], Theorem 6.3).

THEOREM 6.1. *If a function $f(x)$ is (CDS)-integrable on $[a, b]$, then it is (CPS)-integrable on $[a, b]$.*

Proof. Let $F(x)$ be an indefinite (CDS)-integral of $f(x)$ on $[a, b]$. Let K be the set of points x of $[a, b]$ throughout no closed neighbourhood of which $f(x)$ is (CPS)-integrable. Then it is easily seen that K is a closed set. We now show that K is a null set. To prove this we assume that K is not null. Let (α_r, β_r) be any complementary interval of K and let p_r, q_r be two points of S such that $\alpha_r < p_r < q_r < \beta_r$. Then $f(x)$ is (CPS)-integrable on $[p_r, q_r]$ and by Theorem 5.1

$$(CPS)\int_{p_r}^{q_r} f(t)\,d\omega = F(q_r)-F(p_r).$$

Since $F(x)\in\mathcal{U}_1, F(q_r)$ and $F(p_r)$ tend to finite limits as q_r, p_r tend to β_r, α_r respectively when $\beta_r, \alpha_r \in D$ and since $F(x)$ is $(\omega)C$-continuous

$$\lim_{\substack{x\to\beta_r\\ x\in S}} (\omega)C(F;\ \beta_r, x) = F(\beta_r)$$

and

$$\lim_{\substack{x\to\alpha_r\\ x\in S}} (\omega)C(F;\ \alpha_r, x) = F(\alpha_r)$$

and hence by Theorems 3.5 and 3.6, $f(x)$ is (CPS)-integrable on $[\alpha_r, \beta_r]$. Therefore K has no isolated points. Since $F(x)$ is $ACG^*-\omega$ (C-sense] on $[a, b]$, there exist a countable number of closed sets E_n such that $[a, b] = \sum_n E_n$ and $F(x)$ is $AC^*-\omega$ (C-sense) on each E_n. Since $K = \sum KE_n$, there exists, by Baire's theorem, a closed interval $[l, m]$ and a positive integer n such that $K(l, m)$ is not null and $K[l, m] = KE_n[l, m] = Q$ (say). Thus $F(x)$ is $AC^*-\omega$ (C-sense) on Q. Let $[c, d]$ be the smallest closed interval containing Q. Denote the component intervals of $[c, d]-Q$ by $\{(c_n, d_n)\}$. We now define the function $G(x)$ as follows:

$$G(x) = \begin{cases} F(x) & \text{for } x\in Q,\\ \bar{F}(c_n)+\dfrac{\bar{\omega}(x)-\omega(c_n+)}{\omega(d_n-)-\omega(c_n+)}\{\bar{F}(d_n)-\bar{F}(c_n)\} & \\ & \text{for } c_n < x < d_n,\ \omega(d_n-)\neq\omega(c_n+),\\ \bar{F}(c_n) = \bar{F}(d_n) & \text{for } c_n < x < d_n,\ \omega(d_n-)=\omega(c_n+),\\ F(c) & \text{for } x < c,\\ F(d) & \text{for } x > d; \end{cases}$$

where

$$\bar{F}(c_n) = \begin{cases} F(c_n) & \text{when } c_n\in S,\\ F(c_n+) & \text{when } c_n\in D; \end{cases}$$

and

$$\bar{F}(d_n) = \begin{cases} F(d_n) & \text{when } d_n \in S, \\ F(d_n -) & \text{when } d_n \in D. \end{cases}$$

Since $F(x)$ is $AC^* - \omega$ (C-sense) on Q, it is $\overline{AC} - \omega$ on Q and so by Lemma 6.1, it is BV on Q. Therefore (cf. [7], Theorem 3.1) $G(x)$ is BV on $[c, d]$ and so by Lemma 6.2 $G'_\omega(x)$ exists finitely ω-almost every where in $[c, d] S$. Now $G'_\omega(x) = (ap) F'_\omega(x)$ ω-almost everywhere in QS. Therefore by Theorem 2.3 $G'_\omega(x) = CDF'_\omega(x) = f(x)$ ω-almost everywhere in QS. Therefore by Lemma 6.2, $f(x)$ is summable (LS) on QS. Again

$$\frac{G(x+) - G(x-)}{\omega(x+) - \omega(x-)} = \frac{F(x+) - F(x-)}{\omega(x+) - \omega(x-)} = f(x)$$

for $x \in QD$. Therefore $f(x)$ is summable (LS) on QD. It follows that $f(x)$ is summable (LS) on Q. Since $F(x)$ is $AC^* - \omega$ (C-sense) on Q, by Theorem 2.2,

$$\sum_n \overline{\underset{c_n < x \leqslant d_n}{\text{bound}}} |(\omega) C(F;\ c_n, x) - F(c_n)| < \infty$$

and

$$\sum_n \overline{\underset{c_n \leqslant x < d_n}{\text{bound}}} |(\omega) C(F;\ d_n, x) - F(d_n)| < \infty$$

and so

$$\sum_n \overline{\underset{c_n < x \leqslant d_n}{\text{bound}}} |(\omega) C(F_n;\ c_n, x)| < \infty$$

and

$$\sum_n \overline{\underset{c_n \leqslant x < d_n}{\text{bound}}} |(\omega) C(F_n;\ d_n, x) - F_n(d_n)| < \infty,$$

where $F_n(x) = F(x) - F(c_n)$. Therefore by Theorem 3.4, $f(x)$ is (CPS)-integrable in $[c, d]$. This is clearly impossible, since c and d are end-points of a closed subset of K. The set K must therefore be null. This completes the proof of the theorem.

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