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The converse of Lapunov convexity theorem

Abstract. In this note we show that the Lapunov convexity theorem fails for every infinite-dimensional F-space.

The well-known Lapunov convexity theorem states that the range of every non-atomic finite-dimensional vector measure is convex and compact (see e.g. [1]). There are also known some examples of non-atomic vector measures with values in concrete infinite-dimensional locally convex spaces with non-convex or non-closed range (see e.g. [1], [6], [10]). In this note, by a slight modification of the construction in [1], p. 279–280, we prove the following theorem:

THEOREM. *Let E be an arbitrary F-space (i.e., a metrizable complete topological linear space). If every non-atomic measure defined on an arbitrary σ-algebra, with values in E, has either convex or closed range, then E is finite dimensional.*

Proof. Suppose that E is infinite dimensional and let $|\cdot|$ be an F-norm (cf. [8]) defining the topology of E. Let (x_n) be a quasi-basic sequence in E (i.e., $\sum_{n=1}^{\infty} t_n x_n = 0$ implies $t_n = 0$ for every n; cf. [2], [3]). We can assume that $|x_n| \leqslant 2^{-n}$. It is easy to observe that if $(c_k)_{k=1}^{\infty} \subset l^{\infty}$, $\|c_k\|_{l^{\infty}} \leqslant 1$ for every k and $c_k(n) \to c(n)$, then $\sum_{n=1}^{\infty} c_k(n) x_n \to \sum_{n=1}^{\infty} c(n) x_n$ as $k \to \infty$.

Let (g_n) be a sequence in $L^1[0, 1]$ such that $\|g_n\|_{L^1} \leqslant 1$ and the linear span of (g_n) is dense in L^1. For every natural number n we define a measure m_n on the σ-algebra Σ of Lebesgue measurable subsets of the interval $S = [0, 1]$ by

$$m_n(A) = \int_A g_n \, d\mu,$$

where μ denotes the Lebesgue measure. Then the function $m\colon \Sigma \to E$ defined by

$$m(A) = \sum_{n=1}^{\infty} m_n(A) x_n$$

is easily seen to be a countably additive non-atomic measure (the latter property follows from the fact that $m(A) = 0$ iff $\mu(A) = 0$).

Now we show that $m(\Sigma)$ is non-convex and non-closed (cf. [1]). Let (B_j) be a sequence of sets such that $1_{B_j} \to \frac{1}{2}$ star-weakly in $L^\infty[0,1]$. For every n we get:

$$m_n(B_j) = \int_S 1_{B_j} g_n \, d\mu \to \int_S \tfrac{1}{2} g_n \, d\mu = \tfrac{1}{2} m_n(S)$$

so $m(B_j) \to \frac{1}{2} m(S)$.

If $m(\Sigma)$ were a closed (or convex) set, there would exist a set C such that $m(C) = \frac{1}{2} m(S)$; in other words,

$$\int_S 1_C g_n \, d\mu = m_n(C) = \tfrac{1}{2} m_n(S) = \int_S \tfrac{1}{2} g_n \, d\mu$$

for every n, because (x_n) is a quasi-basic sequence. Since the linear span of (g_n) is dense in L^1, $1_C = \frac{1}{2}$ μ almost everywhere which is impossible. Therefore $\frac{1}{2} m(S) \in \overline{m(\Sigma)} \setminus m(\Sigma)$ and $\frac{1}{2} m(S) \in \operatorname{conv} m(\Sigma) \setminus m(\Sigma)$.

Remarks. 1° If E is a Banach space, then m has a finite total variation, if, moreover, each (g_n) is the characteristic function of a subset of S, then m is an indefinite Bochner integral (cf. [7], [9]).

2° $\overline{m(\Sigma)}$ is compact (cf. [4], Proposition 3.1). If, moreover, E is locally convex space, then $\overline{m(\Sigma)}$ is convex because $\overline{m(\Sigma)}^{\sigma} = \overline{\operatorname{conv}} \, m(\Sigma)$ (see e.g. [3], Lemma 5).

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