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On the boundedness of solutions of non-linear differential equations in Banach spaces

The purpose of the paper is to prove the existence and some properties of bounded solutions of the non-linear differential equation

$$\dot{x} = A(t)x + f(t, x)$$

under the assumption that the linear equation

$$\dot{x} = A(t)x + b(t)$$

has at least one bounded solution for each function b belonging to a Banach function space B. Here x represents a function with values in some Banach space and the real independent variable t ranging over $\langle 0, \infty)$. Our results generalize some theorems due to Massera, Schäffer [3], Coppel [1] and Talpalaru [7].

In this paper we use some of the notation, definitions, and results from the book of Massera–Schäffer [4].

Let $J = [0, \infty)$, and let E be a Banach space with the norm $\|\cdot\|$. We introduce the following notations:

$\tilde{E}$ – the space of continuous linear mappings $E \to E$;

$C = C(J, E)$ – the space of bounded continuous functions $u: J \to E$ with the norm $\|u\|_c = \sup\{\|u(t)\|: t \in J\}$;

$L^1 = L^1(J, E)$ – the space of Bochner integrable functions $u: J \to E$ with the norm $\|u\|_1 = \int\limits_0^\infty \|u(s)\|\, ds$;

$L = L(J, E)$ – the space of strongly measurable functions $u: J \to E$, Bochner integrable in every finite subinterval J' of J, with the topology of the convergence in the mean on every such J', i.e. convergence in $L^1(J', E)$ of the restrictions to J'.

Assume that $B(J, R)$ is a Banach function space such that

1° $B(J, R) \subset L(J, R)$ and $B(J, R)$ is stronger than $L(J, R)$;

2° $B(J, R)$ is not stronger than $L^1(J, R)$;

3° $B(J, R)$ contains all essentially bounded function with compact support;

4° if $u \in B(J, R)$ and v is a real-valued measurable function on J such that $|v| \leqslant |u|$, then $v \in B(J, R)$ and $\|v\|_B \leqslant \|u\|_B$.

Denote by $B = B(J, E)$ the Banach space of all strongly measurable functions u: $J \to E$ such that $\|u\| \in B(J, R)$ provided with the norm $\|u\|_B = \| \|u\| \|_B$.

Further, let $A \in L(J, \tilde{E})$, and let E_0 be the set of all points of E which are values for $t = 0$ of bounded solutions of the linear differential equation

$$\dot{x} = A(t)x. \tag{1}$$

We assume that E_0 is closed and has a closed complement E_1, i.e. there exists a closed subspace E_1 such that E is the direct sum of E_0 and E_1. Let P be the projection of E onto E_0. Moreover, let U: $J \to \tilde{E}$ be the solution of the differential equation $\dot{U} = A(t)U$ with the initial condition $U(0) = I$.

Assume that for every $b \in B$ there exists at least one bounded solution of the differential equation

$$\dot{x} = A(t)x + b(t). \tag{2}$$

Then by Theorem 51.E of [4] there exists a constant $k > 0$ such that for every $b \in B$ the equation (2) has a unique bounded solution x with $x(0) \in E_1$, and this solution satisfies $\|x\|_c \leqslant k\|b\|_B$. For any $b \in B$ denote by $T(b)$ the bounded solution x of (2) such that $x(0) \in E_1$. Then T is a mapping of B into C and

1° $\|T(b)\|_c \leqslant k\|b\|_B$ for $b \in B$;

2° $T(\lambda_1 b_1 + \lambda_2 b_2) = \lambda_1 T(b_1) + \lambda_2 T(b_2)$ for $b_1, b_2 \in B$ and $\lambda_1, \lambda_2 \in R$.

Moreover, by [4], Theorem 52.J,

$$T(b)(t) = \int_0^t U(t)PU^{-1}(s)b(s)ds - \int_t^\infty U(t)(I-P)U^{-1}(s)b(s)ds \quad (t \in J) \tag{3}$$

for every $b \in B$ with compact support.

Applying Theorem 62.D of [4] we deduce that there exist a positive-valued function N defined on J and a positive constant α such that every solution x of (1) with $x(0) \in E_0$ satisfies, for all $t \geqslant t_0 \geqslant 0$,

$$\|x(t)\| \leqslant N(t_0)e^{-\alpha(t-t_0)}\|x(t_0)\|,$$

and the fundamental solution U of (1) satisfies

$$\|U(t)P\| \leqslant N(0)\|P\|e^{-\alpha t} \quad \text{for all } t \in J. \tag{4}$$

In what follows we shall make use of the well-known Krasnoselskiĭ theorem [2], p. 57:

Suppose that F is a mapping of a complete metric space $\langle X, d\rangle$ into itself and

$$d\big(F(x), F(y)\big) \leqslant q(a,b)\,d(x,y)$$

for each $x, y\in X$ such that $a \leqslant d(x,y) \leqslant b$, where $q(a,b) < 1$ for $b \geqslant a > 0$. Then there exists a unique $u\in X$ such that $u = F(u)$.

Consider the non-linear differential equation

$$\dot{x} = A(t)x + f(t,x), \tag{5}$$

where $(t,x)\to f(t,x)$ is a function from $J\times E$ into E which is continuous in x for any fixed $t\in J$, and strongly measurable in t for any fixed $x\in E$.

THEOREM 1. *If*

1° $r\colon J\to J$ *is a non-decreasing function such that* $\sup\{r(u)/u\colon a\leqslant u\leqslant b\} < 1$ *for each* $a, b, 0 < a \leqslant b$;

2° *there exists* $h\in B(J,R)$ *such that* $k\|h\|_B \leqslant 1$ *and* $\|f(t,x)-f(t,y)\| \leqslant h(t)r(\|x-y\|)$ *for each* $x, y\in E$ *and* $t\in J$;

3° $f(\cdot, 0)\in B$,

then for any $a\in E_0$ *there exists a unique bounded solution* $x(\cdot, a)$ *of* (5) *with* $Px(0,a) = a$. *Moreover, for any* $\varepsilon > 0$ *there exists* $\mathfrak{z} > 0$ *such that*

$$\|x(\cdot,p)-x(\cdot,a)\|_c \leqslant \varepsilon \quad \text{for each } a, p\in E_0,\ \|p-a\| \leqslant \mathfrak{z}.$$

Proof. For each $x, y\in C$ the function $s\to f(s, x(s))$ is strongly measurable on J and

$$\|f(s,x(s))-f(s,y(s))\| \leqslant h(s)r(\|x-y\|_c) \quad \text{for } s\in J.$$

Therefore

$$\|f(\cdot,x)-f(\cdot,y)\|_B \leqslant \|h\|_B r(\|x-y\|_c) \quad \text{for } x, y\in C.$$

In particular this implies that $f(\cdot,x)\in B$, because

$$f(\cdot,x) = \big(f(\cdot,x)-f(\cdot,0)\big)+f(\cdot,0) \quad \text{and} \quad f(\cdot,0)\in B.$$

Put $H(v) = T\big(f(\cdot,v)\big)$ for $v\in C$. Then H is a mapping of C into C, and

$$\|H(v)-H(z)\|_c = \big\|T\big(f(\cdot,v)-f(\cdot,z)\big)\big\|_c \leqslant k\,\|f(\cdot,v)-f(\cdot,z)\|_B$$
$$\leqslant k\,\|h\|_B\, r(\|v-z\|_c) \leqslant r(\|v-z\|_c) \quad \text{for each } v, z\in C.$$

Fix $a\in E_0$ and put $S(v) = H(v+w)$ for $v\in C$, where $w = U(\cdot)a$. From the above it is clear that S is a mapping of C into C, and

$$\|S(v)-S(z)\|_c \leqslant r(\|v-z\|_c) \quad \text{for } v, z\in C.$$

Thus we can apply Krasnoselskiĭ's theorem which yields the existence of a unique mapping $x\in C$ such that $x = S(x)$, i.e.

$$\dot{x}(t) = A(t)x(t)+f\big(t, x(t)+w(t)\big) \quad \text{for } t\in J.$$

Since $\dot{w}(t) = A(t)w(t)$ for $t \in J$, the function $u = x+w$ is a bounded solution of (5). Moreover, $Pu(0) = Pw(0) = a$, because $x(0) \in E_1$. Suppose that z is a bounded solution of (5) such that $Pz(0) = a$. Then $y = z-w$ is a bounded solution of $\dot{y} = A(t)y+f(t, z(t))$ such that $y(0) \in E_1$. Consequently, $y = H(z) = H(y+w) = S(y)$, which implies $y = x$, and therefore $z = x+w = u$.

LEMMA. *If $f(\cdot, 0) = 0$ and $m = \sup\{\|U(t)P\|: t \in J\}$, then $\|x(\cdot, a)\|_c \leqslant d$ for each $d > 0$ and $a \in E_0$ such that $\|a\| \leqslant (d-r(d))/m$.*

Proof of lemma. If $f(\cdot, 0) = 0$, then $H(0) = 0$. Fix $d > 0$ and $a \in E_0$ such that $\|a\| \leqslant (d-r(d))/m$, and put $w = U(\cdot)a$. We shall show that the function $u \to S(u) = H(u+w)$ maps the ball $K_d = \{x \in C: \|x\|_c \leqslant r(d)\}$ into itself. Indeed, as $\|w\|_c \leqslant m\|a\| \leqslant d-r(d)$, $\|x+w\|_c \leqslant \|x\|_c + \|w\|_c \leqslant d$ for any $x \in K_d$, and hence

$$\|S(x)\|_c = \|H(x+w)-H(0)\|_c \leqslant r(\|x+w\|_c) \leqslant r(d) \quad \text{for } x \in K_d.$$

Applying Krasnoselskiĭ's theorem we deduce that there exists $v \in K_d$ such that $v = S(v)$. Since the equation $u = S(u)$ has exactly one solution in C, and $u = x(\cdot, a) - w$ satisfies this equation, we conclude that $v = x(\cdot, a) - w$, and finally $\|x(\cdot, a)\|_c \leqslant \|v\|_c + \|w\|_c \leqslant r(d)+d-r(d) = d$. This completes the proof of lemma.

For any $\varepsilon > 0$ put $\delta = (\varepsilon - r(\varepsilon))/m$. Then for any $a, p \in E_0$ such that $\|a-p\| \leqslant \delta$ the function $u = x(\cdot, a)-x(\cdot, p)$ is a bounded solution of the equation

$$\dot{x} = A(t)x+g(t, x),$$

where $g(t, y) = f(t, x(t, p)+y)-f(t, x(t, p))$ for $(t, y) \in J \times E$, and $Pu(0) = a-p$. Since $g(t, 0) = 0$ and

$$\|g(t, x)-g(t, y)\| \leqslant h(t)r(\|x-y\|) \quad \text{for } t \in J,\ x, y \in E,$$

from the above lemma it follows that $\|u(t)\| \leqslant \varepsilon$, i.e.

$$\|x(t, a)-x(t, p)\| \leqslant \varepsilon \quad \text{for } t \in J.$$

Remark 1. Theorem 1 is a generalization of an analogous result of Massera–Schäffer [3] for $B = L^p$ and $r(u) = qu, q < 1$.

THEOREM 2. *If*

1° *$r: J \to J$ is a non-decreasing right continuous function such that $r(0) = 0$ and $r(u) < u$ for $u > 0$;*

2° *there exists a $h \in B(J, R)$ such that $k\|h\|_B \leqslant 1$ and*

$$\|f(t, x)\| \leqslant h(t)r(\|x\|) \quad \textit{for each } (t, x) \in J \times E,$$

then every bounded solution x of (5) satisfies $\lim_{t\to\infty} \|x(t)\| = 0$.

Proof. Assume that x is a bounded solution of (5). First we shall show that

$$x = U(\cdot)Px(0)+T(f(\cdot, x)). \tag{6}$$

Let $z = T(f(\cdot, x))$ and $y = x-U(\cdot)Px(0)-z$. Since $z(0)\in E_1$, $y(0) = x(0)-Px(0)-z(0)\in E_1$. Moreover,

$$\begin{aligned}\dot{y}(t) &= \dot{x}(t)-\dot{U}(t)Px(0)-\dot{z}(t)\\ &= A(t)x(t)+f(t, x(t))-A(t)U(t)Px(0)-A(t)z(t)-f(t, x(t))\\ &= A(t)y(t) \quad \text{for } t\in J,\end{aligned}$$

and hence y is a bounded solution of (1) with $y(0)\in E_1$. Therefore $y = 0$ which proves (6).

For any $\tau > 0$ put

$$u_\tau = T(\chi_{[0,\tau]}f(\cdot, x)) \quad \text{and} \quad v_\tau = T(\chi_{[\tau,\infty)}f(\cdot, x)).$$

Because

$$\|\chi_{[\tau,\infty)}(t)f(t, x(t))\| \leqslant h(t)\chi_{[\tau,\infty)}(t)r(\|x(t)\|) \leqslant h(t)r(\sup_{t\geqslant\tau} \|x(t)\|)$$

for $t\in J$, we have

$$\|v_\tau\|_c \leqslant k\|\chi_{[\tau,\infty)}f(\cdot, x)\|_B \leqslant k\|h\|_B r(\sup_{t\geqslant\tau}\|x(t)\|) \leqslant r(\sup_{t\geqslant\tau}\|x(t)\|).$$

On the other hand, by (3),

$$u_\tau(t) = U(t)P\int_0^\tau U^{-1}(s)f(s, x(s))ds \quad \text{for } t \geqslant \tau,$$

and therefore

$$\|u_\tau(t)\| \leqslant \|U(t)P\|\cdot\|\int_0^\tau U^{-1}(s)f(s, x(s))ds\| \quad \text{for } t \geqslant \tau.$$

Let $p = \overline{\lim_{t\to\infty}}\|x(t)\|$. Suppose that $p > 0$. Since $r(p) < p$ and r is right continuous, there exists $\varepsilon > 0$ such that $r(p+\varepsilon) < p$. Moreover, by the definition of p, there exists $\tau > 0$ such that $\|x(t)\| \leqslant p+\varepsilon$ for $t \geqslant \tau$. As

$$x = U(\cdot)Px(0)+T(f(\cdot, x)) = U(\cdot)Px(0)+u_\tau+v_\tau,$$

$$\begin{aligned}\|x(t)\| &\leqslant \|U(t)P\|\cdot\|x(0)\|+r(\sup_{t\geqslant\tau}\|x(t)\|)+\|u_\tau(t)\|\\ &\leqslant \|U(t)P\|\cdot\|x(0)\|+r(p+\varepsilon)+\|U(t)P\|\cdot\|\int_0^\tau U^{-1}(s)f(s, x(s))ds\|\end{aligned}$$

for $t \geqslant \tau$. By (4) this implies $p = \overline{\lim_{t\to\infty}}\|x(t)\| \leqslant r(p+\varepsilon)$ in contradiction with $r(p+\varepsilon) < p$. Consequently, $p = 0$, which ends the proof of Theorem 2.

Remark 2. Theorem 2 generalizes some result of Coppel [1] for $B = L^p$, $E = R^n$ and $r(u) = qu$, $q < 1$.

THEOREM 3. *Assume that*

1° $\lim_{d\to\infty} \|\chi_{[d,\infty)} b\|_B = 0$ *for every* $b \in B(J, R)$;

2° $(t, u) \to h(t, u)$ *is a non-negative function defined for* $t, u \in J$ *such that*

(i) *for any fixed* $t \in J$ *the function* h *is non-decreasing on* u;

(ii) $h(\cdot, u) \in B(J, R)$ *for each fixed* u.

If $\|f(t, x)\| \leqslant h(t, \|x\|)$ *for each* $(t, x) \in J \times E$, *then every bounded solution of* (5) *satisfies* $\lim_{t\to\infty} \|x(t)\| = 0$.

Proof. Let x be a bounded solution of (5). For any $\tau > 0$ let u_τ and v_τ be the same as in the proof of Theorem 2. Since

$$\|\chi_{[\tau,\infty)}(t) f(t, x(t))\| \leqslant \chi_{[\tau,\infty)}(t) h(t, \|x\|_c) \quad \text{for } t \in J,$$

$$\|v_\tau\|_c \leqslant k \|\chi_{[\tau,\infty)} f(\cdot, x)\|_B \leqslant k \|\chi_{[\tau,\infty)} h(\cdot, \|x\|_c)\|_B.$$

By assumption 1°, $\lim_{\tau\to\infty} \|\chi_{[\tau,\infty)} h(\cdot, \|x\|_c)\|_B = 0$, and therefore for any $\varepsilon > 0$ we can choose $\tau > 0$ such that $\|v_\tau\|_c \leqslant \varepsilon/3$. Moreover, by (4), $\lim_{t\to\infty} \|U(t)P\| = 0$. Hence there exists a $t_0 > 0$ such that

$$\|u_\tau(t)\| \leqslant \|U(t)P\| \cdot \Big\| \int_0^\tau U^{-1}(s) f(s, x(s))\, ds \Big\|$$

$$\leqslant \varepsilon/3 \quad \text{and} \quad \|U(t)P\| \cdot \|x(0)\| \leqslant \varepsilon/3$$

for $t \geqslant t_0$. From this, by (6), it follows that

$$\|x(t)\| \leqslant \|U(t)P\| \cdot \|x(0)\| + \|u_\tau(t)\| + \|v_\tau(t)\| \leqslant \varepsilon \quad \text{for } t \geqslant t_0.$$

As ε is arbitrary, this implies $\lim_{t\to\infty} \|x(t)\| = 0$.

Remark 3. Theorem 3 generalizes a similar result of Talpalaru [7] for $E = R^n$ and $B = L^p$.

Remark 4. Our results may be applied to the important case, when B is any Orlicz space L_φ generated by a convex φ-function φ such that

$$\lim_{u\to 0} \varphi(u)/u = 0 \quad \text{and} \quad \lim_{u\to\infty} \varphi(u)/u = \infty$$

(cf. [5], [6]). Theorems 1 and 2 are true for every Orlicz space; Theorem 3 is true only in this case where the function φ satisfies the condition $\varphi(2u) \leqslant c\varphi(u)$for each $u \geqslant 0$.

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