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LECH MALIGRANDA (Poznań)

Interpolation of Lipschitz operators for the pairs of spaces (L^1, L^∞) and (l^1, c_0)

Abstract. Let μ_1 be a σ-finite non-atomic and μ_2 a discrete measure. Let T: $L^1(\mu_i)+L^\infty(\mu_i) \to L^1(\mu_i)+L^\infty(\mu_i)$ be such that T is a Lipschitz operator from $L^1(\mu_i)$ to $L^1(\mu_i)$ and from $L^\infty(\mu_i)$ to $L^\infty(\mu_i)$, where $L^\infty(\mu_2) = c_0$ and $L^\varrho(\mu_i)$ is a maximal rearrangement invariant Banach function space such that $L^\infty(\mu_1) \subset L^\varrho(\mu_1) \subset L^1(\mu_1)$, $L^\varrho(\mu_2) = l^\varrho \subset c_0$, or $L^\varrho(\mu_i)$ is a minimal rearrangement invariant Banach function space. Then T or its extension $\hat{T}$ is Lipschitz from $L^\varrho(\mu_i)$ to $L^\varrho(\mu_i)$ for $i = 1, 2$, and the bound K_ϱ (or respectively $\hat{K}_\varrho$) does not exceed $\max(K_1, K_\infty)$.

0. Introduction. This theorem was proved by W. Orlicz [12] for Orlicz spaces $L^\varphi(0, l)$ in the case where $l < \infty$ for linear operators, and later [13] for Lipschitz operators. In [4], the theorem was proved for maximal spaces $L^\varrho(0, 1)$ and linear operators given by an integral transformation. In [10] Mitjagin proved the theorem for linear operators and a minimal rearrangement invariant Banach function space $L^\varrho(0, 1)$. Moreover, Mitjagin was the first who proved this theorem for linear operators and minimal sequence spaces l^ϱ.

Calderón [3] established this theorem in 1966 for quasi-linear operators and every $L^\varrho(\mu)$ having the majorant property. Next, Lorentz and Shimogaki [5], [6] proved it in the case of Lipschitz operators and for every maximal $L^\varrho(0, l)$ when $l < \infty$, and for minimal $L^\varrho(0, \infty)$.

In this paper we show that this theorem holds if μ is an adequate measure and $L^\varrho(\mu)$ is a maximal rearrangement invariant Banach function space such that $L^\infty(\mu) \subset L^\varrho(\mu) \subset L^1(\mu)$ if μ non-atomic and $l^\varrho \subset c_0$ if μ is discrete, or $L^\varrho(\mu)$ is a minimal rearrangement invariant Banach function space. The idea of the proof is taken from Lorentz–Shimogaki [6] in the case of non-atomic measures. We give also another proof of this theorem for a maximal rearrangement invariant Banach function space, applying the theory of Bennett [1] and Peetre [14].

1. Preliminaries. In the sequel, (S, Σ, μ) is a σ-finite measure space with a countably additive non-negative measure μ on a σ-algebra Σ of subsets of an abstract set S. By $M(P)$ we denote the space of all real-valued (resp. non-negative) μ-measurable functions on S, finite a.e. on S.

A mapping ϱ: $P \to [0, \infty]$ is called a *function norm* if ϱ satisfies the following conditions for all f, f_n $(n = 1, 2, \ldots)$ and g in P:

(i) $\varrho(f) = 0$ iff $f = 0$ a.e.

$$\varrho(f+g) \leqslant \varrho(f)+\varrho(g), \quad \varrho(\lambda f) = \lambda\varrho(f) \quad (\lambda \geqslant 0);$$

(ii) $f \leqslant g$ a.e. implies $\varrho(f) \leqslant \varrho(g)$;

If we identify μ-almost equal functions, then

$$L^{\varrho}(\mu) = \{f \in M: \varrho(|f|) < \infty\}$$

is a normed linear space with the norm $\|f\| = \varrho(|f|)$. Such a space is usually called a *normed Köthe space* or *normed function space* (for the theory of normed Köthe spaces, see [18]).

For each ϱ, $\varrho'(f) = \sup\{\int_S |fg|\,d\mu: \varrho(g) \leqslant 1\}$ is called the *associated norm* to ϱ; it satisfies conditions (i) and (ii). The *associated space*, denoted by $(L^{\varrho}(\mu))'$ or $L^{\varrho'}(\mu)$, is defined as

$$L^{\varrho'}(\mu) = \{f \in M: \varrho'(|f|) < \infty\}.$$

A function norm ϱ is called:

continuous if $f_n \in L^{\varrho}(\mu)$, $0 \leqslant f_n \downarrow 0$, imply $\varrho(f_n) \downarrow 0$,
semi-continuous if $0 \leqslant f_n \uparrow f$, $f \in L^{\varrho}(\mu)$, imply $\varrho(f_n) \uparrow \varrho(f)$,
monotone-complete if $0 \leqslant f_n \uparrow f$ and $\sup_n \varrho(f_n) < \infty$ imply $f \in L^{\varrho}(\mu)$.

If ϱ satisfies *Fatou property*, i.e. $f_0, f_1, \ldots \in P$ and $f_n \uparrow f$ a.e. implies $\varrho(f_n) \uparrow \varrho(f)$ (this is equivalent to semi-continuity and monotone-completeness of the norm), then $L^{\varrho}(\mu)$ is called a *maximal normed function space*, and if ϱ is a continuous norm, then $L^{\varrho}(\mu)$ is called a *minimal normed function space*.

In the sequel we consider *Banach function spaces* (complete normed function spaces) such that

$$L^1(\mu) \cap L^{\infty}(\mu) \subset L^{\varrho}(\mu) \subset L^1(\mu)+L^{\infty}(\mu).$$

It is clear that if μ is a non-atomic measure and $\mu(S) \leqslant \infty$ or μ is a discrete measure (i.e. purely atomic with atoms of equal measure 1), then

$$L^1(\mu) \cap L^{\infty}(\mu) \subset L^{\varrho}(\mu) \subset L^1(\mu)+L^{\infty}(\mu)$$

if and only if

(iii) $\mu(E) < \infty$ implies that there exists A_E independent of f such that $\int_E |f|\,d\mu \leqslant A_E\varrho(f)$,

(iv) $\mu(E) < \infty$ implies $\varrho(\chi_E) < \infty$.

For each μ-measurable function f on S, the function $d_f(y) = \mu\{x \in S:$

$|f(x)| > y\}$, $y > 0$, is called the *distribution function* of f. f_1 and f_2 are called *equimeasurable* if $d_{f_1} = d_{f_2}$ holds; we denote this by writing $f_1 \sim f_2$.

A function norm ϱ is called *rearrangement-invariant* if

(v) $f_1 \sim f_2$ implies $\varrho(f_1) = \varrho(f_2)$;

$L^{\varrho}(\mu)$ is then called a *rearrangement-invariant Banach function space.*

Examples of rearrangement invariant Banach function spaces are Lebesgue spaces L^q $(1 \leqslant p \leqslant \infty)$, Lorentz spaces $\Lambda, M, L^{pq}, \Lambda_\alpha$ $(0 < \alpha \leqslant 1)$ and Orlicz spaces L^{φ}.

The smallest and the largest of the rearrangement invariant maximal Banach function spaces with non-atomic or discrete measure are respectively $L^1(\mu) \cap L^{\infty}(\mu)$ and $L^1(\mu)+L^{\infty}(\mu)$, this in the sense that the continuous embeddings

$$L^1(\mu) \cap L^{\infty}(\mu) \subset L^{\varrho}(\mu) \subset L^1(\mu)+L^{\infty}(\mu)$$

hold for any rearrangement-invariant maximal Banach function space $L^{\varrho}(\mu)$ satisfying (iii) and (iv).

2. Majorant property. For each $f \in M$, f^* denotes the non-increasing rearrangement of f, which is the right continuous inverse of the function d_f, i.e.

$$f^*(t) \doteq \inf\{y > 0\colon d_f(y) \leqslant t\}, \qquad 0 \leqslant t < \mu(S) \leqslant \infty.$$

It is clear that

$$\begin{aligned} f^*(t) &= d^m_{d_f}(t) = |\{\lambda > 0\colon d_f(\lambda) > t > 0\}| \\ &= \sup\{\lambda > 0\colon d_f(\lambda) > t > 0\} \quad \text{for } 0 < t < \mu(S), \end{aligned}$$

and $f^*(0) = \operatorname{ess\,sup}_{x \in S} |f(x)|$. We write $g \prec f$ for f, $g \in L^1(\mu)+L^{\infty}(\mu)$, if

$$\int_0^t g^*(s)\,ds \leqslant \int_0^t f^*(s)\,ds \quad \text{for any } 0 < t < \mu(S).$$

LEMMA 1. *If* $h_n \prec f$, $n = 1, 2, \ldots$, *then* $\sum_{n=1}^{\infty} 2^{-n} h_n \prec f$.

Proof. By the inequality $\int_0^t (f_1+f_2)^*(u)\,du \leqslant \int_0^t f_1^*(u)\,du + \int_0^t f_2^*(u)\,du$ (see [7], p. 108), we get

$$\int_0^t \Big(\sum_{n=1}^{N} 2^{-n} h_n\Big)^*(s)\,ds \leqslant \sum_{n=1}^{N} 2^{-n} \int_0^t h_n^*(s)\,ds \leqslant \int_0^t f^*(s)\,ds \quad \text{for every } N \geqslant 1.$$

Since $\sum_{n=1}^{N} 2^{-n} h_n \uparrow \sum_{n=1}^{\infty} 2^{-n} h_n$ implies $\big(\sum_{n=1}^{N} 2^{-n} h_n\big)^* \uparrow \big(\sum_{n=1}^{\infty} 2^{-n} h_n\big)^*$, by Beppo–

Levi theorem we obtain

$$\lim_{N\to\infty}\int_0^t \Big(\sum_{n=1}^{N} 2^{-n}h_n\Big)^*(s)\,ds = \int_0^t \Big(\sum_{n=1}^{\infty} 2^{-n}h_n\Big)^*(s)\,ds.$$

Hence $\sum_{n=1}^{\infty} 2^{-n}h_n \prec f$.

If $f\in L^\varrho(\mu)$ and $g\prec f$ imply $g\in L^\varrho(\mu)$, then we shall say that $L^\varrho(\mu)$ has the *majorant property* ($L^\varrho(\mu)\in \mathrm{MP}$); if, moreover, $\varrho(g)\leqslant \varrho(f)$, then we shall say that $L^\varrho(\mu)$ has the *strong majorant property* ($L^\varrho(\mu)\in\mathrm{SMP}$).

LEMMA 2. *If $L^\varrho(\mu)\in\mathrm{MP}$, then there exists an equivalent rearrangement invariant function norm ϱ_0 such that $L^{\varrho_0}(\mu)\in\mathrm{SMP}$.*

Proof (see also [7], Theorem 16.1). Let us put, for $f\in L^\varrho(\mu)$,

$$\varrho_0(f) = \sup\{\varrho(g)\colon g\in\Omega(f)\},$$

where $\Omega(f) = \{h\in L^1(\mu)+L^\infty(\mu)\colon h\prec f\}$. There exists a constant $C>0$ such that $g\prec f$, $f\in L^\varrho(\mu)$, imply $\varrho(g)\leqslant C\varrho(f)$. Suppose that this condition does not hold for each $C>0$. Then there exist positive functions f_n, g_n, $n=1,2,\ldots$, such that

$$g_n\prec f_n,\qquad \varrho(g_n)\geqslant n\qquad\text{and}\qquad \varrho(f_n)=2^{-2n}.$$

Putting $f=\sum_{n=1}^{\infty}2^n f_n$, we have $f\in L^\varrho(\mu)$ and $2^n g_n\prec 2^n f_n\prec \sum_{n=1}^{\infty}2^n f_n = f$ for $n\geqslant 1$. By Lemma 1, we get

$$g=\sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty}2^{-n}(2^n f_n)\prec f;$$

hence $g\in L^\varrho(\mu)$. On the other hand, $0\leqslant g_n\leqslant g$, therefore $\varrho(g)\geqslant\varrho(g_n)\geqslant n$ for all $n\geqslant 1$, a contradition. From the above we get $\varrho(f)\leqslant\varrho_0(f)\leqslant C\varrho(f)$ for $f\in L^\varrho(\mu)$. ϱ_0 is a rearrangement-invariant function norm.

For example, we shall show the triangle inequality. Let $f_1, f_2\in L^\varrho(\mu)$ and $\varepsilon>0$. Then there exists $g\in L^\varrho(\mu)$ such that $g\prec f_1+f_2$, $\varrho_0(f_1+f_2)\leqslant\varrho(g)+\varepsilon$. There exist g_1, g_2 such that $g_i\prec f_i$, $i=1,2$, $g=g_1+g_2$ (see [5]). Since $L^\varrho(\mu)\in\mathrm{MP}$, therefore $g_1,g_2\in L^\varrho(\mu)$. We have

$$\varrho_0(f_1+f_2)\leqslant\varrho(g)+\varepsilon\leqslant\varrho(g_1)+\varrho(g_2)+\varepsilon\leqslant\varrho_0(f_1)+\varrho_0(f_2)+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we conclude the triangle inequality.

A measure space (S,Σ,μ) is called *adequate* if for any $f,g\in P$

$$\sup\Big\{\int_S fg'\,d\mu\colon g'\sim g\Big\} = \int_0^{\mu(S)} f^*(t)g^*(t)\,dt.$$

Non-atomic measure spaces and discrete measure spaces are adequate (see Luxemburg [7], Mills [9], Silverman [16]).

LUXEMBURG THEOREM. *If μ is an adequate measure and $L^\varrho(\mu)$ is maximal, then $L^\varrho(\mu) \in$ SMP.*

If μ is not adequate, then this theorem may be false (counter-example — see [7], p. 118).

If ϱ is monotone-complete, then by the Amemiya theorem (see [18]), $\gamma\varrho(f) \leqslant \varrho''(f) \leqslant \varrho(f)$ for $f \in M$ and $0 < \gamma \leqslant 1$. Hence, if μ is adequate, then $f \in L^\varrho(\mu)$ and $g \prec f$ implies $g \in L^\varrho(\mu)$ and $\varrho(g) \leqslant 1/\gamma\varrho(f)$.

MITJAGIN–CALDERÓN THEOREM. *If μ is a separable non-atomic measure or a discrete measure and $L^\varrho(\mu)$ is minimal, then $L^\varrho(\mu) \in$ SMP.*

G. I. Russu [15] has given examples of rearrangement-invariant Banach function spaces $L^\varrho(m)$ and $L^\zeta(m)$, where $S = [0, 1]$ and m is the Lebesgue measure, such that $L^\varrho(m) \notin$ MP and $L^\zeta(m) \in$ MP, $L^\zeta(m) \notin$ SMP.

We shall say that $L^\varrho(\mu)$ has *universal majorant property* ($L^\varrho(\mu) \in$ UMP) if each closed rearrangement-invariant Banach function subspace $L^{\varrho_0}(\mu)$ of the space $L^\varrho(\mu)$ has the majorant property, i.e. $L^{\varrho_0}(\mu) \in$ MP.

Let $f \in L^1(m) + L^\infty(m)$, and let

$$(Hf)(t) = 1/t \int_0^t f^*(s)\,ds \cdot \chi_{[0,l]}(t) \quad \text{if } S = [0, l], \ l \leqslant \infty.$$

We shall say that $L^\varrho(m)$ has *Hardy property* ($L^\varrho(m) \in$ HP) if $f \in L^\varrho(m)$ implies $Hf \in L^\varrho(m)$. If $L^\varrho(m) \in$ HP, then each of its closed rearrangement invariant subspaces $L^{\varrho_0}(m)$ belongs to HP. Since $L^\varrho(m) \in$ HP implies $L^\varrho(m) \in$ MP, therefore if $L^\varrho(m) \in$ HP, then $L^\varrho(m) \in$ UMP. Note that the converse is false: $L^1(m) \in$ UMP and $L^1(m) \notin$ HP. A. A. Siedajev has given even an example of $L^\varrho(m)$ without continuous norm such that $L^\varrho(m) \in$ UMP and $L^\varrho(m) \notin$ HP.

3. Non-atomic case. An operator T which maps a Banach space X into a Banach space Y is called a *Lipschitz operator* if $T0 = 0$ and if $\|Tf - Tg\|_Y \leqslant K\|f - g\|_X$, $f, g \in X$, for some $K > 0$. The smallest K in this inequality is called the *bound* of T.

By Lip$(X, Y; K)$ ($B(X, Y; K)$, $\alpha(X, Y; K)$) we denote the class of all Lipschitz (bounded, linear and bounded) operators T from X to Y with bound not exceeding K. If $X = Y$, we shall write Lip$(X: K)$ ($B(X; K)$, $\alpha(X; K)$).

LEMMA 3. *Let μ be a non-atomic measure. If $\mu(S) \geqslant t > 0$, then there exists a set $E_t \in \Sigma$ such that $\mu(E_t) = t$ and*

$$\int_{E_t} |f|\,d\mu = \int_0^t f^*(u)\,du, \qquad |f(x)| \geqslant f^*(t) \quad \textit{for } x \in E_t \textit{ a.e.},$$

$$|f(x)| \leqslant f^*(t) \quad \textit{for } x \in S \setminus E_t \textit{ a.e.}$$

Proof (see [11], Theorem 5.4.7 or [17], Lemma 3.17).

For $\alpha > 0$, $f^{(\alpha)}$ will denote the *α-truncation* of f, that is, the function

$$f^{(\alpha)}(x) = \min\big(|f(x)|, \alpha\big) \operatorname{sgn} f(x).$$

Lemma 4. *Let μ be a non-atomic measure. If T: $L^1(\mu)+L^\infty(\mu) \to L^1(\mu)+L^\infty(\mu)$ is such that $T \in \mathrm{Lip}\big(L^1(\mu);\ 1\big) \cap B\big(L^\infty(\mu);\ 1\big)$, then $Tf \prec f$ for any $f \in L^1(\mu)$.*

Proof. For each a, $0 < a < \mu(S)$, we can find measurable sets $E_{1,a}$ and $E_{2,a}$ such that $\mu(E_{1,a}) = \mu(E_{2,a}) = a$,

$$\int_0^a (Tf)^*(u)\,du = \int_{E_{1,a}} |Tf(t)|\,d\mu, \qquad \int_0^a f^*(u)\,du = \int_{E_{2,a}} |f(t)|\,d\mu,$$

$$|f(t)| \geqslant f^*(a) \quad \text{for } t \in E_{2,a} \text{ a.e.},$$

$$|f(t)| \leqslant f^*(a) \quad \text{for } t \in S \backslash E_{2,a} \text{ a.e.}$$

Putting $\alpha = f^*(a)$, we have for the truncation $f^{(\alpha)}$ of f

$$\begin{aligned}
\int_0^a (Tf)^*(u)\,du &= \int_{E_{1,a}} |Tf(t)|\,d\mu \leqslant \int_{E_{1,a}} |Tf(t) - (Tf^{(\alpha)})(t)|\,d\mu + \int_{E_{1,a}} |Tf^{(\alpha)}(t)|\,d\mu \\
&\leqslant \int_S |f(t) - f^{(\alpha)}(t)|\,d\mu + \int_{E_{1,a}} \|f^{(\alpha)}\|_\infty\,d\mu \\
&= \int_{E_{2,a}} |f(t) - f^{(\alpha)}(t)|\,d\mu + a\,\|f^{(\alpha)}\|_\infty \\
&\leqslant \int_{E_{2,a}} |f(t) - \alpha \operatorname{sgn} f(t)|\,d\mu + \alpha a \\
&= \int_{E_{2,a}} \big(|f(t)| - \alpha\big)\,d\mu + \alpha a = \int_{E_{2,a}} |f(t)|\,d\mu = \int_0^a f^*(u)\,du.
\end{aligned}$$

Hence $Tf \prec f$.

Proposition (Calderón theorem). *If T: $L^1(\mu)+L^\infty(\mu) \to L^1(\mu)+L^\infty(\mu)$ is such that $T \in \alpha\big(L^1(\mu);\ K_1\big) \cap \alpha\big(L^\infty(\mu);\ K_\infty\big)$ and $L^\varrho(\mu) \in \mathrm{MP}$, then $T \in \alpha\big(L^\varrho(\mu);\ K_\varrho\big)$, where $K_\varrho \leqslant C \max(K_1, K_\infty)$ and $C > 0$ is the constant from Lemma 2.*

Theorem 1. *Let μ be a non-atomic measure and let T: $L^1(\mu)+L^\infty(\mu) \to L^1(\mu)+L^\infty(\mu)$ be such that $T \in \mathrm{Lip}\big(L^1(\mu);\ K_1\big) \cap \mathrm{Lip}\big(L^\infty(\mu);\ K_\infty\big)$.*

(a) *If $L^\varrho(\mu)$ is maximal and $L^\infty(\mu) \subset L^\varrho(\mu) \subset L^1(\mu)$ or $L^\varrho(\mu)$ is minimal and μ is a separable finite measure, then $T \in \mathrm{Lip}\big(L^\varrho(\mu);\ K_\varrho\big)$, where $K_\varrho \leqslant \max(K_1, K_\infty)$;*

(b) *If $L^\varrho(\mu)$ is minimal and $\mu(S) = \infty$, then T can be extended uniquely to $\hat{T}$ on $L^\varrho(\mu)$ belonging to $\mathrm{Lip}\big(L^\varrho(\mu);\ \hat{K}_\varrho\big)$, where $\hat{K}_\varrho \leqslant \max(K_1, K_\infty)$.*

Proof. We fix $h \in L^1(\mu) \cap L^\infty(\mu)$ and define the operator S by

$$Sf = \frac{T(f+h) - Th}{\max(K_1, K_\infty)}, \quad f \in L^1(\mu) \cup L^\infty(\mu).$$

Then $S \in \mathrm{Lip}(L^1(\mu);\ 1) \cap \mathrm{Lip}(L^\infty(\mu);\ 1)$. Hence, by Lemma 4, $Sf \prec f$ for each $f \in L^\varrho(\mu) \cap L^1(\mu)$ and $\varrho(Sf) \leqslant \varrho(f)$, because $L^\varrho(\mu) \in \mathrm{SMP}$. This means that

$$\varrho(Tf-Th) = \varrho(\max(K_1, K_\infty) S(f-h)) \leqslant \max(K_1, K_\infty)\varrho(f-h).$$

($a_{\max}$) For arbitrary $f, g \in L^\varrho(\mu)$, we consider the truncations $f^{(n)}, g^{(n)}$. Then $T(f^{(n)})$ and $T(g^{(n)})$ converge to Tf and Tg, respectively, in the $L^1(\mu)$-norm since $T \in \mathrm{Lip}(L^1(\mu);\ K_1)$ and $L^\varrho(\mu) \subset L^1(\mu)$. Consequently, the same holds in the measure μ. Therefore, for a properly chosen sequence n_i, $T(f^{(n_i)})$ and $T(g^{(n_i)})$ converge almost everywhere to Tf and Tg. Since $f^{(n)} \in L^\infty(\mu)$ for any $f \in L^\varrho(\mu)$ and $|f^{(n)} - g^{(n)}| \leqslant |f-g|$, we have

$$\varrho\big(T(f^{(n_i)}) - T(g^{(n_i)})\big) \leqslant \max(K_1, K_\infty)\varrho(f-g).$$

Hence, by virtue of Fatou property, we get

$$\varrho(Tf-Tg) \leqslant \varliminf_{i\to\infty} \varrho\big(T(f^{(n_i)}) - T(g^{(n_i)})\big) \leqslant \max(K_1, K_\infty)\varrho(f-g).$$

($a_{\min}$) For arbitrary $f, g \in L^\varrho(\mu)$ we consider $h_n \in L^\infty(\mu)$ $(n = 1, 2, \ldots)$, such that $\varrho(g-h_n) \to 0$ as $n \to \infty$. We have

$$\varrho(Tf-Tg) \leqslant \lim_{n\to\infty} \varrho(Tf-Th_n) + \lim_{n\to\infty} \varrho(Th_n - Tg) \leqslant \max(K_1, K_\infty),$$

$$\lim_{n\to\infty} \big(\varrho(f-h_n) + \varrho(h_n - g)\big) = \max(K_1, K_\infty)\varrho(f-g).$$

(b) Since ϱ is a continuous norm, T can be extended from $L^\varrho(\mu) \cap \cap L^1(\mu) \cap L^\infty(\mu)$ to $L^\varrho(\mu)$ uniquely, because $L^1(\mu) \cap L^\infty(\mu)$ is dense in $L^\varrho(\mu)$.

Remarks. (1) If $S = (0, l)$, $l < \infty$ and $L^\infty(m) \subset L^\varrho(m) \subset L^1(m)$ is maximal or $l = \infty$ and $L^\varrho(m)$ is minimal, then Theorem 1 is the Lorentz–Shimogaki theorem (see [5] and [6]).

(2) Lorentz and Shimogaki gave the exact value of K_ϱ. There is

$$K_\varrho \leqslant K_\infty \sup\left\{\frac{\varrho(\sigma_a f)}{\varrho(f)} : 0 \neq f \in L^\varrho\right\}, \quad \text{where } a = \frac{K_\infty}{K_1}$$

and

$$(\sigma_a f)(t) = f(at)\chi_{[0,\mu(S)]}(at)$$

(see [6]).

(3) In Theorem 1 we can assume that $T \in \mathrm{Lip}(L^1(\mu);\ K_1) \cap B(L^\infty(\mu);\ K_\infty)$ instead of $T \in \mathrm{Lip}(L^1(\mu);\ K_1) \cap \mathrm{Lip}(L^\infty(\mu);\ K_\infty)$.

Problem. Is Theorem 1 true for a non-atomic measure such that $\mu(S) = \infty$ and the space $L^1(\mu) + L^\infty(\mu)$, i.e. does there exist an extension $\hat{T} \in \mathrm{Lip}(L^1(\mu) + L^\infty(\mu);\ K)$?

The space $L^1(\mu) + L^\infty(\mu)$ is identical with Gould space L^G (see [8]) for which $L^1 \subset L^G_a \subset L^G = L^1 + L^\infty$.

4. Sequential case. If μ is a discrete measure and $f \in c_0$, then f^* is a step function constant on $[n, n+1)$, $n = 0, 1, 2, \ldots$; thus we shall sometimes regard f^* as the sequence $(f_n^*)_{n=0}^{\infty}$, where $(f_n^*)_{n=0}^{\infty}$ is a non-increasing sequence obtained from $(|f_n|)_{n=0}^{\infty}$ by a suitable permutation of indices. In the case $0 \leqslant t < 1$ we have

$$\int_0^t f^*(s)\,ds = \int_0^t f^*(0)\,ds = tf_0^*;$$

moreover, if $t \geqslant 1$, then

$$\int_0^t f^*(s)\,ds = \int_0^{[t]} f^*(s)\,ds + \int_{[t]}^t f^*(s)\,ds = \sum_{n=0}^{[t]-1} \int_n^{n+1} f^*(s)\,ds + \int_{[t]}^t f^*([t])\,ds$$
$$= \sum_{n=0}^{[t]-1} f^*(n) + (t-[t])\, f^*([t]) = \sum_{n=0}^{[t]-1} f_n^* + (t-[t])\, f_{[t]}^*.$$

Hence

$$\int_0^t f^*(s)\,ds = \begin{cases} tf_0^* & \text{if } 0 \leqslant t < 1, \\ \sum_{n=0}^{[t]-1} f_n^* + (t-[t])\, f_{[t]}^* & \text{if } \quad t \geqslant 1. \end{cases}$$

LEMMA 5. *If* $\sum_{n=0}^{k} |b_n| \leqslant \sum_{n=0}^{k} |a_n|$ *for* $0 \leqslant k \leqslant N$ *and* $\alpha_0 \geqslant \alpha_1 \geqslant \ldots \geqslant \alpha_N > 0$, *then* $\sum_{n=0}^{N} \alpha_n |b_n| \leqslant \sum_{n=0}^{N} \alpha_n |a_n|$.

Proof. If $A_k = \sum_{n=0}^{k} |a_n|$ and $B_k = \sum_{n=0}^{k} |b_n|$ $(A_{-1} = B_{-1} = 0)$, then, using Abel's transformation, we have

$$\sum_{n=0}^{N} \alpha_n |a_n| = \alpha_N A_N + \sum_{n=0}^{N-1} A_n(\alpha_n - \alpha_{n+1}) \geqslant \alpha_N B_N + \sum_{n=0}^{N-1} B_n(\alpha_n - \alpha_{n+1}) = \sum_{n=0}^{N} \alpha_n |b_n|.$$

LEMMA 6. *If* $T \in \mathrm{Lip}(l^1;\ 1) \cap B(c_0;\ 1)$, *then* $Tf \prec f$ *for* $f \in l^1$.

Proof. Let $f^* = (f_0^*, f_1^*, \ldots) = (|f_{\sigma(0)}|, |f_{\sigma(1)}|, \ldots)$ and

$$(Tf)^* = \big((Tf)_0^*, (Tf)_1^*, \ldots\big) = \big(|(Tf)_{\pi(0)}|, |(Tf)_{\pi(1)}|, \ldots\big),$$

where σ, π are permutations of non-negative indices. Let $\alpha = f_k^*$ for $k \geqslant 0$. Then we have

$$\sum_{n=0}^{k} (Tf)_n^* = \sum_{n=0}^{k} |(Tf)_{\pi(n)}| \leqslant \sum_{n=0}^{k} |(Tf)_{\pi(n)} - (Tf^{(\alpha)})_{\pi(n)}| + \sum_{n=0}^{k} |(Tf^{(\alpha)})_{\pi(n)}|$$
$$\leqslant \sum_{n=0}^{\infty} |f_{\pi(n)} - f_{\pi(n)}^{(\alpha)}| + \sum_{n=0}^{k} \sup_n |f_{\pi(n)}^{(\alpha)}|$$
$$= \sum_{n=0}^{\infty} |f_{\pi(n)} - f_{\pi(n)}^{(\alpha)}| + (k+1) \sup_n |f_{\pi(n)}^{(\alpha)}| = A.$$

Let $A_k = \{n \geqslant 0\colon |f_{\pi(n)}| \geqslant f_k^*\}$. Evidently, $\mu(A_k) \geqslant k+1$, Then

$$\begin{aligned} A &= \sum_{n\in A_k} |f_{\pi(n)} - f_{\pi(n)}^{(\alpha)}| + (k+1)\sup_n |f_{\pi(n)}^{(\alpha)}| \\ &\leqslant \sum_{n\in A_k} |f_{\pi(n)} - \alpha \operatorname{sgn} f_{\pi(n)}| + (k+1) f_k^* \\ &= \sum_{n\in A_k} (|f_{\pi(n)}| - f_k^*) + (k+1) f_k^* \\ &= \sum_{n\in A_k} |f_{\pi(n)}| - f_k^*(\mu(A_k)-k-1) \leqslant \sum_{n=0}^{\mu(A_k)} |f_{\sigma(n)}| - f_k^*(\mu(A_k)-k-1) \\ &= \sum_{n=0}^{k} |f_{\sigma(n)}| + \sum_{n=k+1}^{\mu(A_k)} |f_{\sigma(n)}| - f_k^*(\mu(A_k)-k-1) \\ &\leqslant \sum_{n=0}^{k} |f_{\sigma(n)}| + |f_{\sigma(k+1)}|(\mu(A_k)-k-1) - |f_{\sigma(k)}|(\mu(A_k)-k-1) \\ &\leqslant \sum_{n=0}^{k} |f_{\sigma(n)}| = \sum_{n=0}^{k} f_n^*. \end{aligned}$$

If $0 < t < 1$, then $(Tf)_0^* = \|Tf\|_{c_0} \leqslant \|f\|_{c_0} = f_0^*$. If $t > 1$ is arbitrarily fixed, then

$$\sum_{n=0}^{k} (Tf)_n^* \leqslant \sum_{n=0}^{k} f_n^* \quad \text{for } 0 \leqslant k \leqslant [t].$$

Let $\alpha_n = 1$ if $n \leqslant [t]-1$ and $\alpha_n = t-[t]$ if $n = [t]$; then by Lemma 5

$$\sum_{n=0}^{[t]-1} (Tf)_n^* + (t-[t])(Tf)_{[t]}^* = \sum_{n=0}^{[t]} \alpha_n (Tf)_n^* \leqslant \sum_{n=0}^{[t]} \alpha_n f_n^* = \sum_{n=0}^{[t]-1} f_n^* + (t-[t]) f_{[t]}^*.$$

Hence $Tf \prec f$.

THEOREM 2. *If* $T \in \operatorname{Lip}(l^1; K_1) \cap \operatorname{Lip}(c_0; K_\infty)$, $l^\varrho \subset c_0$ *and* l^ϱ *is maximal or minimal, then* $T \in \operatorname{Lip}(l^\varrho; K_\varrho)$, *where* $K_\varrho \leqslant \max(K_1, K_\infty)$.

Proof. We have $\varrho(Tf - Th) \leqslant \max(K_1, K_\infty)\, \varrho(f-h)$ for $f \in l^1$ and $h \in l^1$, by the proof of Theorem 1.

If l^ϱ is maximal and if $f_{(n)}(k) = f(k)\chi_{\{0,1,\ldots,n\}}(k)$, then $f_{(n)} \uparrow f$, $|f_{(n)} - g_{(n)}| \leqslant |f-g|$, $f_{(n)} \in l^1$ and $T(f_{(n)}) \to T(f)$ in c_0-norm, since $T \in \operatorname{Lip}(c_0; K_\infty)$ and $\|f_{(n)} - f\|_{c_0} \to 0$ as $n \to \infty$. We have

$$\varrho(T(f_{(n)}) - T(g_{(n)})) \leqslant \max(K_1, K_\infty)\varrho(f_{(n)} - g_{(n)}) \leqslant \max(K_1 K_\infty)\varrho(f-g).$$

Hence, by virtue of the Fatou property, we obtain $T \in \operatorname{Lip}(l^\varrho; K_\varrho)$, where $K_\varrho \leqslant \max(K_1, K_\infty)$. In case of l^ϱ minimal, the proof is performed similarly as in Theorem 1.

5. Alternative proofs. Applying the theory of Bennett and Peetre, we give here alternative proofs of Theorems 1 and 2 for maximal rearrangement invariant spaces.

Let (X_1, X_2) be a *Banach couple*, i.e. there is a Hausdorff topological vector space X and continuous embedings $X_1 \subset X, X_2 \subset X$. The K-functional of Peetre is defined on X_1+X_2 for each $t > 0$ by

$$K(t;f) \equiv K(t;f;X_1, X_2) = \inf_{f=f_1+f_2} (\|f_1\|_1+t\|f_2\|_2), \quad f \in X_1+X_2.$$

Since $K(t; f)$ is a continuous concave function of $t > 0$, we have

$$K(t;f) = K(0^+;f)+\int_0^t k(s;f)ds, \quad f \in X_1+X_2,$$

where $k(s; f)$ is non-negative, right-continuous and non-increasing for $s > 0$. We shall restrict our attention to these $f \in X_1+X_2$ for which $K(0^+;f) \equiv \lim_{t\to 0^+} K(t;f) = 0$, which is equivalent to the fact that $f \in \overline{X_1 \cap X_2}^{X_1}+X_2$ (see [2], p. 8).

If (X_1, X_2) is a Banach couple and ϱ is a rearrangement invariant norm on $(0, \mu(S))$, we denote by $(X_1, X_2)_{\varrho;k}$ the space of elements $f \in \overline{X_1 \cap X_2}^{X_1}+X_2$ for which $\varrho(k(s;f))$ is finite. The space $(X_1, X_2)_{\varrho;k}$ is Banach space with norm $\|f\|_{\varrho;k} = \varrho(k(t;f))$ and

$$X_1 \cap X_2 \subset (X_1, X_2)_{\varrho;k} \subset \overline{X_1 \cap X_2}^{X_1}+X_2 \subset X_1+X_2$$

(see [1], p. 420).

THEOREM 3. *Let (X_1, X_2) and (Y_1, Y_2) be two Banach couples, and E_1, E_2 subsets of X_1+X_2, and let ϱ be a rearrangement invariant norm such that $L^\varrho \in$ SMP. If $T: X_1+X_2 \to Y_1+Y_2$ is such that*

(a) *$K(t; Tf-Tg) \leqslant C_1 K(C_2 t;f-g)$ for all $f \in E_1, g \in E_2, t \in (0, \infty)$,*

(b) *$\|Tf-Th\|_{Y_1} \leqslant K_1 \|f-h\|_{X_1}$ if $f-h \in X_1$, $f \in E_1$, $\|Th-Tg\|_{Y_2} \leqslant K_2 \|h-g\|_{X_2}$ if $h-g \in X_2, g \in E_2$,*

then there holds

$$\|Tf-Tg\|_{\varrho;k} \leqslant K \|f-g\|_{\varrho;k} \quad \text{if } f \in E_1, g \in E_2, f-g \in (X_1, X_2)_{\varrho;k},$$

where $K \leqslant \max(C_1, C_1 C_2)$ or $K \leqslant \max(K_1, K_2)$, respectively.

Proof. (a) We note first that if $f-g \in \overline{X_1 \cap X_2}^{X_1}+X_2$, then $Tf-Tg \in \overline{Y_1 \cap Y_2}^{Y_1}+Y_2$. Hence we have

$$K(t; Tf-Tg) = \int_0^t k(s; Tf-Tg)ds \quad \text{for any } f-g \in (X_1 X_2)_{\varrho;k}.$$

In this case, the inequality $K(t; Tf-Tg) \leqslant C_1 K(C_2 t; f-g)$ reduces to

$$\int_0^t k(s; Tf-Tg)ds \leqslant C_1 C_2 \int_0^t k(C_2 s;f-g)ds, \quad 0 < t < \infty.$$

Hence $\varrho(k(s;\, Tf-Tg)) \leqslant \varrho(C_1\, C_2\, k(C_2\, s; f-g))$, and so

$$\begin{aligned}\|Tf-Tg\|_{\varrho;k} &\leqslant \varrho(C_1\, C_2\, k(C_2\, s; f-g)) \\ &\leqslant C_1\, C_C \sup_{f-g\in L^\varrho} \frac{\varrho(k(C_2\, s; f-g))}{\varrho(k(s; f-g))} \cdot \varrho(k(s; f-g)) \\ &\leqslant \max(C_1, C_1\, C_2)\, \varrho(k(s; f-g)) = \max(C_1, C_1\, C_2)\, \|f-g\|_{\varrho;k}.\end{aligned}$$

(b) If $f-g = f_1+f_2$ is any decomposition, we have a decomposition $Tf-Tg = g_1+g_2$ if we set

$$f_1 = f-h, \quad f_2 = h-g, \quad g_1 = Tf-Th, \quad g_2 = Th-Tg.$$

Using (b), we obtain

$$\begin{aligned}K(t;\, Tf-Tg) &\leqslant \|g_1\|_1 + t\, \|g_2\|_2 = \|Tf-Th\|_1 + t\, \|Th-Tg\|_2 \\ &\leqslant K_1\, \|f-h\|_1 + K_2\, t\, \|h-g\|_2 = K_1(\|f_1\|_1 + K_2/K_1, t\, \|f_2\|_2),\end{aligned}$$

which clearly implies $K(t;\, Tf-Tg) \leqslant K_1\, K(K_2/K_1, t; f-g)$. Hence and by (a), we complete the proof.

Remark. In the case of a linear operator T and $E_1 = X_1+X_2$, $E_2 = \{0\}$, (a) is Bennett's interpolation theorem (see [1], Theorem 5.3), and (b) is Peetre's theorem (see [14], Theorem 2.1).

We say that a Banach function space Y has the (*) *property* in relation to a subspace X and some classes of operators $\mathscr{T}$ if

(*) for any $f, g\in Y$ there exist $(f_n), (g_n) \in X$ such that $f_n \uparrow f$, $g_n \uparrow g$ a.e., $|f_n - g_n| \leqslant |f-g|$ and there is a subsequence (n_k) for which $T(f_{n_k}) \to Tf$, $T(g_{n_k}) \to Tg$ a.e. for $T\in \mathscr{T}$.

For example: (a) L^ϱ such that $L^\infty \subset L^\varrho \subset L^1$ has the (*) property in relation to L^∞ and $T\in \mathrm{Lip}(L^1;\, K_1)$;

(b) l^ϱ such that $l^\varrho \subset c_0$ has the (*) property in relation to l^1 and $T\in \mathrm{Lip}(c_0;\, K_\infty)$.

THEOREM 4. *Let (X_1, X_2) be a Banach function couple, $L^\varrho \in$ SMP and let T: $X_1+X_2 \to X_1+X_2$ be such that $T\in \mathrm{Lip}(X_1; K_1) \cap \mathrm{Lip}(X_2, K_2)$. If L^ϱ is maximal and $(X_1, X_2)_{\varrho;k}$ has the (*) property in relation to $X_1 \cap X_2$ and to the above T or if L^ϱ is minimal, then T or $\hat{T} \in \mathrm{Lip}((X_1, X_2)_{\varrho;k};\, K_\varrho$ or $\hat{K}_\varrho)$, where K_ϱ or $\hat{K}_\varrho \leqslant \max(K_1, K_2)$.*

Proof. Let $f_0 \in X_1 \cap X_2$ and $Sf = T(f+f_0) - Tf_0, f\in X_1\cap X_2$. Then

$$\|Sf\|_1 \leqslant K_1\, \|f\|_1 \quad \text{for } f\in X_1,$$
$$\|Sf-Sg\|_2 \leqslant K_2\, \|f-g\|_2 \quad \text{for } f, g\in X_2.$$

Hence, by Theorem 3 (b) with $E_1 = \{0\}$ and $E_2 = X_2$, we have

$$\|Tf-Tf_0\|_{\varrho;k} = \|S(f-f_0)\|_{\varrho;k} \leqslant \max(K_1, K_2)\, \|f-f_0\|_{\varrho;k}$$

for $f \in (X_1, X_2)_{\varrho;k} \cap X_1 \cap X_2$ and $f_0 \in X_1 \cap X_2$. Hence the theorem follows, by the same argument as in the proof of Theorem 1.

COROLLARY 1. *Let* (X_1, X_2) *be a Banach function couple,* $\|f\|_{X_2} \leqslant C \cdot \|f\|_{X_1}$ *for* $f \in X_1$, *and let* L^ϱ *be maximal. If* $T \in \mathrm{Lip}(X_1; K_1) \cap \mathrm{Lip}(X_2; K_2)$ *and* $(X_1, X_2)_{\varrho;k}$ *has the* (*) *property in relation to* X_1, *then*

$$T \in \mathrm{Lip}\big((X_1, X_2)_{\varrho;k};\ K\big), \quad \textit{where } K \leqslant \max(K_1, K_2).$$

We consider the Lebesgue spaces $L^1(\mu)$ and $L^\infty(\mu)$ over a σ-finite adequate measure space (S, Σ, μ). If $L^{\varrho_0}(\mu)$ is a maximal rearrangement invariant space, then there is a maximal rearrangement invariant norm ϱ on $(0, \mu(S))$ such that $\varrho_0(f) = \varrho(f^*)$ (see [7], Theorem 12.2). Since

$$K\big(t; f; L^1(\mu), L^\infty(\mu)\big) = \int_0^t f^*(s)\,ds, \quad 0 < t < \infty$$

(see [17], p. 240), $k(s; f)$ is just $f^*(s)$. Therefore, the norm on $(L^1, L^\infty)_{\varrho;k}$ is

$$\|f\|_{\varrho;k} = \varrho\big(k(s; f)\big) = \varrho(f^*) = \varrho_0(f).$$

Hence

$$\big(L^1(\mu), L^\infty(\mu)\big)_{\varrho;k} = L^{\varrho_0}(\mu).$$

COROLLARY 2. *If* $T \in \mathrm{Lip}\big(L^1(0, l); K_1\big) \cap \mathrm{Lip}\big(L^\infty(0, l); K_\infty\big)$, where $l < \infty$, or $T \in \mathrm{Lip}(l^1; K_1) \cap \mathrm{Lip}(c_0; K_\infty)$, *then* $T \in \mathrm{Lip}\big(L^\varrho(0, l); K\big)$ *or* $T \in \mathrm{Lip}(l^{\varrho_0}; K)$, *respectively, where* $K \leqslant \max(K_1, K_\infty)$.

The proof is a consequence of Corollary 1 and the fact that $\big(L^1(\mu); L^\infty(\mu)\big)_{\varrho;k} = L^{\varrho_0}(\mu)$.

Remark. If $a = (a_n)_{n=1}^\infty \in c_0$, then $K(t; a; l^1, c_0) = ta_1^* = t\|a\|_{c_0}$ if $0 \leqslant t < 1$, and

$$K(t, a; l^1, c_0) = \sum_{n=1}^{[t]} a_n^* + (t - [t])\, a_{[t]+1}^* \quad \text{if } t \geqslant 1.$$

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