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On a differential inequality of the type $X'' + f(t)X \geq 0$

The aim of this paper is to prove the following

**Theorem.** Let $x(t), f(t)$ be real functions determined on $[a, b]$ such that

1. $x(t)$ is twice differentiable on $[a, b]$, $f(t)$ is bounded on $[a, b]$,
2. $x'' + f(t)x \geq 0$ on $(a, b)$,
3. $x(a) = x(b) = 0$, $x(t)$ is positive on $(a, b)$,

Let $t_0 \in (a, b)$ be a point such that

$$x(t_0) = \max_{t \in [a, b]} x(t)$$

and let $M = \sup_{t \in [a, b]} f(t)$. Then $M > 0$ and the following inequalities hold:

$$\frac{b + a - \sqrt{(b-a)^2 - 8/M}}{2} \leq t_0 \leq \frac{b + a + \sqrt{(b-a)^2 - 8/M}}{2},$$

$$t_0 \leq b - \sqrt{2/M}.$$ 

**Proof.** At the beginning we shall show that $M > 0$. Namely, we have $x'(a) \geq 0$, $x'(b) \leq 0$, so for a certain $\xi \in (a, b)$, $x''(\xi) < 0$. From condition (2) it follows that

$$f(\xi)x(\xi) \geq -x''(\xi) > 0;$$

hence $f(\xi) > 0$ and therefore $M = \sup_{t \in [a, b]} f(t) \geq f(\xi) > 0$.

Let $h(t) = x(t) + \frac{1}{2} M x(t_0) t^2$ for $t \in [a, b]$. Then $h''(t) \geq 0$ on $(a, b)$ because $h''(t) = x''(t) + M x(t_0) + x''(t) + f(t)x(t) \geq 0$. Thus the function $h(t)$ is convex on $[a, b]$. By the Jensen inequality,

$$h(t_0) = x(t_0) + \frac{M}{2} x(t_0) t_0^2$$

$$= h \left( \frac{b - t_0}{b - a} a + \frac{t_0 - a}{b - a} b \right) \leq \frac{b - t_0}{b - a} h(a) + \frac{t_0 - a}{b - a} h(b)$$

$$= \frac{b - t_0}{b - a} \frac{M}{2} x(t_0) a^2 + \frac{t_0 - a}{b - a} \frac{M}{2} x(t_0) b^2,$$
whence we obtain
\[
\frac{2}{M} + t_0^2 \leq \frac{b-t_0}{b-a} a^2 + \frac{t_0-a}{b-a} b^2, \quad \text{or} \quad t_0^2 - (a+b)t_0 + ab + \frac{2}{M} \leq 0;
\]
this completes the proof of (4).

In the proof of (5) we shall use the Taylor formula. We have
\[
x(b) = x(t_0) + (b-t_0)x'(t_0) + \frac{(b-t_0)^2}{2} x''(\theta),
\]
where \( \theta \in (t_0, b) \). But \( x'(t_0) = 0 \) and \( x(b) = 0 \); hence \( x(t_0) + \frac{(b-t_0)^2}{2} x''(\theta) = 0 \).

We know, however, that \( h'' \geq 0 \). Then \( x''(\theta) \geq -Mx(t_0), \quad \frac{(b-t_0)^2}{2} x''(\theta) \geq -M \frac{(b-t_0)^2}{2} x(t_0) \). Hence
\[
x(t_0) - \frac{M}{2} \frac{(b-t_0)^2}{2} x(t_0) \leq 0, \quad \frac{2}{M} \leq (b-t_0)^2, \quad t_0 \leq b - \sqrt[2]{\frac{2}{M}}.
\]
Let us mention that (4) gives the estimation
\[
(6) \quad b-a \geq 2\sqrt{2}/\sqrt{M}.
\]

From the paper of Z. Opial\(^{(1)}\) it is easy to obtain the similar estimation
\[
(7) \quad b-a \geq \pi/\sqrt{k},
\]
where \( k = \sup \{ |f(t)| \} \). In spite of the fact that \( 2 \sqrt{2} < \pi \). In general, however, \( k > m \), so both results (6) and (7) are incomparable.