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On a differential inequality of the type $X''+f(t)X \geqslant 0$

The aim of this paper is to prove the following

THEOREM. *Let $x(t)$, $f(t)$ be real functions determined on $[a,b]$ such that*

(1) $x(t)$ *is twice differentiable on* $[a,b]$, $f(t)$ *is bounded on* $[a,b]$,

(2) $$x''+f(t)x \geqslant 0 \quad \text{on } (a,b),$$

(3) $$x(a) = x(b) = 0, \quad x(t) \text{ is positive on } (a,b),$$

Let $t_0 \in (a,b)$ be a point such that

$$x(t_0) = \max_{t\in[a,b]} x(t)$$

and let $M = \sup_{t\in[a,b]} f(t)$. Then $M > 0$ and the following inequalities hold:

(4) $$\frac{b+a-\sqrt{(b-a)^2-8/M}}{2} \leqslant t_0 \leqslant \frac{b+a+\sqrt{(b-a)^2-8/M}}{2},$$

(5) $$t_0 \leqslant b-\sqrt{2}/\sqrt{M}.$$

Proof. At the beginning we shall show that $M > 0$. Namely, we have $x'(a) \geqslant 0$, $x'(b) \leqslant 0$, so for a certain $\xi \in (a,b)$, $x''(\xi) < 0$. From condition (2) it follows that

$$f(\xi)x(\xi) \geqslant -x''(\xi) > 0;$$

hence $f(\xi) > 0$ and therefore $M = \sup_{t\in[a,b]} f(t) \geqslant f(\xi) > 0$.

Let $h(t) = x(t)+\frac{1}{2}Mx(t_0)t^2$ for $t \in [a,b]$. Then $h''(t) \geqslant 0$ on (a,b) because $h''(t) = x''(t)+Mx(t_0) \geqslant x''(t)+f(t)x(t) \geqslant 0$. Thus the function $h(t)$ is convex on $[a,b]$. By the Jensen inequality,

$$\begin{aligned} h(t_0) &= x(t_0)+\frac{M}{2}x(t_0)t_0^2 \\ &= h\left(\frac{b-t_0}{b-a}a+\frac{t_0-a}{b-a}b\right) \leqslant \frac{b-t_0}{b-a}h(a)+\frac{t_0-a}{b-a}h(b) \\ &= \frac{b-t_0}{b-a}\frac{M}{2}x(t_0)a^2+\frac{t_0-a}{b-a}\frac{M}{2}x(t_0)b^2, \end{aligned}$$

whence we obtain

$$\frac{2}{M}+t_0^2 \leqslant \frac{b-t_0}{b-a}a^2+\frac{t_0-a}{b-a}b^2, \quad \text{or} \quad t_0^2-(a+b)t_0+ab+\frac{2}{M} \leqslant 0;$$

this completes the proof of (4).

In the proof of (5) we shall use the Taylor formula. We have

$$x(b) = x(t_0)+(b-t_0)x'(t_0)+\frac{(b-t_0)^2}{2}x''(\theta),$$

where $\theta \in (t_0, b)$. But $x'(t_0) = 0$ and $x(b) = 0$; hence $x(t_0)+\frac{(b-t_0)^2}{2}x''(\theta) = 0$. We know, however, that $h'' \geqslant 0$. Then $x''(\theta) \geqslant -Mx(t_0)$, $\frac{(b-t_0)^2}{2}x''(\theta) \geqslant -M\frac{(b-t_0)^2}{2}x(t_0)$. Hence

$$x(t_0)-\frac{M}{2}(b-t_0)^2x(t_0) \leqslant 0, \quad \frac{2}{M} \leqslant (b-t_0)^2, \quad t_0 \leqslant b-\sqrt{\frac{2}{M}}.$$

Let us mention that (4) gives the estimation

$$b-a \geqslant 2\sqrt{2}/\sqrt{M}. \tag{6}$$

From the paper of Z. Opial(1) it is easy to obtain the similar estimation

$$b-a \geqslant \pi/\sqrt{k}, \tag{7}$$

where $k = \sup_{t\in[a,b]} |f(t)|$. In spite of the fact that $2\sqrt{2} < \pi$. In general, however, $k > m$, so both results (6) and (7) are incomparable.

(1) Z. Opial, *Sur une inégalité de C. de la Vallée Poussin*, Ann. Polon. Math. 6 (1959), p. 87–91.