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On a certain boundary problem for the equation $\Delta^2 u - C^4 u = f(X)$

1. In this paper we construct the solution of the equation

$$\Delta^2 u(X) - C^4 u(X) = f(X), \tag{1}$$

where $X = (x_1, x_2)$ and C is a positive constant, for the half-plane

$$D = \{X: |x_1| < \infty, x_2 > 0\}$$

with the Riquier conditions:

$$u(x_1, 0) = f_1(x_1), \quad \Delta u(x_1, 0) = f_2(x_1).$$

The functions f_i $(i = 1, 2)$ are defined for $x_1 \in \partial D$, where ∂D denotes the boundary of D. f is a given function on the set D.

2. We give now some formulae and theorems which will be needed later. Let X and Y be two different points in 2-dimensional Euclidean space E_2. Let

$$r = XY = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}.$$

As we know [5], the fundamental solution of the equation

$$\Delta^2 u(X) - C^4 u(X) = 0 \tag{1a}$$

is the function

$$U(r) = \pi Y_0(Cr) + 2K_0(Cr), \tag{2}$$

where $Y_0(z)$ is the Bessel function of order zero of the second kind and $K_0(z)$ is the Mac Donald function of order zero [3]. Let $X \in D$, $Y \in \bar{D} = D \cup \partial D$. Let us denote by $X_1 = (x_1, -x_2)$ the symmetric image of the point X with respect to the axis $y_2 = 0$. Further let us write

$$r_1 = X_1 Y = [(x_1 - y_1)^2 + (x_2 + y_2)^2]^{1/2}, \quad \varrho = [(x_1 - y_1)^2 + x_2^2]^{1/2}.$$

The following theorems are true (see [1]).

THEOREM 1. *The function*

$$G(X, Y) = U(r) - U(r_1), \tag{3}$$

where $U(r)$ is given by (2), is a Green function with the pole X for equation (1a) *and for D with the boundary conditions*

$$G(X, Y) = 0, \quad \Delta_Y G(X, Y) = 0 \quad \textit{for} \quad Y \in \partial D.$$

THEOREM 2. *Let $f_i (i = 1, 2)$ be two functions measurable in ∂D and continuous at the point $\bar{x}_1 \in \partial D$. Let $\int_{-\infty}^{\infty} |f_i(y_1)| \, dy_1 < \infty$ $(i = 1, 2)$. Then the function*

$$(4) \qquad u_1(X) = (4\pi C)^{-1} x_2 \int_{-\infty}^{\infty} [f_1(y_1) C^2 N_1(\varrho) + f_2(y_1) N_2(\varrho)] \, dy_1,$$

where

$$N_i(\varrho) = \varrho^{-1} [(-1)^i \pi Y_1(C\varrho) + 2K_1(C\varrho)] \quad (i = 1, 2)$$

is the solution of equation (1a) *in the domain D and satisfies the following boundary conditions:*

$$\lim u_1(X) = f_1(\bar{x}_1), \quad \lim \Delta u_1(X) = f_2(\bar{x}_1) \quad \textit{as} \quad X \to (\bar{x}_1, 0+).$$

3. We shall now examine some properties of the function $U(r)$ given by formula (2) and prove some lemmas justifying differentiation of the integral of the form $V(X) = \int_D f(Y) U(r) dY$.

If we use the formulae [3]

$$\frac{d}{dz} [z^{-n} K_n(z)] = -z^{-n} K_{n+1}(z), \qquad \frac{d}{dz} [z^{-n} Y_n(z)] = -z^{-n} Y_{n+1}(z),$$

the identities

$$\Delta_X Y_0(Cr) + C^2 Y_0(Cr) = 0, \quad \Delta_X K_0(Cr) - C^2 K_0(Cr) = 0,$$

and asymptotic properties of the Bessel functions Y_n and Mac Donald functions K_n, the following will easily be obtained:

$$(5) \qquad \begin{aligned} U(r) &= o(r^{2-\gamma}), & D_{x_i} U(r) &= o(r^{1-\gamma}), \\ D_{x_1} D_{x_2} U(r) &= O(1), & D^2_{x_i} U(r) &= o(r^{-\gamma}) \end{aligned} \quad (i = 1, 2;\ 0 < \gamma < 1) \text{ as } r \to 0,$$

$$(6) \qquad \Delta_X U(r) = o(r^{-\gamma}), \quad D_{x_i} \Delta_X U(r) = O(r^{-1}) \quad (i = 1, 2;\ 0 < \gamma < 1) \text{ as } r \to 0,$$

$$(7) \qquad D^{\alpha}_{x_1} D^{\beta}_{x_2} U(r) = O(1) \quad \text{as } r \to \infty \ (\alpha, \beta = 0, 1, 2, 3, 4;\ \alpha + \beta \leqslant 4),$$

$$(8) \qquad \frac{d}{dr} \Delta_X U(r) = C^3 [\pi Y_1(Cr) - 2K_1(Cr)] = -4C^2 r^{-1} + h(r),$$

where $h(r) = O(1)$ as $r \to 0$ $(r \to +\infty)$.

We shall prove the following

LEMMA 1. *If the function f is measurable and bounded in D and $\int_D |f(Y)|\,dY < \infty$, then the integrals*

$$V_{\alpha\beta}(X) = \int_D f(Y) D_{x_1}^{\alpha} D_{x_2}^{\beta} U(r)\,dY \quad (\alpha, \beta = 0, 1, 2;\ \alpha+\beta \leqslant 2),$$

$$W_i(X) = \int_D f(Y) D_{x_i} \Delta_X U(r)\,dY \quad (i = 1, 2)$$

are uniformly convergent at every point $X_0 \in \bar{D}$.

Proof. It follows from (5) and (7) that there exists a number $M > 0$ such that

$$|D_{x_1}^{\alpha} D_{x_2}^{\beta} U(r)| \leqslant M \quad \text{for } r > 0\ (\alpha, \beta = 0, 1).$$

Therefore

$$|V_{\alpha\beta}(X)| \leqslant M \int_D |f(Y)|\,dY \quad \text{for } X \in \bar{D}\ (\alpha, \beta = 0, 1)$$

and the common majorant of integrals $V_{\alpha\beta}(X)$ $(\alpha, \beta = 0, 1)$ is the convergent integral $M \int_D |f(Y)|\,dY$.

We shall now prove that the integral $W_1(X)$ is uniformly convergent at the point $X_0 \in \bar{D}$. The proof for the integrals $W_2(X)$, $V_{20}(X)$, and $V_{02}(X)$ is similar. It follows from (6) and (7) that there exist numbers $R > 0$, $M_i > 0$ $(i = 1, 2)$ such that

$$(9) \qquad |D_{x_i} \Delta_X U(r)| \leqslant \begin{cases} M_1 r^{-1} & \text{for } 0 < r < 4R, \\ M_2 & \text{for } r \geqslant 2R. \end{cases}$$

Let $K(X_0, 3R)$ denote a circle with the centre $X_0 \in \bar{D}$ and radius $3R$. We shall write $W_1(X)$ in the form:

$$W_1(X) = W_1^1(X) + W_1^2(X),$$

where

$$W_1^1(X) = \int_{D \cap K(X_0, 3R)} f(Y) D_{x_1} \Delta_X U(r)\,dY,$$

$$W_1^2(X) = \int_{D \setminus K(X_0, 3R)} f(Y) D_{x_1} \Delta_X U(r)\,dY.$$

For $X \in K(X_0, R)$ and $Y \in K(X_0, 3R)$ we have $r = XY \leqslant XX_0 + YX_0 \leqslant R + 3R = 4R$. If $X \in K(X_0, R)$ and $Y \in D \setminus K(X_0, 3R)$, then $r = XY = X_0Y - X_0X \geqslant 2R$. From these results and from formula (9) we obtain

$$|W_1(X)| \leqslant |W_1^1(X)| + |W_1^2(X)| \leqslant M_1 \int_{D \cap K(X_0, 3R)} |f(Y)| r^{-1}\,dY + M_2 \int_{D \setminus K(X_0, 3R)} |f(Y)|\,dY$$

for $X \in K(X_0, R)$ (cf. [4]).

It follows from the above inequalities that the integral $W_1(X)$ is uniformly convergent at the point $X_0 \in \bar{D}$.

We get as corollaries of Lemma 1 the following

COROLLARY 1. *If a function f satisfies the assumptions of Lemma 1, then the function*

$$V(X) = \int_D f(Y) U(r) dY \tag{10}$$

is of class C^2 in $\bar{D}$ and

$$D^{\alpha}_{x1} D^{\beta}_{x2} V(X) = V_{\alpha\beta}(X) \quad \textit{for } X \in \bar{D} \ (\alpha, \beta = 0, 1, 2;\ \alpha+\beta \leqslant 2).$$

COROLLARY 2. *If a function f satisfies the assumptions of Lemma 1, then the function*

$$W(X) = \int_D f(Y) \Delta_X U(r) dY \tag{11}$$

is of class C^1 in $\bar{D}$ and

$$D_{x_i} W(X) = W_i(X) \quad \textit{for } X \in \bar{D} \ (i = 1, 2).$$

We now prove

LEMMA 2. *If a function f is bounded and of class C^1 in D and $\int_D |f(Y)| dY < \infty$, then the function $W(X)$ defined by formula* (11) *is of class C^2 in D.*

Proof. Let the circle $K(X_0, 3R) \subset D$ and conditions (9) hold true. We now present the function $W(X)$ given by formula (11) in the form

$$W(X) = L(X) + H(X), \tag{11a}$$

where

$$L(X) = \int_{K(X_0, 3R)} f(Y) \Delta_X U(r) dY, \quad H(X) = \int_{D \setminus K(X_0, 3R)} f(Y) \Delta_X U(r) dY.$$

Let $X \in K(X_0, R)$. Then by (7) the function $H(X)$ is of class C^2 in $K(X_0, R)$ and its derivatives up to the order two may be found by differentiation under the sign of the integral. Taking into consideration the above properties and the fact that the function $U(r)$ as a function of the point X $(X \neq Y)$ satisfies equation (1a) we have

$$\Delta H(X) = C^4 \int_{D \setminus K(X_0, 3R)} f(Y) U(r) dY \quad \text{for } X \in K(X_0, R). \tag{12}$$

Using the formula

$$D_{x_i} \Delta_X U(r) = -D_{y_i} \Delta_X U(r) \quad (i = 1, 2)$$

and Corollary 1, we get

$$D_{x_i} L(X) = -\int_{K(X_0, 3R)} f(Y) D_{y_i} \Delta_X U(r) dY \quad (i = 1, 2).$$

From the formula for integration by parts [2] we obtain

(13)

$$D_{x_i} L(X) = \int_{K(X_0,3R)} \Delta_X U(r) D_{y_i} f(Y) dY + \int_{\partial K(X_0,3R)} f(Y) \Delta_X U(r) \cos(n_Y, y_i) dS_Y$$

for $X \in K(X_0, R)$ $(i = 1, 2)$, where n_Y denotes the inward normal to $\partial K(X_0, 3R)$. Formulae (13), Lemma 1, and Corollary 2 imply that the function $L(X)$ is of class C^2 in $K(X_0, R)$. Thus the function $W(X)$ is of class C^2 in $K(X_0, R)$ and hence also at the point X_0.

LEMMA 3. *If the function f satisfies the assumptions of Lemma 2, then*

(14) $$\lim \Delta L(X_0) = -8\pi C^2 f(X_0) \quad \text{as } R \to 0,\ X_0 \in D.$$

Proof. By Lemma 2 and (13) we have

$$D_{x_i}^2 L(X) = \int_{K(X_0,3R)} D_{y_i} f(Y) D_{x_i} \Delta_X U(r) dY + \int_{\partial K(X_0,3R)} f(Y) D_{x_i} \Delta_X U(r) \cos(n_Y, y_i) dS_Y$$

for $X \in K(X_0, R)$ $(i = 1, 2)$. For $\sum_{i=1}^{2} D_{x_i}^2 L(X_0)$, according to the formulae

$$D_{x_i} \Delta_X U(r) = -D_{y_i} \Delta_X U(r) \quad (i = 1, 2),$$

we obtain

$$\Delta L(X_0) = \sum_{i=1}^{2} D_{x_i}^2 L(X_0) = B_1(X_0) + B_2(X_0),$$

where

$$B_1(X_0) = \int_{K(X_0,3R)} \sum_{i=1}^{2} D_{y_i} f(Y) D_{x_i} \Delta_X U(r)|_{X=X_0} dY,$$

$$B_2(X_0) = \int_{\partial K(X_0,3R)} f(Y) \frac{\partial}{\partial n_Y} \Delta_X U(r)|_{X=X_0} dS_Y.$$

The integral $B_1(X_0)$ is an integral of the type $W_1^1(X_0)$ and can be made arbitrarily small by selecting the sufficiently small radius $3R$. It is enough to show that

(15) $$\lim B_2(X_0) = -8\pi C^2 f(X_0) \quad \text{as } R \to 0.$$

Since on the boundary $\partial K(X_0, 3R)$ of the circle $K(X_0, 3R)$ we have

$$\frac{\partial}{\partial n_Y} \Delta_X U(r)|_{X=X_0, Y\in\partial K(X_0,3R)} = -\frac{d}{dr} \Delta_X U(r)|_{r=3R},$$

we get by (8)

$$B_2(X_0) = \int_{\partial K(X_0,3R)} f(Y) \frac{dU(r)}{dr}\bigg|_{X=X_0} dS_Y$$
$$= \int_{\partial K(X_0,3R)} f(Y)\left[\frac{-4C^2}{r}+h(r)\right]\bigg|_{r=3R} dS_Y.$$

Applying the mean value theorem to the last integral, we obtain

$$B_2(X_0) = 2\pi 3Rf(Q)\left[\frac{-4C^2}{3R}+h(3R)\right], \quad \text{where } Q\in \partial K(X_0, 3R).$$

Then, by the continuity of f for $X = X_0$, we get (15).

LEMMA 4. *If a function f satisfies the assumptions of Lemma 2, then the function $-(8\pi C^2)^{-1} V(X)$, where $V(X)$ is given by formula* (10), *satisfies equation* (1) *in D.*

Proof. From Corollary 1 and Lemma 2 it follows, by formula (11a),

$$\Delta^2[aV(X_0)] = a\Delta W(X_0)$$
$$= a\Delta L(X_0)+a\Delta H(X_0)\pm C^4 \int_{K(X_0,3R)} af(Y)U(r)|_{X=X_0}dY,$$

where $K(X_0, 3R) \subset D$ and $a = -(8\pi C^2)^{-1}$. By (12) we get

$$\Delta^2[aV(X_0)] = a\Delta L(X_0)+aC^4 V(X_0)-C^4 a \int_{K(X_0,3R)} f(Y)U(r)|_{X=X_0}dY.$$

Since

$$\lim a \int_{K(X_0,3R)} f(Y)U(r)|_{X=X_0}dY = 0 \quad \text{as } R\to 0,$$

we have by (14)

$$\Delta^2[aV(X_0)] = f(X_0)+aC^4 V(X_0) \quad \text{for every } X_0\in D.$$

LEMMA 5. *Let f be function measurable and bounded in D. Let $\int_D |f(Y)|dY < \infty$. Then the function $A(X) = \int_D f(Y)U(r_1)dY$ is of class C^4 in D and satisfies equation* (1a) *in this set.*

Proof. We shall prove that the integrals

$$A_{\alpha\beta}(X) = \int_D f(Y)D^{\alpha}_{x1}D^{\beta}_{x2}U(r_1)dY \quad (\alpha, \beta = 0,1,2,3,4;\ \alpha+\beta \leqslant 4)$$

are uniformly convergent at every point $X_0 \in D$. Let $K(X_0, R) \subset D$. For $X\in K(X_0, R)$ and $Y\in D$ we have $r_1 \geqslant x_2 \geqslant \delta > 0$, where δ is a positive constant. From formulae (7) we see that the functions $D^{\alpha}_{x1}D^{\beta}_{x2}U(r_1)$ $(\alpha, \beta = 0,1,2,3,4;\ \alpha+\beta \leqslant 4)$ are bounded for $r_1 \geqslant \delta$. From these results we can obtain the inequalities

$$|A_{\alpha\beta}(X)| \leqslant M_{\alpha\beta}\int_D |f(Y)|dY \quad (\alpha, \beta = 0,1,2,3,4;\ \alpha+\beta \leqslant 4)$$

for $X \in K(X_0, R)$, where $M_{\alpha\beta}$ are positive constants. It follows from the above inequalities that integrals $A_{\alpha\beta}(X)$ ($\alpha, \beta = 0, 1, 2, 3, 4$; $\alpha+\beta \leqslant 4$) are uniformly convergent at the point $X_0 \in D$. Thus the function $A(X)$ is of class C^4 in D and $D^{\alpha}_{x1} D^{\beta}_{x2} A(X) = A_{\alpha\beta}(X)$ for $X \in D$ ($\alpha, \beta = 0, 1, 2, 3, 4$; $\alpha+\beta \leqslant 4$). Taking into consideration the above properties and the fact that the function $U(r_1)$ as a function of the point X satisfies equation (1a) in D, we have

$$\Delta^2 A(X) - C^4 A(X) = \int_D f(Y)[\Delta^2_X U(r_1) - C^4 U(r_1)]\,dY = 0 \qquad \text{for } X \in D.$$

4. As an immediate corollary of Theorem 1 and Lemmas 4, 5 we get

THEOREM 3. *If a function f is bounded and of class C^1 in D and $\int_D |f(Y)|\,dY < \infty$, then the function*

$$u_2(X) = -(8\pi C^2)^{-1} \int_D f(Y) G(X, Y)\,dY \tag{16}$$

is the solution of equation (1) *in the domain D with the boundary conditions:*

$$\lim u_2(X) = 0, \quad \lim \Delta u_2(X) = 0 \quad \text{as } X \to \bar{X} \in \partial D,\ X \in D.$$

As a consequence of Theorems 2, 3 we get the following

THEOREM 4. *If functions f_i ($i = 1, 2$) satisfy the assumptions of Theorem* 2 *and a function f satisfies the assumptions of Theorem* 3, *then the function*

$$u(X) = u_1(X) + u_2(X),$$

where $u_1(X), u_2(X)$ are defined by formulas (4) *and* (16), *respectively, is the solution of equation* (1) *in the domain D with the boundary conditions*:

$$\lim u(X) = f_1(\bar{x}_1), \quad \lim \Delta u(X) = f_2(\bar{x}_1) \quad \text{as } X \to (\bar{x}_1, 0+),\ X \in D.$$

References

[1] M. Filar, *On a certain boundary problem for the equation $\Delta^2 u - C^4 u = 0$*, Comm. Math. 16 (1972), p. 83–90.

[2] M. Krzyżański, *Równania różniczkowe cząstkowe rzędu drugiego*, cz. I, Warszawa 1957 (in Polish).

[3] N. N. Lebiediew, *Funkcje specjalne i ich zastosowania*, Warszawa 1957 (in Polish).

[4] H. Marcinkowska, *Wstęp do teorii równań różniczkowych*, Warszawa 1972, p. 189 (in Polish).

[5] J. Musiałek, *Construction of the fundamental solution for the equation $\Delta^2 u + ku = 0$*, Comm. Math. 9 (1965), p. 213–236.