ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXI (1979)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXI (1979)

Z. POLNIAKOWSKI

On some ordinary differential equations

The purpose of this paper is to prove asymptotic properties (for $x \to \infty$) of integrals of some ordinary, non-linear differential equations. In Section 1 we consider a system of differential equations (Theorem 1) and the differential equation $y' = F(x, y)$ (Theorem 2). Theorem 3 is an application of Theorem 2. In Section 2 we prove Theorem 4 concerning the differential equation $y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$, $n \geqslant 2$, and Theorem 5 as an application of Theorem 4. The functions occurring in this paper are complex valued.

1. We shall formulate the following

THEOREM 1. *Suppose that* $n \geqslant 1$, *the functions* $a_\nu(x)$ *are continuous,* $a_\nu(x) \neq 0$ *for* $x \geqslant x_0$ *and* $\nu = 1, \ldots, n$, *and*

(1) $$|a_\nu(x)| \leqslant K|\operatorname{re} a_\nu(x)| \quad \text{for } x \geqslant x_0,\ \nu = 1, \ldots, n \text{ and some } K \geqslant 1,$$

(2) $$\int_{x_0}^{\infty} \frac{dx}{|a_\nu(x)|} = \infty \quad \text{for } \nu = 1, \ldots, n.$$

Moreover, suppose that there exists $\eta > 0$ *and complex values* c_ν *such that* $f_\nu(x, y_1, \ldots, y_n)$, $\nu = 1, \ldots, n$, *are continuous functions of the variable* x *and satisfy the inequalities*

(3) $$|f_\nu(x, y_1^*, \ldots, y_n^*) - f_\nu(x, y_1^{**}, \ldots, y_n^{**})| \leqslant \lambda \sum_{j=1}^{n} |y_j^* - y_j^{**}|$$

for $x \geqslant x_0$, $|y_j - c_j| \leqslant \eta$, $|y_j^* - c_j| \leqslant \eta$ *and* $|y_j^{**} - c_j| \leqslant \eta$ $(j = 1, \ldots, n)$ *with some* λ *satisfying the inequality* $0 < \lambda < 1/nK$ (*the variables* y_j *are complex*);

(4) $$\lim_{x \to \infty} f_\nu(x, c_1, \ldots, c_n) = c_\nu, \quad \nu = 1, \ldots, n.$$

Then the system (or the differential equation in the case $n = 1$)

(5) $$a_\nu(x) y_\nu' + y_\nu = f_\nu(x, y_1, \ldots, y_n), \quad \nu = 1, \ldots, n,$$

has for sufficiently large x an integral $\{\bar{y}_1, \ldots, \bar{y}_n\}$ *such that* $\lim_{x\to\infty} \bar{y}_\nu(x) = c_\nu$ *for* $\nu = 1, \ldots, n$.

Remark 1. We obtain from Theorem 1 l'Hospital's rule if (5) is replaced by (23) and if $f_\nu(x, y_1, \ldots, y_n) = f_\nu(x)$. Then this theorem can be treated as a generalization of l'Hospital's rule.

Remark 2. System (23) can be generalized in the form

$$(*)\qquad y_\nu = \varphi_\nu(x) + \int_{x_0}^{\infty} N_\nu(x, t) f_\nu\big(x, y_1(t), \ldots, y_n(t)\big)\, dt, \qquad \nu = 1, \ldots, n,$$

where $\lim_{x\to\infty} \varphi_\nu(x) = 0$ and the functions $N_\nu(x, t)$ satisfy conditions of Töplitz:

(i) $\overline{\lim_{x\to\infty}} \int_{x_0}^{\infty} |N_\nu(x, t)|\, dt \leqslant K < \infty$,

(ii) $\lim_{x\to\infty} N_\nu(x, t) = 0$ almost uniformly for $t \geqslant x_0$,

(iii) $\lim_{x\to\infty} \int_{x_0}^{\infty} N_\nu(x, t)\, dt = 1$.

As in the proof of Theorem 1 one can prove the existence, for large x, of a solution $\{\bar{y}_1, \ldots, \bar{y}_n\}$ of system $(*)$ such that $\lim_{x\to\infty} \bar{y}_\nu(x) = c_\nu$.

THEOREM 2. *Suppose that*

(6) $r(x) \neq 0$ *and there exists a continuous* $r'(x)$ *for* $x \geqslant x_0$,

Moreover, suppose that there exists an $\eta > 0$ *such that*

(7) $F(x, y)$ *is a continuous function of the variable* x *for* $x \geqslant x_0$ *and* $|y - r(x)| \leqslant \eta |r(x)|$,

$F_y\big(x, r(x)\big)$ *is continuous, different from* 0 *and*

$$(8)\qquad |F_y(x, r)| \leqslant K |\mathrm{re}\, F_y(x, r)| \quad \textit{for } x \geqslant x_0 \textit{ and some } K \geqslant 1,$$

$$(9)\qquad \int_{x_0}^{\infty} \big|F_y\big(x, r(x)\big)\big|\, dx = \infty,$$

$$(10)\qquad \lim_{x\to\infty} F(x, r)/\{rF_y(x, r)\} = 0,$$

$$(11)\qquad \lim_{x\to\infty} r'/\{rF_y(x, r)\} = 0,$$

$$(12)\qquad |\ln F_y(x, y) - \ln F_y(x, r)| \leqslant \lambda_1/2$$

for $x \geqslant x_0$, $|y - r(x)| \leqslant \eta |r(x)|$ *and for some* λ_1 $(0 < \lambda_1 < 1/K)$.

Then the differential equation

$$(13)\qquad y' = F(x, y)$$

has for sufficiently large x an integral $\bar{y}(x)$ *such that* $\bar{y}(x) \sim r(x)$ *for* $x \to \infty$.

THEOREM 3. *Suppose that the functions* $b_\nu(x)$, $\nu = 1, \ldots, p$, *are continuous for* $x \geqslant x_0$, α *and* β_ν, $\nu = 1, \ldots, p$, *are re-mbers and* $\alpha \neq 0$. *Moreover, suppose that*

(14) *there exists a continuous derivative* $b'(x)$ *for* $x \geqslant x_0$,

(15) $$b(x) \neq 0 \quad \text{and} \quad |b^{(\alpha-1)/\alpha}(x)| \leqslant K |\operatorname{re} \tau_k^{-1} b^{(\alpha-1)/\alpha}(x)|$$

for $x \geqslant x_0$ *and some* $K \geqslant 1$, *where* $\tau_k = e^{2k\pi i/\alpha}$ *with some integer* k,

(16) $$\lim_{x\to\infty} b'(x) b^{1/\alpha-2}(x) = 0,$$

(17) $$\lim_{x\to\infty} b_\nu(x) b^{\beta_\nu/\alpha-1}(x) = 0 \quad \text{for } \nu = 1, \ldots, p.$$

Then the differential equation $y' = y^\alpha - b(x) + \sum_{\nu=1}^{p} b_\nu(x) y^{\beta_\nu}$ *has, for sufficiently large* x, *an integral* $\bar{y}(x)$ *such that* $\bar{y}(x) \sim \tau_k b^{1/\alpha}(x)$ *for* $x \to \infty$.

We assume here $\operatorname{Arg} b^{1/\alpha}(x) = (1/\alpha) \operatorname{Arg} b(x)$. Let us notice that hypotheses of Theorem 3 are satisfied by the functions $b(x) = x^\sigma$ and $b_\nu(x) = x^{\sigma_\nu}$, where we have $\sigma(1-\alpha)/\alpha < 1$ and $\sigma_\nu < \sigma(1-\beta_\nu/\alpha)$ for $\nu = 1, \ldots, p$.

LEMMA. *Suppose that* $n \geqslant 1$ *and for every* ν $(1 \leqslant \nu \leqslant n)$ *the function* $g_\nu(x)$ *satisfies the condition*

(18a) $$\lim_{x\to\infty} |g_\nu(x)| = \infty \quad \text{and} \quad \int_{x_0}^{x} |g'_\nu(t)|\, dt \leqslant K |g_\nu(x)|$$

or

(18b) $$\lim_{x\to\infty} g_\nu(x) = 0, \quad g_\nu(x) \neq 0 \quad \text{and} \quad \int_{x}^{\infty} |g'_\nu(t)|\, dt \leqslant K |g_\nu(x)|$$

for $x \geqslant x_0$, *with some* $K \geqslant 1$;

(19) $\varphi_\nu(x)$ *are continuous for* $x \geqslant x_0$ *and* $\lim_{x\to\infty} \varphi_\nu(x) = 0$ *for* $\nu = 1, \ldots, n$.

We set $J_\nu^{(0)}(z) = |\lambda/g_\nu(x)| \int_{\alpha_\nu}^{\beta_\nu} |g'_\nu(t)| z(t)\, dt$, *where* $\alpha_\nu = x_0$, $\beta_\nu = x$ *in the case* (18a), $\alpha_\nu = x$, $\beta_\nu = \infty$ *in the case* (18b) *and* λ *satisfies the inequality* $0 < \lambda < 1/nK$.

Then the system (or the integral equation in the case $n = 1$):

(20) $$z_\nu = \varphi_\nu(x) + J_\nu^{(0)}\Big(\sum_{j=1}^{n} z_j\Big), \quad \nu = 1, \ldots, n,$$

has for $x \geqslant x_0$ *a solution* $\{\bar{z}_1, \ldots, \bar{z}_n\}$, *defined by* (22), *such that* $\lim_{x\to\infty} \bar{z}_\nu(x) = 0$ *for* $\nu = 1, \ldots, n$.

Proof. We set for $x \geqslant x_0$, $\nu = 1, \ldots, n$ and $m = 0, 1, \ldots$, $J_\nu^{(m+1)}(z) = \sum_{j=1}^{n} J_\nu^{(0)}(J_j^{(m)}(z))$ and we shall prove by induction for $m = 0, 1, \ldots$, the inequality

$$(21) \qquad 0 \leqslant J_\nu^{(m)}(1) \leqslant (n\lambda K)^{m+1}, \quad x \geqslant x_0 \text{ and } \nu = 1, \ldots, n.$$

By (18a) or (18b) we have $0 \leqslant J_\nu^{(0)}(1) \leqslant \lambda K$ and inequality (21) holds for $m = 0$. Suppose that it is true for some $m \geqslant 0$. Then $J_\nu^{(m+1)}(1) \leqslant (n\lambda K)^{m+1} \cdot nJ_\nu^{(0)}(1) \leqslant (n\lambda K)^{m+2}$.

There exists an $M > 0$ such that $|\varphi_\nu(x)| \leqslant M$ for $x \geqslant x_0$ and $\nu = 1, \ldots, n$. We set

$$(22) \qquad \bar{z}_\nu(x) = \varphi_\nu(x) + \sum_{m=0}^{\infty} \sum_{j=1}^{n} J_\nu^{(m)}(\varphi_j), \quad \nu = 1, \ldots, n,$$

the series being uniformly convergent for $x \geqslant x_0$, by (21). Moreover, we have $|\bar{z}_\nu(x)| \leqslant M + Mn^2 \lambda K/(1 - n\lambda K)$ for $x \geqslant x_0$ and $\nu = 1, \ldots, n$. The functions $\bar{z}_\nu(x)$ satisfy system (20). Namely we have

$$\begin{aligned}
\varphi_\nu(x) + J_\nu^{(0)}\Big(\sum_{j=1}^{n} \bar{z}_j(x)\Big) &= \varphi_\nu(x) + \sum_{j=1}^{n} J_\nu^{(0)}(\varphi_j) + \sum_{j=1}^{n} J_\nu^{(0)}\Big(\sum_{m=0}^{\infty} \sum_{k=1}^{n} J_j^{(m)}(\varphi_k)\Big) \\
&= \varphi_\nu(x) + \sum_{j=1}^{n} J_\nu^{(0)}(\varphi_j) + \sum_{m=0}^{\infty} \sum_{k=1}^{n} \sum_{j=1}^{n} J_\nu^{(0)}(J_j^{(m)}(\varphi_k)) \\
&= \varphi_\nu(x) + \sum_{j=1}^{n} J_\nu^{(0)}(\varphi_j) + \sum_{m=0}^{\infty} \sum_{k=1}^{n} J_\nu^{(m+1)}(\varphi_k) = \bar{z}_\nu(x).
\end{aligned}$$

Setting $L_\nu = \overline{\lim_{x \to \infty}} |\bar{z}_\nu(x)|$ and assuming $x \to \infty$ in (20) we obtain $L_\nu \leqslant \lambda K \sum_{j=1}^{n} L_j$. Then $\sum_{\nu=1}^{n} L_\nu \leqslant n\lambda K \sum_{j=1}^{n} L_j$ and $(1 - n\lambda K) \sum_{\nu=1}^{n} L_\nu \leqslant 0$. We obtain form this $L_\nu = 0$ for $\nu = 1, \ldots, n$.

Proof of Theorem 1. We set

$$g_\nu(x) = \exp \int_{x_0}^{x} \frac{dt}{a_\nu(t)} \quad \text{for } x \geqslant x_0$$

and $\nu = 1, \ldots, n$. Since

$$|g_\nu|'/|g_\nu| = \frac{d}{dx} \ln |g_\nu| = \operatorname{re}(g_\nu'/g_\nu),$$

we obtain

$$|g_\nu'|/|g_\nu|' = \left|\frac{1}{a_\nu}\right| \Big/ \operatorname{re} \frac{1}{a_\nu} = |a_\nu|/\operatorname{re} a_\nu.$$

By (1) the functions $\operatorname{re} a_\nu$ and $|g_\nu|'$ have a constant (the same) sign for $x \geqslant x_0$ and we have $|g_\nu'| \leqslant K \big||g_\nu|'\big|$. By (1) and (2) for every ν $(1 \leqslant \nu \leqslant n)$

the function $g_\nu(x)$ satisfies hypothesis (18a) in the case $\operatorname{re} a_\nu > 0$ or (18b) in the case $\operatorname{re} a_\nu < 0$.

First suppose that $c_\nu = 0$ for $\nu = 1, \ldots, n$. We choose an $x_1 \geqslant x_0$ such that we have for $x \geqslant x_1$

$$|f_\nu(x, 0, \ldots, 0)| \leqslant \eta(1-n\lambda K)/4K \quad \text{for } \nu = 1, \ldots, n$$

and

$$|1/g_\nu(x)| \leqslant \eta(1-n\lambda K)/4 \quad \text{if} \quad \operatorname{re} a_\nu(x) > 0.$$

We obtain formally from (5) the system of integral equations

$$y_\nu = \gamma_\nu/g_\nu(x) + \delta_\nu P_\nu\big(f_\nu(x, y_1, \ldots, y_n)\big), \quad \nu = 1, \ldots, n, \tag{23}$$

where $P_\nu(z) = \big(1/g_\nu(x)\big) \int_{\alpha_\nu}^{\beta_\nu} g'_\nu(t) z(t)\, dt$, $\delta_\nu = 1$, $\alpha_\nu = x_1$, $\beta_\nu = x$ in the case $\operatorname{re} a_\nu > 0$ and $\delta_\nu = -1$, $\alpha_\nu = x$, $\beta_\nu = \infty$ in the case $\operatorname{re} a_\nu < 0$; γ_ν are constants. We assume $\gamma_\nu = 1$ if $\operatorname{re} a_\nu > 0$ and $\gamma_\nu = 0$ if $\operatorname{re} a_\nu < 0$.

We set, for $x \geqslant x_1$ and $\nu = 1, \ldots, n$, $y_{\nu 0}(x) = \gamma_\nu/g_\nu(x)$,

$$y_{\nu, m+1}(x) = y_{\nu 0}(x) + \delta_\nu P_\nu\big(f_\nu(x, y_{1m}, \ldots, y_{nm})\big), \quad m = 0, 1, \ldots$$

By (4) and (3) we obtain $|f_\nu(x, y_{10}, \ldots, y_{n0})| \leqslant |f_\nu(x, 0, \ldots, 0)| + \lambda \sum_{\nu=1}^{n} |y_{\nu 0}| \to 0$ for $x \to \infty$, $\nu = 1, \ldots, n$. Then there exist integrals $P_\nu\big(f_\nu(x, y_{10}(x), \ldots, y_{n0}(x))\big)$. Applying l'Hospital's rule, we obtain

$$\begin{aligned}\overline{\lim_{x\to\infty}} \big|P_\nu\big(f_\nu(x, y_{10}, \ldots, y_{n0})\big)\big| &\leqslant \overline{\lim_{x\to\infty}} |f_\nu(x, y_{10}, \ldots, y_{n0}) g'_\nu(x)| / \big||g_\nu(x)|'\big| \\ &\leqslant K \lim_{x\to\infty} f_\nu(x, y_{10}, \ldots, y_{n0}) = 0.\end{aligned}$$

We obtain from this that $\lim_{x\to\infty} y_{\nu 1}(x) = 0$. Similarly we prove by induction for $m = 1, 2, \ldots$ the existence of integrals $P_\nu\big(f_\nu(x, y_{1m}, \ldots, y_{nm})\big)$.

We shall show by induction for $m = 0, 1, \ldots$ that

$$|y_{\nu m} - y_{\nu 0}| \leqslant \eta/2 \quad \text{and} \quad |y_{\nu m}| \leqslant \eta \tag{24}$$

for $x \geqslant x_1$ and $\nu = 1, \ldots, n$. We have $|y_{\nu 0}| \leqslant \eta(1-n\lambda K)/4 < \eta/2$ and (24) holds for $m = 0$. Suppose that it is true for some m ($m \geqslant 0$). Then by (3) we get

$$\begin{aligned}|f_\nu(x, y_{1m}, \ldots, y_{nm})| &\leqslant |f_\nu(x, 0, \ldots, 0)| + \lambda \sum_{j=1}^{n} |y_{j0}| + \lambda \sum_{j=1}^{n} |y_{jm} - y_{j0}| \\ &\leqslant \eta(1-n\lambda K)/4K + n\lambda\eta(1-n\lambda K)/4 + n\lambda\eta/2 \\ &\leqslant \eta(1-n\lambda K)/2K + n\lambda\eta/2 = \eta/2K,\end{aligned}$$

$$|y_{\nu, m+1} - y_{\nu 0}| = \big|P_\nu\big(f_\nu(x, y_{1m}, \ldots, y_{nm})\big)\big| \leqslant (\eta/2K)|P_\nu(1)| \leqslant \eta/2.$$

We obtain $|y_{\nu, m+1}| < \eta$, since $|y_{\nu 0}| < \eta/2$.

Suppose that $J_\nu^{(m)}(z)$ are defined as in lemma, with x_1 instead of x_0. We set $\varphi_j(x) = \sum_{\nu=1}^{n} |f_\nu(x, 0, \ldots, 0)|/n\lambda + |y_{j0}|$. We shall prove by induction for $m = 0, 1, \ldots$ and $x \geqslant x_1$, $\nu = 1, \ldots, n$ the inequalities

$$|y_{\nu,m+1} - y_{\nu m}| \leqslant J_\nu^{(m)}\Big(\sum_{j=1}^{n} \varphi_j\Big). \tag{25}$$

We obtain for $m = 0$

$$|y_{\nu 1} - y_{\nu 0}| = \big|P_\nu\big(f_\nu(x, y_{10}, \ldots, y_{n0})\big)\big| \leqslant J_\nu^{(0)}\big(|f_\nu(x, y_{10}, \ldots, y_{n0})|/\lambda\big)$$
$$\leqslant J_\nu^{(0)}\Big(|f_\nu(x, 0, \ldots, 0)|/\lambda + \sum_{j=1}^{n} |y_{j0}|\Big) = J_\nu^{(0)}\Big(\sum_{j=1}^{n} \varphi_j\Big).$$

Suppose that (25) holds for some m $(m \geqslant 0)$. Then

$$|y_{\nu,m+2} - y_{\nu,m+1}| = \big|P_\nu\big(f_\nu(x, y_{1,m+1}, \ldots, y_{n,m+1}) - f_\nu(x, y_{1m}, \ldots, y_{nm})\big)\big|$$
$$\leqslant J_\nu^{(0)}\Big(\sum_{j=1}^{n} |y_{j,m+1} - y_{jm}\Big)$$
$$\leqslant J_\nu^{(0)}\Big(\sum_{j=1}^{n} J_j^{(m)}\Big(\sum_{s=1}^{n} \varphi_s\Big)\Big) = J_\nu^{(m+1)}\Big(\sum_{s=1}^{n} \varphi_s\Big).$$

By (4) we have $\lim_{x\to\infty} \varphi_j(x) = 0$ for $j = 1, \ldots, n$. Since $\varphi_j(x)$ are continuous for $x \geqslant x_1$, they are bounded for these x. By (25) and (21) we infer that the series $\sum_{m=0}^{\infty} |y_{\nu,m+1} - y_{\nu m}|$ are uniformly convergent for $x \geqslant x_1$, $\nu = 1, \ldots, n$, and there exist the limits functions $\bar{y}_\nu = \lim_{m\to\infty} y_{\nu m} = y_{\nu 0} + \sum_{m=0}^{\infty} (y_{\nu,m+1} - y_{\nu m})$. Since the functions $y_{\nu m}$ are continuous for $x \geqslant x_1$, then $\bar{y}_\nu$ are also continuous for these x. Moreover, they satisfy the inequalities $|\bar{y}_\nu| \leqslant \eta$ for $x \geqslant x_1$. By lemma we have

$$|\bar{y}_\nu| \leqslant |y_{\nu 0}| + \sum_{m=0}^{\infty} |y_{\nu,m+1} - y_{\nu m}| \leqslant \varphi_\nu + \sum_{m=0}^{\infty} J_\nu^{(m)}\Big(\sum_{j=1}^{n} \varphi_j\Big) = \bar{z}_\nu \to 0 \quad \text{for } x \to \infty.$$

We shall prove that the equalities

$$\lim_{m\to\infty} P_\nu\big(f_\nu(x, y_{1m}, \ldots, y_{nm})\big) = P_\nu\big(f_\nu(x, \bar{y}_1, \ldots, \bar{y}_n)\big)$$

hold uniformly for $x \geqslant x_1$, $\nu = 1, \ldots, n$. For a given $\varepsilon > 0$ we choose an index M such that $|y_{\nu m} - \bar{y}_\nu| \leqslant \varepsilon$ for $m \geqslant M$, $\nu = 1, \ldots, n$ and $x \geqslant x_1$. Then

$$\big|P_\nu\big(f_\nu(x, y_{1m}, \ldots, y_{nm})\big) - P_\nu\big(f_\nu(x, \bar{y}_1, \ldots, \bar{y}_n)\big)\big| \leqslant \lambda \Big|P_\nu\Big(\sum_{j=1}^{n} |y_{jm} - \bar{y}_j|\Big)\Big|$$
$$\leqslant n\lambda\varepsilon |P_\nu(1)| \leqslant n\lambda K\varepsilon < \varepsilon.$$

We obtain from this that $\bar{y}_\nu$ satisfy system (23). It is easy to see that the functions $f_\nu(x, \bar{y}_1(x), \ldots, \bar{y}_n(x))$ are continuous for $x \geqslant x_1$. Then by (23) the functions $\bar{y}_\nu(x)$ are differentiable and satisfy system (5).

In the case $c_\nu \neq 0$ we substitute into (5) $y_\nu(x) = u_\nu(x)+c_\nu, f_\nu(x, y_1, \ldots, y_n) = f_\nu^*(x, u_1, \ldots, u_n)+c_\nu$ and we obtain the system

$$(26) \qquad a_\nu(x)u_\nu'+u_\nu = f_\nu^*(x, u_1, \ldots, u_n), \qquad \nu = 1, \ldots, n,$$

where the functions $f_\nu^*(x, u_1, \ldots, u_n)$ (instead of $f_\nu(x, y_1, \ldots, y_n)$) satisfy hypotheses of Theorem 1 for $c_\nu = 0$. By the proved part of this theorem we obtain that there exists for $x \geqslant x_1$ an integral $\{\bar{u}_1, \ldots, \bar{u}_n\}$ of system (26) such that $\lim\limits_{x\to\infty} \bar{u}_\nu(x) = 0$ for $\nu = 1, \ldots, n$. We set $\bar{y}_\nu(x) = \bar{u}_\nu(x)+c_\nu$.

Proof of Theorem 2. We substitute $y = r(x)u$ into (13) and we obtain the differential equation

$$(27) \qquad hu'+u = f(x, u),$$

where $h(x) = -1/F_y(x, r)$ and $f(x, u) = u+hF(x, ru)/r-(hr'/r)u$.

We shall show that the functions $h(x) = a_1(x)$ and $f(x, u) = f_1(x, u)$ satisfy hypotheses of Theorem 1 for $n = c_1 = 1$. By (8) and (9), hypotheses (1) and (2) are satisfied. By (12) and the inequality $|e^z-1| \leqslant 2|z|$, true for $|z| \leqslant 1/2$, we get $F_y(x, y)/F_y(x, r) = e^{\theta\lambda_1}$, where

$$|\theta| \leqslant 1/2, \quad \text{and} \quad |F_y(x, y)/F_y(x, r)-1| = |e^{\theta\lambda_1}-1| \leqslant \lambda_1.$$

We choose λ satisfying the inequality $\lambda_1 < \lambda < 1/K$ and x_1 ($x_1 \geqslant x_0$) such that $|hr'/r| \leqslant \lambda-\lambda_1$ for $x \geqslant x_1$, what by (11) is possible. Then

$$\begin{aligned}|f_u(x, u)| &= |1+hF_y(x, ru)-hr'/r| \leqslant |F_y(x, y)/F_y(x, r)-1|+|hr'/r| \\ &\leqslant \lambda_1+\lambda-\lambda_1 = \lambda\end{aligned}$$

for $x \geqslant x_1$ and $|u-1| \leqslant \eta$, and hypothesis (3) is satisfied. Finally, by (10) and (11) we get

$$f(x, 1) = 1-F(x, r)/\{rF_y(x, r)\}-hr'/r \to 1 \quad \text{for } x \to \infty$$

and hypothesis (4) is satisfied. We complete the proof by applying Theorem 1.

Proof of Theorem 3. We shall show that hypotheses of Theorem 2 are satisfied by the functions $F(x, y) = y^\alpha-b(x)+\sum\limits_{\nu=1}^{p} b_\nu(x)y^{\beta_\nu}$ and $r(x) = \tau_k b^{1/\alpha}(x)$. We obtain $F_y(x, r) = \alpha r^{\alpha-1}+\sum\limits_{\nu=1}^{p} \beta_\nu b_\nu r^{\beta_\nu-1} \sim \alpha r^{\alpha-1}$ for $x \to \infty$, by (17). By (16) we get $b^{(1-\alpha)/\alpha}(x) = o(x)$ for $x \to \infty$ if $\alpha \neq 1$, and $\int\limits_{x_0}^{\infty} |b^{(\alpha-1)/\alpha}(x)|\,dx = \infty$. Then the function $F_y(x, r)$ satisfies hypothesis (9).

By the equality $\operatorname{re} C = \operatorname{re} D \operatorname{re}(C/D) - \operatorname{im} D \operatorname{im}(C/D)$ for $C = F_y(x, r)$ and $D = \alpha r^{\alpha-1}$ we obtain $\operatorname{re} C \sim \operatorname{re} D$ for $x \to \infty$, since by (15) we have $|\operatorname{im} D/\operatorname{re} D| \leqslant K$. Then hypothesis (8) is satisfied for some K_1 $(K_1 > K)$ and for sufficiently large x. By (17) we obtain $\lim_{x\to\infty} F(x, r)/\{rF_y(x, r)\} = \lim_{x\to\infty} \sum_{\nu=1}^{p} b_\nu r^{\beta_\nu}/\alpha r^\alpha = 0$. By (16) we get $\lim_{x\to\infty} r'/\{rF_y(x, r)\} = \lim_{x\to\infty} r'/\alpha r^\alpha = \tau_k \alpha^{-2} \lim_{x\to\infty} b' b^{1/\alpha-2} = 0$ and hypotheses (10) and (11) are satisfied.

We choose λ_2 satisfying the inequality $0 < \lambda_2 < 1/K_1$. Suppose that y satisfies the condition

$$|y/r(x) - 1| \leqslant \eta, \quad x \geqslant x_1, \tag{28}$$

where $\eta = \min(1/2, \lambda_2/8|\alpha-1|)$ if $\alpha \neq 1$ and $\eta = 1/2$ if $\alpha = 1$. Then there exist constants M_1 and M_2 such that $0 < M_1 \leqslant |y/r| \leqslant M_2$. We obtain $\ln\{F_y(x, y)/\alpha r^{\alpha-1}\} = \ln\{(y/r)^{\alpha-1} + \varrho(x, y)\}$, where by (17) we have

$$\varrho(x, y) = \sum_{\nu=1}^{p} \sigma_\nu b_\nu b^{\beta_\nu/\alpha-1} (y/r)^{\beta_\nu-1} \to 0 \quad \text{for } x \to \infty$$

and y satisfying (28); σ_ν are constants.

By the inequality $|\ln(A+B)| \leqslant |\ln A| + 2|B/A|$ $(|B/A| \leqslant 1/2,\ A \neq 0)$ for $A = (y/r)^{\alpha-1}$ and $B = \varrho(x, y)$, we obtain

$$|\ln\{F_y(x, y)/\alpha r^{\alpha-1}\}| \leqslant |\ln(y/r)^{\alpha-1}| + 2M|\varrho(x, y)| \tag{29}$$

for $x \geqslant x_1$ and y satisfying (28), where $M = \max_{x \geqslant x_1} |y/r|^{1-\alpha}$ and $x_1 \geqslant x_0$ is so chosen that $|\varrho(x, y)(y/r)^{1-\alpha}| \leqslant 1/2$ for these x and y.

We choose $x_2 \geqslant x_1$ such that

$$|\varrho(x, y)| \leqslant \lambda_2/16M \quad \text{and} \quad |\ln\{F_y(x, r)/\alpha r^{\alpha-1}\}| \leqslant \lambda_2/8$$

for $x \geqslant x_2$ and y satisfying (28). By (29) we obtain for these x and y

$$\begin{aligned} |\ln F_y(x, y) - \ln F_y(x, r)| &\leqslant |\ln\{F_y(x, y)/\alpha r^{\alpha-1}\}| + |\ln\{F_y(x, r)/\alpha r^{\alpha-1}\}| \\ &\leqslant |(\alpha-1)\ln(y/r)| + 2M|\varrho(x, y)| + |\ln\{F_y(x, r)/\alpha r^{\alpha-1}\}| \\ &\leqslant \lambda_2/4 + \lambda_2/8 + \lambda_2/8 = \lambda_2/2, \end{aligned}$$

since by (28) we have $y/r = 1 + \theta\eta$ $(|\theta| \leqslant 1)$, and in the case $\alpha \neq 1$ we get $|\ln(y/r)| = |\ln(1+\theta\eta)| \leqslant 2\eta \leqslant \lambda_2/4|\alpha-1|$.

Then hypothesis (12) is satisfied. We complete the proof by applying Theorem 2.

2. In this section we shall prove two theorems.

THEOREM 4. *Suppose that* $n \geqslant 2$ *and*

(30) $r(x) \neq 0$ *and there exists a continuous* $r'(x)$ *for* $x \geqslant x_0$,

there exists $\eta_1 > 0$ *such that* $F(x, y_1, \ldots, y_n)$, $\frac{d}{dx} F_{y_1}(x, r, 0, \ldots, 0)$ *and* $F_{y_\nu}(x, r, 0, \ldots, 0)$, $\nu = 2, \ldots, n$, *are continuous functions of the variable* x *and* $F_{y_1}(x, r, 0, \ldots, 0) \neq 0$ *for* $x \geqslant x_0$, $|y_1 - r| \leqslant \eta_1 |r|$, $|y_\nu| \leqslant \eta_1 |rh^{1-\nu}|$, $\nu = 2, \ldots$ $\ldots, n$, *where* $h(x) = \{F_{y_1}(x, r, 0, \ldots, 0)\}^{-1/n}$. *The variables* y_ν *are complex. Moreover, suppose that*

(31) $$|h(x)| \leqslant K |\mathrm{re}\, \varepsilon_m h(x)| \quad \textit{for } m = 1, \ldots, n,$$

and $x \geqslant x_0$, *where* $\varepsilon_m = e^{2m\pi i/n}$;

(32) $$\lim_{x\to\infty} h' = \lim_{x\to\infty} hr'/r = 0,$$

(33) $$\lim_{x\to\infty} F(x, r, 0, \ldots, 0)/\{rF_{y_1}(x, r, 0, \ldots, 0)\} = 0.$$

Let us suppose that we have for $x \geqslant x_0$, $|y_1 - r| \leqslant \eta_1 |r|$, $|y_\nu| \leqslant \eta_1 |rh^{1-\nu}|$, $\nu = 2, \ldots, n$, *and with some* λ_1 $(0 < \lambda_1 < 1/nK)$:

(34) $$|\ln F_{y_1}(x, y_1, \ldots, y_n) - \ln F_{y_1}(x, r, 0, \ldots, 0)| \leqslant \lambda_1/2,$$

(35) $$|F_{y_\nu}(x, y_1, \ldots, y_n) h^{n-\nu+1}| \leqslant \lambda_1, \quad \nu = 2, \ldots, n.$$

Then the differential equation

(36) $$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$$

has for sufficiently large x *an integral* $\bar{y}(x)$ *such that* $\bar{y}(x) \sim r(x)$ *and* $\bar{y}^{(\nu)}(x) = o(rh^{-\nu})$, $\nu = 1, \ldots, n-1$, *for* $x \to \infty$.

Let us notice that hypothesis (31) is satisfied if, for example, $h(x) = \sigma h^*(x)$, where σ is a complex number, $h^*(x)$ is a real valued function and we have $\mathrm{re}(\varepsilon_m \sigma) \neq 0$ for $m = 1, \ldots, n$.

THEOREM 5. *Suppose that* $n \geqslant 2$ *and the functions* $b_{\nu s}(x)$ *are continuous for* $x \geqslant x_0$, α and $\beta_{\nu s}$ *are real numbers*, $\alpha \neq 0$, $\beta_{\nu s} \geqslant 1$ *for* $\nu = 2, \ldots, n$ *and* $s = 1, \ldots, p_\nu$. *Moreover, suppose that*

(37) *there exist continuous derivatives* $b'(x)$ *and* $b'_{1s}(x)$ *for* $x \geqslant x_0$ *and* $s = 1, \ldots, p_1$,

(38) $b(x) \neq 0$ *and* $|b^{(1-\alpha)/n\alpha}| \leqslant K |\mathrm{re}(\varrho \varepsilon_m \tau_k^{(1-\alpha)/n} b^{(1-\alpha)/n\alpha})|$ *for* $x \geqslant x_0$, $m = 1, \ldots, n$ *and some integer* k, *where* $\varepsilon_m = e^{2m\pi i/n}$, $\tau_k = e^{2k\pi i/\alpha}$, $\varrho = 1$ *if* $\alpha > 0$ *and* $\varrho = e^{-\pi i/n}$ *if* $\alpha < 0$,

(39) $$\lim_{x\to\infty} b' b^{(1-\alpha)/n\alpha - 1} = 0,$$

(40) $$\lim_{x\to\infty} b_{1s} b^{\beta_{1s}/\alpha - 1} = 0 \quad \textit{for } s = 1, \ldots, p_1,$$

(41) $$\lim_{x\to\infty} b'_{1s} b^{\beta_{1s}/\alpha + (1-\alpha)/n\alpha - 1} = 0 \quad \textit{if } \alpha \neq 1,$$
$$\lim_{x\to\infty} \frac{d}{dx}(b_{1s} b^{\beta_{1s} - 1}) = 0 \quad \textit{if } \alpha = 1,\ s = 1, \ldots, p_1,$$

$$\lim_{x\to\infty} b_{\nu s} b^{(\nu-1)(\alpha-1)\beta_{\nu s}/n\alpha+\beta_{\nu s}/\alpha-1} = 0 \quad \textit{for } \nu = 2, \ldots, n, \tag{42}$$

and $s = 1, \ldots, p_\nu$.

Then the differential equaiion

$$y^{(n)} = y^\alpha - b(x) + \sum_{\nu=1}^{n} \sum_{s=1}^{p_s} b_{\nu s}(x)(y^{(\nu-1)})^{\beta_{\nu s}} \tag{43}$$

has for sufficiently large x *an integral* $\bar{y}(x)$ *such that* $\bar{y}(x) \sim \tau_k b^{1/\alpha}(x)$ *and* $\bar{y}^{(\nu)}(x) = o(b^{\nu(\alpha-1)/n\alpha+1/\alpha})$, $\nu = 1, \ldots, n-1$, *for* $x \to \infty$.

Let us observe that hypothesis (38) is satisfied if, for example, $b^{(1-\alpha)/n\alpha}(x) = \sigma B(x)$, where σ is a complex number, $B(x)$ is a real valued function and we have $\operatorname{re}(\varrho\varepsilon_m \tau_k^{(1-\alpha)/n}\sigma) \neq 0$ for $m = 1, \ldots, n$.

Proof of Theorem 4. The differential equation (36) can be written in the form

$$y'_\nu = y_{\nu+1}, \quad \nu = 1, \ldots, n-1, \quad y'_n = F(x, y_1, \ldots, y_n),$$

where $y_\nu = y^{(\nu-1)}$. Substituting $y_\nu = (1/n)\sum_{j=1}^{n} \varepsilon_{j-1}^{\nu-1} u_j r h^{1-\nu}$, $\nu = 1, \ldots, n$, where $\varepsilon_k = e^{2k\pi i/n}$, we obtain the system

$$\begin{aligned} &h\sum_{j=1}^{n} \varepsilon_{j-1}^{\nu-1} u'_j + (hr'/r + (1-\nu)h')\sum_{j=1}^{n} \varepsilon_{j-1}^{\nu-1} u_j = \sum_{j=1}^{n} \varepsilon_{j-1}^{\nu} u_j, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nu = 1, \ldots, n-1, \\ &h\sum_{j=1}^{n} \varepsilon_{j-1}^{n-1} u'_j + (hr'/r + (1-n)h')\sum_{j=1}^{n} \varepsilon_{j-1}^{n-1} u_j = nh^n F/r. \end{aligned} \tag{44}$$

For a given m $(1 \leqslant m \leqslant n)$ we multiply the ν-th equation in (44) by $\varepsilon_{m-1}^{n-\nu}$, $\nu = 1, \ldots, n$, and we add the obtained n equations. It is easy to show that

$$\sum_{\nu=1}^{n} \varepsilon_{m-1}^{n-\nu} \sum_{j=1}^{n} \varepsilon_{j-1}^{\nu-1} u_j = n\varepsilon_{m-1}^{n-1} u_m$$

and

$$\sum_{\nu=1}^{n-1} \varepsilon_{m-1}^{n-\nu} \sum_{j=1}^{n} \varepsilon_{j-1}^{\nu} u_j = \sum_{j=1}^{n} u_j \sum_{\nu=1}^{n} \varepsilon_\nu^{j-m} - \sum_{j=1}^{n} u_j = nu_m - \sum_{j=1}^{n} u_j.$$

Then we obtain

$$\varepsilon_{m-1}^{n-1} hu'_m + \varepsilon_{m-1}^{n-1}(hr'/r)u_m + h'\sum_{j=1}^{n} \sigma_{mj} u_j = u_m - (1/n)\sum_{j=1}^{n} u_j + h^n F/r,$$

$$-\varepsilon_{m-1}^{n-1} hu'_m + u_m = (1/n)\sum_{j=1}^{n} u_j - h^n F/r + \sum_{j=1}^{n} d_{mj}(x) u_j, \tag{45}$$

$m = 1, \ldots, n$, where σ_{mj} are constants and $\lim_{x\to\infty} d_{mj}(x) = 0$, by (32). System

(45) can be written in the form

$$a_\nu(x)u_\nu' + u_\nu = f_\nu(x, u_1, \ldots, u_n), \qquad \nu = 1, \ldots, n, \tag{46}$$

where $a_\nu(x) = -\varepsilon_{\nu-1}^{n-1}h(x)$ and $f_\nu(x, u_1, \ldots, u_n) = (1/n)\sum_{j=1}^{n} u_j - h^n F/r + \sum_{j=1}^{n} d_{\nu j}u_j$.
We shall prove that the functions a_ν and f_ν satisfy hypotheses of Theorem 1.

By (31) hypothesis (1) is satisfied. By (32) we obtain $h(x) = o(x)$ for $x \to \infty$ and hypothesis (2) is also satisfied. We have $u_m = \sum_{\nu=1}^{n} \varepsilon_{m-1}^{n-\nu+1} y_\nu h^{\nu-1}/r$, $m = 1, \ldots, n$, and the domain defined by the inequalities $|y_1 - r| \leqslant \eta_1|r|$, $|y_\nu| \leqslant \eta_1|rh^{1-\nu}|$, $\nu = 2, \ldots, n$, is transformed into the domain $|u_m - 1| \leqslant n\eta_1$ for $m = 1, \ldots, n$. We have namely

$$|u_m - 1| = \Big|\sum_{\nu=1}^{n} \varepsilon_{m-1}^{n-\nu+1} y_\nu h^{\nu-1}/r - 1\Big| \leqslant |y_1/r - 1| + \sum_{\nu=2}^{n} |y_\nu h^{\nu-1}/r| \leqslant n\eta_1.$$

We set $n\eta_1 = \eta$. Then $f_\nu(x, u_1, \ldots, u_n)$ are continuous functions of the variable x for $x \geqslant x_0$ and $|u_\nu - 1| \leqslant \eta$.

Next, we have

$$\frac{\partial}{\partial u_m} f_\nu(x, u_1, \ldots, u_n) = 1/n - h^n F_{y_1}/n - \sum_{j=2}^{n} \varepsilon_{m-1}^{j-1} h^{n-j+1} F_{y_j}/n + d_{\nu m}.$$

By (34) we show, as in the proof of Theorem 2, that $|1 - h^n F_{y_1}| \leqslant \lambda_1$. We choose values λ ($\lambda_1 < \lambda < 1/nK$) and $x_1 \geqslant x_0$ such that we have $|d_{\nu m}(x)| \leqslant \lambda - \lambda_1$ for $x \geqslant x_1$ and $\nu, m = 1, \ldots, n$. By (35) we get

$$\left|\frac{\partial}{\partial u_m} f_\nu(x, u_1, \ldots, u_n)\right| \leqslant \lambda_1/n + (n-1)\lambda_1/n + \lambda - \lambda_1 = \lambda \quad \text{for } x \geqslant x_1 \text{ and } |u_\nu - 1| \leqslant \eta.$$

Then hypothesis (3) is satisfied for $c_\nu = 1$. Finally, we have $f_\nu(x, 1, \ldots, 1) = 1 - h^n F(x, r, 0, \ldots, 0)/r + \sum_{j=1}^{n} d_{\nu j} \to 1$, by (33) and (32).

By Theorem 1 system (46) has an integral $\{\bar{u}_1, \ldots, \bar{u}_n\}$ such that $\lim_{x\to\infty} \bar{u}_\nu(x) = 1$, $\nu = 1, \ldots, n$. We set for $x \geqslant x_1$: $\bar{y}_\nu = (1/n)\sum_{j=1}^{n} \varepsilon_{j-1}^{\nu-1} \bar{u}_j rh^{1-\nu}$, $\nu = 1, \ldots, n$, and $\bar{y}(x) = \bar{y}_1(x)$.

Proof of Theorem 5. We shall show that hypotheses of Theorem 4 are satisfied by the functions

$$F(x, y_1, \ldots, y_n) = y_1^\alpha - b(x) + \sum_{\nu=1}^{n}\sum_{s=1}^{p_\nu} b_{\nu s}(x)\, y_\nu^{\beta_{\nu s}} \quad \text{and} \quad r(x) = \tau_k b^{1/\alpha}(x).$$

We have $F_{y_1}(x, r, 0, \ldots, 0) = \alpha r^{\alpha-1} + \sum_{s=1}^{p_1} \beta_{1s} b_{1s} r^{\beta_{1s}-1}$ and $F_{y_\nu}(x, r, 0, \ldots, 0) = \sum_{s=1}^{p_\nu} \beta_{\nu s} b_{\nu s} r^{\beta_{\nu s}-1}$ for $\nu = 2, \ldots, n$.

By (40) we get $F_{y_1}(x, r, 0, \ldots, 0) \sim \alpha r^{\alpha-1}$ and $h(x) = \{F_{y_1}(x, r, 0, \ldots, 0)\}^{-1/n} \sim \alpha^{-1/n(1-\alpha)/n}$ for $x \to \infty$. As in the proof of Theorem 3 we show that $\operatorname{re} h(x) \sim \operatorname{re}(\alpha^{-1/n} r^{(1-\alpha)/n})$ for $x \to \infty$. By (38) there exists K_1 ($K_1 > K$) such that the function $h(x)$ satisfies hypothesis (31) for sufficiently large x and K_1 instead of K.

If $\alpha \neq 1$, then

$$\begin{aligned} -nh'h^{-n-1} &= \frac{d}{dx} F_{y_1}(x, r, 0, \ldots, 0) = \alpha(\alpha-1) r' r^{\alpha-2} + \sum_{s=1}^{p_1} \beta_{1s}(b_{1s} r^{\beta_{1s}-1})' \\ &= r' r^{\alpha-2}\Big\{\alpha(a-1) + \sum_{s=1}^{p_1} \beta_{1s}(\beta_{1s}-1) b_{1s} r^{\beta_{1s}-\alpha}\Big\} + \sum_{s=1}^{p_1} \beta_{1s} b'_{1s} r^{\beta_{1s}-1} \\ &= O(r' r^{\alpha-2}) + \sum_{s=1}^{p_1} \beta_{1s} b'_{1s} r^{\beta_{1s}-1} \end{aligned}$$

for $x \to \infty$, by (40). Moreover, we have for $x \to \infty$

$$h^{n+1} r' r^{\alpha-2} \sim \alpha^{-(n+1)/n} r' r^{(1-\alpha)/n-1} \sim cb' b^{(1-\alpha)/n\alpha-1} \to 0, \quad \text{by (39)},$$

$$h^{n+1} b'_{1s} r^{\beta_{1s}-1} \sim c_1 b'_{1s} b^{\beta_{1s}/\alpha+(1-\alpha)/n\alpha-1} \to 0, \quad \text{by (41)},$$

and we obtain that $\lim_{x\to\infty} h'(x) = 0$.

If $\alpha = 1$, then $\lim_{x\to\infty} h(x) = 1$ and

$$-nh'h^{-n-1} = \frac{d}{dx} F_{y_1}(x, r, 0, \ldots, 0) = \sum_{s=1}^{p_1} \beta_{1s}(b_{1s} r^{\beta_{1s}-1})' \to 0, \quad \text{by (41)}.$$

By (39) we get

$$hr'/r \sim \alpha^{-1/n} r' r^{(1-\alpha)/n-1} \sim c_2 b' b^{(1-\alpha)/n\alpha-1} \to 0 \quad \text{for } x \to \infty$$

and hypothesis (32) is satisfied.

By (40) we obtain for $x \to \infty$

$$F(x, r, 0, \ldots, 0)/\{rF_{y_1}(x, r, 0, \ldots, 0)\} \sim \sum_{s=1}^{p_1} b_{1s} r^{\beta_{1s}-\alpha}/\alpha \to 0$$

and hypothesis (33) is satisfied.

We choose λ_1 satisfying the inequality $0 < \lambda_1 < 1/nK$. Suppose that the variables y_ν satisfy the inequalities

$$(47) \qquad |y_1/r - 1| \leqslant \eta, \quad |y_\nu| \leqslant \eta\, |rh^{1-\nu}| \quad \text{for } \nu = 2, \ldots, n,\ x \geqslant x_0,$$

where $\eta = \min\{1/2, \lambda_1/8|\alpha-1|\}$ if $\alpha \neq 1$ and $\eta = 1/2$ if $\alpha = 1$. We obtain for $x \geqslant x_0$ and y_ν satisfying (47)

$$\begin{aligned}
&|\ln F_{y_1}(x, y_1, \ldots, y_n) - \ln F_{y_1}(x, r, 0, \ldots, 0)| \\
&\quad = |\ln\{F_{y_1}(x, y_1, \ldots, y_n)/\alpha r^{\alpha-1}\} - \ln\{F_{y_1}(x, r, 0, \ldots, 0)/\alpha r^{\alpha-1}\}| \\
&\quad \leqslant |\ln\{(y_1/r)^{\alpha-1} + (1/\alpha)\sum_{s=1}^{p_1} \beta_{1s} b_{1s} b^{\beta_{1s}/\alpha-1}(y_1/r)^{\beta_{1s}-1}\}| + \\
&\qquad + |\ln\{F_{y_1}(x, r, 0, \ldots, 0)/\alpha r^{\alpha-1}\}|
\end{aligned}$$

and as in the proof of Theorem 3 we show that hypothesis (34) is satisfied for some $x_1 \geqslant x_0$ instead of x_0. Finally, we obtain for $x \geqslant x_0$ and y_ν, $\nu = 2, \ldots, n$, satisfying (47):

$$\begin{aligned}
|F_{y_\nu}(x, y_1, \ldots, y_n) h^{n-\nu+1}| &= \Big|h^{n-\nu+1} \sum_{s=1}^{p_\nu} \beta_{\nu s} b_{\nu s} y_\nu^{\beta_{\nu s}-1}\Big| \\
&\leqslant \Big|h^{n-\nu+1} \sum_{s=1}^{p_\nu} \beta_{\nu s} \eta^{\beta_{\nu s}-1} b_{\nu s} r^{\beta_{\nu s}-1} h^{(1-\nu)(\beta_{\nu s}-1)}\Big|.
\end{aligned}$$

Since

$$\begin{aligned}
h^{n-\nu+1} b_{\nu s} r^{\beta_{\nu s}-1} h^{(1-\nu)(\beta_{\nu s}-1)} &= b_{\nu s} r^{\beta_{\nu s}-1} h^{n+(1-\nu)\beta_{\nu s}} \sim \\
&\sim c_3 b_{\nu s} b^{(\nu-1)(\alpha-1)\beta_{\nu s}/n\alpha + \beta_{\nu s}/\alpha - 1} \to 0, \quad \text{by (42)},
\end{aligned}$$

then hypothesis (35) is satisfied for sufficiently large x and y_ν satisfying (47). We complete the proof applying Theorem 4.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES