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On the measures of non-compactness

In [1] K. Kuratowski defined a measure of non-compactness and proved the theorem of non-empty intersection for it.

We give here a shorter proof of this theorem.

Theorem 1. *Let* (M, d) *be a complete metric space and let* $b\colon 2^M \to \bar{R}$ *be a measure of non-compactness satisfying the following conditions:*

1° $b(E) = 0$ *iff* $\bar{E}$ *is compact,*

2° *if* $E \subset F$, *then* $b(E) \leqslant b(F)$,

3° $b(E) = b(E \cup \{x\})$ *for* $x \in M$.

Then, if for a chain $(\mathscr{F}, C)$ *of closed non-empty subsets of* M *we have* $\inf\big(b(F)\colon F \in \mathscr{F}\big) = 0$, *then* $\bigcap_{F \in \mathscr{F}} F \neq \emptyset$.

Proof. Let us consider a subsequence $(F_n)_{n \in N}$ such that, for all $n \in N$, $F_{n+1} \subset F_n$ and $\lim_{n \to \infty} b(F_n) = 0$. We can choose elements $x_n \in F_n$. Let us consider the set $D = \bigcup_{n=1}^{\infty} \{x_n\}$. By 3° for all $k \in N$ we have

$$b(D) = b\Big(\bigcup_{n=1}^{\infty} \{x_n\}\Big) = b\Big(\bigcup_{n=1}^{\infty} \{x_n\}\Big) \leqslant b(F_k)$$

and therefore $b(D) = 0$. All the limits of the convergent subsequences of $(x_n)_{n \in N}$ are contained in $F\colon = \bigcap_{n=1}^{\infty} F_n$, which is closed and compact. Now we may consider only a chain consisting of compact sets, which, as it is known, have a non-empty intersection. Q.E.D.

Let (X, T) be a linear topological space.

Definition 1. We say that for $Y \subset X$ a mapping $f\colon Y \to Y$ is *generalized condensing* if for $A \subset Y$ and such that $f(A) \subset A$ from the compactness of $\overline{A - \overline{\text{co}}\, f(A)}$ it follows that $\bar{A}$ is compact (cf. [2]).

We present below a modification of the result obtained in [2].

THEOREM 2. *If* $A = \overline{\mathrm{co}}\, A \subset X$ and $f\colon A \to A$ *is a generalized condensing mapping, then there exists a compact convex set* C *such that* $\overline{\mathrm{co}}\, f(C) = C$.

LEMMA. *If* $A = \overline{\mathrm{co}}\, A$, $f\colon A \to A$ *and there exists a compact set* $B \subset A$ *such that* $f(B) \subset B$, *then there exists a* $C = \overline{\mathrm{co}}\, C \subset A$ *such that* $\overline{\mathrm{co}}\, f(C) = C$.

Proof. Let us consider a family $\mathcal{F} = \big(F = \overline{\mathrm{co}}\, F \subset A\colon \overline{\mathrm{co}}\, f(F) \subset F\big)$. It is non-empty, as $A \in \mathcal{F}$. By the Hausdorff theorem there exists a maximal chain $\mathcal{H} \subset \mathcal{F}$ containing a set that has a non-empty intersection with B. Let $\mathcal{G} \subset \mathcal{H}$ consist of all sets G such that $G \cap B \neq \emptyset$. The set $C := \bigcap_{G \in \mathcal{G}} G$ is non-empty because, by the compactness of B, $\bigcap_{G \in \mathcal{G}} G \cap B \neq \emptyset$. Moreover, $C = \overline{\mathrm{co}}\, C$, $\overline{\mathrm{co}}\, f(C) \subset C$ and $\overline{\mathrm{co}}\, f(C) \in \mathcal{G}$, as $f(B \cap C) \subset B$. By virtue of the definition of C it must be $\overline{\mathrm{co}}\, f(C) = C$.

Proof of the theorem. In view of the lemma and the definition of the generalized condensing mapping, the theorem will be proved if we find a compact set B such that $f(B) \subset B$.

Let D be any compact subset of A, and $B = \bar{B}$ the minimal set containing D such that $f(B) \subset B$. We see that $B - \overline{f(B)} \subset D$ because $B - \overline{f(B)} \cap (X - D)$ is relatively open in B and would be rejected when non-empty. Now, by the definition of f, B is compact.

COROLLARY. *If* (X, T) *is a topological space and for a mapping* $f\colon X \to X$ *there exists a compact set* B *such that* $f(B) \subset B$, *then we can find a compact set* $C \subset X$ *for which* $\overline{f(C)} = C$.

Proof. We may follow the procedure that was used in the proof of our lemma, with substituting the operation "$\overline{\mathrm{co}}$" by "$\overline{}$" (closure), and considering a maximal chain containing B.

DEFINITION 2. We say that a mapping $f\colon X \to X$ is *b-condensing* ($b\colon 2^X \to \bar{R}$) if for all $A \subset X$ such that $b\big(f(A)\big) < \infty$ and $b(A) \leqslant b\big(f(A)\big)$, $\overline{f(A)}$ is compact.

We see that if b satisfies 1°, 2° and

3°′ $b(E) = b(\overline{\mathrm{co}}\, E \cup Z)$ for all $Z \subset A$ and $b(Z) = 0$,

then for $Y \subset X$ and $b(Y) < \infty$, any b-condensing mapping $f\colon Y \to Y$ is generalized condensing.

Now, let $(B, \|\ \|)$ be a Banach space and let $s\colon 2^B \to \bar{R}$ denote the Hausdorff measure of non-compactness (for $A \subset B$: $s(A) = \inf(r \in \bar{R}$: there exists a finite r-net in A)). It is known that for $E, F \subset B$ s satisfies 1° and the following conditions:

4° $s(E \cup F) = \max\big(s(E), s(F)\big)$,

5° $s(\overline{\mathrm{co}}\, E) = s(E)$,

6° if $N_e(E) := \big(x \in B\colon \operatorname{dist}(x, E) \leqslant e\big)$, then there exists a function $k\colon R^+ \to \bar{R}$ such that $\lim_{e \to 0^+} k(e) = 0$, $s\big(N_e(E)\big) \leqslant s(E) + k(e)$.

THEOREM 3. *The measure of non-compactness* $b\colon 2^B \to \bar{R}$ *satisfies* 1°, 4°–6° *iff there exists a non-decreasing continuous function* $g\colon \bar{R} \to \bar{R}$ *such that* $b = g \circ s$.

Proof. (i) For $x \in B$, $r \in R^+$ and the balls $K \subset B$ we have $\lim_{e \to 0} b(K(x, r+e)) = b(K(x, r))$.

Let $0 < e$; then in view of 4° and, on the other hand, by 6°,

$$b(K(x, r)) \leqslant b(K(x, r+e)) \leqslant b(K(x, r)) + k(e),$$

and similarly, for $-r \leqslant e < 0$,

$$b(K(r-e)) \leqslant b(K(x, r)) \leqslant b(K(x, r-e)) + k(e)$$

which proves (i).

(ii) For all $x \in B$, $r \in R^+$ $b(K(x, r)) = b(K(0, r))$.

For each $0 < e < r$ we can choose $t \in R^+$ such that $K(x, r-e) \subset \overline{\text{co}}\,\{K(0, r) \cup \{tx\}\}$ and therefore by 4° and 5°, $b(K(x, r-e)) \leqslant b(K(0, r))$ and similarly $b(K(0, r-e)) \leqslant b(K(x, r))$. Now (ii) is a consequence of (i).

(iii) Let $E \subset B$, $s(E) = r$. From the definition of s and by 4°, (ii) and 6° we have:

$$b(E) = b\Big(\bigcup_{i \in I_e} K(x_{\pm}, r+e)\Big) = b(K(0, r+e)) \leqslant b(E) + k(e),$$

where $E \subset \Big(\bigcup_{i \in I_e} K(x_i, r+e)\Big)$ and I_e is finite. Therefore

$$b(E) = b(K(0, r)) =: g(r) = g \circ s(K(0, r)) = g \circ s(E)$$

which with (i) and 4° gives the thesis of our theorem.

COROLLARY. *If* $f\colon B \to B$ *is a b-condensing mapping and* b *satisfies* 1°, 4°–6°, *then* f *is s-condensing.*

Proof. Obviously, if $s(f(A)) < s(A)$, then $b(f(A)) \leqslant b(A)$ and equvalently, because of the properties of g, from $b(A) \leqslant b(f(A))$ it follows that $s(A) \leqslant s(f(A))$.

References

[1] K. Kuratowski, *Sur les espaces completes*, Fund. Math. 15 (1930), p. 301–309.

[2] E. A. Lifschic, B. N. Sadovskij, Theory of fixed point for generalized condensing operators (in Russian). D.A.N. SSSR 183.2 (1968), p. 278–279.