ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXI (1979)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXI (1979)

G. MAJCHER and T. STYRYLSKA (Kraków)

On a multistage optimization problem (I)

1. Introduction and formulation of the problem. Symbols:
j, m, m_j, n, r – natural numbers;
A, B – $m\times r$, $m\times 1$ (resp.) block matrices;
A_j, B_j – $m_j\times r$, $m_j\times 1$ (resp.) given numerical matrices, $j = 1, 2, \ldots, n$;
V – $r\times 1$ matrix of the unknowns;
ϱ_A (ϱ_B, ϱ_V) – rank of the matrix A $(B, V$, resp);

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}.$$

We assume that

$$(Z_1) \qquad 1 \leqslant m < r, \quad 1 \leqslant n \leqslant m, \quad \varrho_A = m = \sum_{j=1}^{n} m_j,$$

which gives

$$(1) \qquad 0 < \varrho_{A_j} = m_j < r \quad \text{for } j = 1, 2, \ldots, n.$$

Let $V = V_0$ be the solution of the equation system

$$(2) \qquad AV = B$$

realizing

$$(3) \qquad \min V^T V.$$

This solution is unique (given e.g. by the method of Lagrange's factors) and has the form

$$(4) \qquad V_0 = A^T (AA^T)^{-1} B,$$

where the $m\times m$ matrix AA^T is symmetric and non-singular (see (Z_1)).

The purpose of this paper is to solve problem (2), (3) using a multistage method in a general case, i.e. for $n > 2$. The two-stage case ($n = 2$) was solved in connection with some problems of the compensating computation ([1], [4], cf. also Section 2 of this paper).

Thus we consider, instead of system (2), the system of subsystem of (2), namely the system

$$A_j V = B_j \quad \text{for } j = 1, 2, \ldots, n, \ n \geqslant 2. \tag{5}$$

First, we find the solution $V_1 = V^{(1)}$ of the problem P_1, i.e. problem (3), (5) with $j = 1$ (1-st stage); then such a matrix $V^{(2)}$ is found as the sum $V_2 = V^{(1)} + V^{(2)}$ be the solution of the problem P_2: (3), (5) with $j = 1, 2$ (2-nd stage); then the next matrix $V^{(3)}$ is found so as the sum $V_3 = V^{(1)} + V^{(2)} + V^{(3)}$ to be the solution of the problem P_3: (3), (5) with $j = 1, 2, 3$ (3-rd stage), etc. An attempt will be made of finding the $r \times 1$ matrices $V^{(j)}$, $j = 2, 3, \ldots, n$, as the solutions of specially for this purpose transformed equations (5); thus

$$\tilde{A}_j V = \tilde{B}_j, \quad j = 2, 3, \ldots, n. \tag{5'}$$

If $V^{(j)}$ are required to fulfil still condition (3) (which is not necessary) all the $V^{(j)}$ will be of the same structure.

Finally, for an arbitrary k-th stage the following problem is formulated:

PROBLEM. We are looking for non-zero matrices $\tilde{A}_j$, $\tilde{B}_j$, $j = 2, 3, \ldots, k$ such that the sum

$$V_k = \sum_{j=1}^{k} V^{(j)}, \quad 1 < k \leqslant n, \tag{6}$$

be the solution of the problem P_k (i.e. (3), (5) with $j = 1, 2, \ldots, k$), where $V^{(j)}$ ($j = 1, 2, \ldots, k$) satisfy the extremum (3) under the condition $A_1 V = B_1$ for $V^{(1)}$ and under conditions (5′) for $j = 2, 3, \ldots, k$.

After finding matrices $\tilde{A}_j$, $\tilde{B}_j$, we get (see formula (4))

$$V^{(j)} = \tilde{A}_j^T (\tilde{A}_j \tilde{A}_j^T)^{-1} \tilde{B}_j \quad \text{for } j = 2, 3, \ldots, k. \tag{7}$$

$V^{(1)}$ will be of an analogous form, i.e.

$$V^{(1)} = A_1^T (A_1 A_1^T)^{-1} B_1. \tag{8}$$

2. Genesis of the problem. 1° In the compensating computation the so-called compensation of the direct conditioned observations leads to looking for corrections V fulfilling conditions (3), (2) (possibly after the preceding linearization of the conditions if they were non-linear) ([1]). It happens, however, that system (2) is not at once fully known and the particular conditions (5) are obtained successively as investigations develop, often with new unknowns (then the previous matrices must be completed with a suitable number of zero columns what, of course, does not change the matrix rank).

In this way new equation systems with new unknowns can be practically added endlessly and solved in stages. The above-presented idea of the multistage procedure gives the advantage that when considering further (of higher j) systems (5) everything need not be computed from the beginning and the previously obtained results can only be corrected by suitably chosen matrices.

2° The equation system (2) may be known, but out of practical reasons: it is sometimes more convenient to divide it into subsystems of the form (5). It is possible e.g. to join into group $j = 1$ all the homogenous equations. Then, of course, $V^{(1)} = 0$, $V^{(2)} = A_2^T(A_2 A_2^T)^{-1} B_2$ and only A_j, B_j $(j = 3, \ldots, n)$ matrices are modified, moreover, all the B_j matrices $(j = 2, 3, \ldots, n)$ do not contain any zero elements.

3. THEOREM 1. *Let assumption* (Z_1) *be fulfilled and let* $V_{k-1} = \sum_{j=1}^{k-1} V^{(j)}$ (k constant $= 2, 3, \ldots, n$) *be a solution of the problem* P_{k-1}. *If the sum*

$$V_k = V_{k-1} + V^{(k)} \tag{9}$$

is also a solution of the problem P_k, *then* $V = V^{(k)}$ *fulfils the relations*

$$\begin{aligned} A_j V &= 0 \quad \textit{for } j = 1, 2, \ldots, k-1, \\ A_k V &= B_k - A_k V_{k-1}, \end{aligned} \tag{10}$$

and conversely. Among the solutions of system (10) *there exists a* $V = V^{(k)}$ *such that* V_k *of the form* (9) *is a solution of the problem* P_k.

Proof. Let us suppose that the sum (9) is a solution of the problem P_k. On the other hand, this solutions is of the form (see formula (4))

$$V_k = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}^T \left(\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}^T \right)^{-1} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{bmatrix}. \tag{11}$$

This and (9) give an equality which multiplied left-hand sided by $\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}$ implies the equality

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{k-1} \\ B_k \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{k-1} \\ A_k V_{k-1} \end{bmatrix} + \begin{bmatrix} A_1 V^{(k)} \\ A_2 V^{(k)} \\ \vdots \\ A_{k-1} V^{(k)} \\ A_k V^{(k)} \end{bmatrix}.$$

Hence $V^{(k)}$ fulfils equations (10).

Conversely, suppose that $V = V^{(k)}$ fulfils equations (10). This and the fact that $A_j V_{k-1} = B_j$ $(j = 1, 2, \ldots, k-1)$ imply that V_k of the form (9) satisfies the system (5) for $j = 1, 2, \ldots, k$. But system (10) has an infinite number of solutions (because $m_1+m_2+\ldots+m_k < r$), among them also $V = V_k - V_{k-1}$, where V_k is of the form (11). Taking just as this solution $V^{(k)}$, we obtain the solution of the problem P_k in the form (9) (which is the same as the form (6)). This completes the proof.

4. Additional conditions on $\tilde{A}_j$. Theorem 1 and formula (7) show that the matrices $\tilde{A}_k$, $\tilde{B}_k$ should be looked for among non-zero $\tilde{m}_k \times r$, $\tilde{m}_k \times 1$ (resp.), $\tilde{m}_k \geqslant 1$, matrices such that

$$\begin{aligned} A_j \tilde{A}_k^T (\tilde{A}_k \tilde{A}_k^T)^{-1} \tilde{B}_k &= 0 \quad \text{for } j = 1, 2, \ldots, k-1, \\ A_k \tilde{A}_k^T (\tilde{A}_k \tilde{A}_k^T)^{-1} \tilde{B}_k &= B_k - A_k V_{k-1}. \end{aligned} \tag{12}$$

Then in the system of $m_1+m_2+\ldots+m_k$ linearly independent equations (12) (see (Z_1) and (1)) there are $\tilde{m}_k(r+1)$ unknowns, where $m_1+m_2+\ldots+m_k < r < \tilde{m}_k(r+1)$. Hence $\tilde{A}_k$, $\tilde{B}_k$ can fulfil some additional conditions, e.g. the following ones:

$$\begin{aligned} A_j \tilde{A}_k^T &= 0 \quad \text{for } j = 1, 2, \ldots, k-1, \\ \tilde{A}_k &= A_k - \sum_{i=1}^{k-1} F_{ki} A_i, \end{aligned} \tag{13}$$

where F_{ki} denote non-zero, and at the moment unknown, $m_k \times m_i$ matrices.

Then the following conclusions result

(14) (i) $\tilde{m}_k = m_k$,

(ii) conditions (12) concerning $j = 1, 2, \ldots, k-1$ are fulfilled,

(iii) $A_k = \tilde{A}_k + \sum_{i=1}^{k-1} F_{ki} A_i$ (by (13)).

This together with the last of relations (12) and with the first $k-1$ relations (13) give

$$\tilde{B}_k = B_k - A_k V_{k-1}. \tag{15}$$

It is clear that the class of matrices A_k fulfilling (13) is a subclass of matrices fulfilling (12).

5. On some matrices and their properties. Two following properties result from (Z_1) and from the Cauchy–Binet theorem ([5]) on the determinant of the matrix product:

(W_1) The $m_j \times m_j$ matrices $A_j A_j^T$ $(j = 1, 2, \ldots, n)$ are symmetric and non-singular.

(W_2) The $r \times r$ matrices $A_j^T (A_j A_j^T)^{-1} A_j$ $(j = 1, 2, \ldots, n)$ are symmetric and singular.

Define the matrix

(16) $$\Delta_j = \begin{cases} I & \text{for } j = 0, \\ \Delta_{j-1}[I - A_j^T(A_j\Delta_{j-1}A_j^T)^{-1}A_j\Delta_{j-1}] & \text{for } j = 1, 2, \ldots, n \end{cases}$$

(I — the identity $r \times r$ matrix). We shall prove that:

(W$_3$) there exist matrices Δ_j ($j = 1, 2, \ldots, n$) given by formula (16); they are $r \times r$, symmetric, singular, $\varrho_{\Delta_j} = r - m_j$; moreover,

(16′) $$\Delta_j = I - \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_j \end{bmatrix}^T \left(\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_j \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_j \end{bmatrix}^T \right)^{-1} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_j \end{bmatrix}$$

and $\Delta_j^p = \Delta_j$ for any arbitrary natural number p and for $j = 1, 2, \ldots, n$.

Indeed, if the matrices Δ_j ($j = 1, 2, \ldots, n$) exist, they are $r \times r$ and symmetric. Singularity and the rank of Δ_j result from the fact that $A_j\Delta_j = 0$, so (see (1)) every matrix Δ_j ($j = 1, 2, \ldots, n$) has $r - m_j < r$ lineary independent rows only.

The proof of formula (16′) is by induction. The existence of the matrix Δ_1 results from property (W$_1$) for $j = 1$. In turn, the rank of the matrix

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^T = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \Delta_0 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^T$$

is equal to its dimension, i.e. $m_1 + m_2$ (see (Z$_1$)). Hence its determinant $\neq 0$, which implies (by the theorem on block calculation of determinants) that $|A_1A_1^T|\,|A_2\Delta_1A_2^T| \neq 0$ and $\varrho_{A_2\Delta_1A_2^T} = m_2$. Assuming that the ranks of the matrices

$$\begin{bmatrix} A_k \\ A_{k+1} \end{bmatrix} \Delta_{k-1} \begin{bmatrix} A_k \\ A_{k+1} \end{bmatrix}^T$$

and $A_k\Delta_{k-1}A_k^T$ are equal to $m_k + m_{k+1}$, m_k, respectively, we get in an analogous manner that $|A_k\Delta_{k-1}A_k^T|\,|A_{k+1}\Delta_kA_{k+1}^T| \neq 0$ which proves the existence of all matrices Δ_j ($j = 1, 2, \ldots, n$).

The proof that $\Delta_j^p = \Delta_j$ for a natural number p and for $j = 1, 2, \ldots, n$ is simple. First it is proved that $\Delta_j^2 = \Delta_j$ (induction with respect to j) and from it the above property follows evidently.

We obtained here also the property

(W$_4$) the matrices $A_j\Delta_{j-1}A_j^T$ for $j = 1, 2, \ldots, n$ are $m_j \times m_j$, symmetric and non-singular.

If we write

(17) $$D_j = I - \Delta_{j-1}A_j^T(A_j\Delta_{j-1}A_j^T)^{-1}A_j \quad \text{for } j = 1, 2, \ldots, n,$$

then (16) and (17) imply $\Delta_j = \Delta_{j-1} D_j^T = \Delta_0 D_1^T D_2^T \dots D_j^T$ which gives (by $\Delta_j = \Delta_j^T$)

$$\Delta_j = D_j D_{j-1} \dots D_2 D_1 \quad \text{for } j = 1, 2, \dots, n. \tag{18}$$

6. Calculation of the matrices F_{ki} and $\tilde{A}_k$. Having eliminated the matrix $\tilde{A}_k$ in relations (13), we obtain the system of $k-1$ equations

$$A_j \sum_{i=1}^{k-1} A_i^T F_{ki}^T = A_j A_k^T \quad (j = 1, 2, \dots, k-1). \tag{19}$$

Let us apply the method of a successive elimination of the unknowns. First of equations (19) is multiplied left-hand sided successively by the coefficients $\psi_{1j} = -A_j A_1^T (A_1 A_1^T)^{-1}$ and added to the j-th ($j = 2, 3, \dots, k-1$) equation. In these equations at F_{k1}^T a zero matrix will appear, whereas at F_{ki}^T ($i = 2, 3, \dots, k-1$) the matrix $A_j \Delta_1 A_i^T$, respectively. The right-hand side will be of the form $A_j \Delta_1 A_k^T$ (cf. notations (16)). Leaving now the first and second equation unchanged, we multiply the second equation left-hand sided by matrices $\psi_{2j} = -A_j \Delta_1 A_2^T (A_2 \Delta_1 A_2^T)^{-1}$ (cf. property (W_4)) add it to the j-th ($j = 3, 4, \dots, k-1$) equation. In the equations of numbers $j = 3, 4, \dots$ $\dots, k-1$ a zero matrix will appear at F_{k2}^T, whereas, at F_{ki}^T the matrix $A_j \Delta_2 A_i^T$ ($i, j = 3, 4, \dots, k-1$), respectively. Proceeding further in an analogous way the equation system

$$\sum_{i=0}^{k-1-j} A_j \Delta_{j-1} A_{j+i}^T F_{k(j+i)}^T = A_j \Delta_{j-1} A_k^T \quad \text{for } j = 1, 2, \dots, k-1,$$

will be obtained, and hence

$$\sum_{i=0}^{k-1-j} F_{k(j+i)} A_{j+i} \Delta_{j-1} A_j^T = A_k \Delta_{j-1} A_j^T.$$

So the solution is

$$F_{ki} = \begin{cases} A_k D_{k-1} D_{k-2} \dots D_{i+1} \Delta_{i-1} A_i^T (A_i \Delta_{i-1} A_i^T)^{-1} & \text{for } i = 1, 2, \dots, k-2, \\ A_k \Delta_{k-2} A_{k-1}^T (A_{k-1} \Delta_{k-2} A_{k-1}^T)^{-1} & \text{for } i = k-1. \end{cases} \tag{20}$$

It results from (13), (17) and (20) that

$$\tilde{A}_k = A_k - \sum_{i=1}^{k-1} F_{ki} A_i = A_k D_{k-1} - \sum_{i=1}^{k-2} F_{ki} A_i$$
$$= A_k D_{k-1} D_{k-2} - \sum_{i=1}^{k-3} F_{ki} A_i = \dots = A_k D_{k-1} D_{k-2} \dots D_2 D_1,$$

and finally,

$$\tilde{A}_k = A_k \Delta_{k-1} \quad \text{for } k = 2, 3, \dots, n. \tag{21}$$

7. The matrix $\tilde{B}_k$. Making use of formulae (6)–(8), (16)–(18), and (21), the matrix $\tilde{B}_k$ can be expressed by means of the matrices A_j and B_j.

First notice that from (21) and (W_1) we get

$$\tilde{A}_k \tilde{A}_k^T = A_k \Delta_{k-1} A_k^T = A_k \tilde{A}_k^T \quad (k \text{ constant} = 2, 3, \ldots, n). \tag{22}$$

Write

$$\tilde{\Delta}_{kj} = D_{k-1} D_{k-2} \ldots D_{j+1} D_{j-1} \ldots D_2 D_1 A_j^T (A_j \Delta_{j-1} A_j^T)^{-1} \quad \text{for } j = 1, 2, \ldots, k-1,\ k = 2, 3, \ldots, n. \tag{23}$$

These are $r \times m_j$ matrices which do not contain the factor D_j. Then

$$\tilde{B}_k = B_k - A_k \sum_{j=1}^{k-1} \tilde{\Delta}_{kj} B_j. \tag{24}$$

8. THEOREM 2. *If assumption* (Z_1) *is fulfilled and the matrices* $\tilde{A}_j$, $\tilde{B}_j$ *are of the form* (21) *and* (24), *respectively, then for every* $k = 1, 2, \ldots, n$ *the sum of the form* (6), *where* $V^{(j)}$ *are of the form* (7), (8) *is the* (*unique*) *solution of the problem* P_k.

Indeed, the solution V_k can be reduced to the form (11). The proof is by induction (see also formula (16′)).

Remark. In another form (see (22)) formula (16) is

$$\Delta_j = \Delta_{j-1} - L_j \tilde{A}_j \quad (j = 1, 2, \ldots, n),$$

where $\tilde{A}_1 = A_1$ and $L_j = \tilde{A}_j^T (A_j \tilde{A}_j^T)^{-1}$. Thus we have (see (7)) $V^{(j)} = L_j \tilde{B}_j$ ($\tilde{B}_1 = B_1$).

Starting to calculate the k-th stage ($k = 2, 3, \ldots, n$) we have already: L_{k-1}, $\tilde{A}_{k-1}$, $\tilde{B}_{k-1}$, Δ_{k-2} and $V^{(j)}$ for $j = 1, 2, \ldots, k-1$. So we must calculate
1° $\Delta_{k-1} = \Delta_{k-2} - L_{k-1} \tilde{A}_{k-1}$, 2° $\tilde{A}_k = A_k \Delta_{k-1}$, 3° $\tilde{B}_k = B_k - A_k \sum_{j=1}^{k-1} V^{(j)}$, 4° $A_k \tilde{A}_k^T$, 5° $(A_k \tilde{A}_k^T)^{-1}$, 6° $L_k = \tilde{A}_k^T (A_k \tilde{A}_k^T)^{-1}$, 7° $V^{(k)} = L_k \tilde{B}_k$.

9. EXAMPLE. The given equations system (5) is:

$$\left.\begin{aligned} 2v_1 + 3v_2 + 11 &= 0 \\ v_1 + v_7 - 110 &= 0 \end{aligned}\right\} \text{stage 1}$$

. . . .

$$v_2 - 2v_3 - 2v_4 + 3 = 0 \quad \text{stage 2}$$

. . . .

$$\left.\begin{aligned} -6v_2 + v_5 + 5v_7 - 542 &= 0 \\ 7v_3 + 8v_4 + v_6 + 104 &= 0 \end{aligned}\right\} \text{stage 2}$$

. . . .

Stage 1 (see (25)).

$$A_1 = \tilde{A}_1 = \left[\begin{array}{cc|c|c} 2 & 3 & & 0 \\ 1 & 0 & 0_{2\times 4} & 1 \end{array}\right], \quad B_1 = \tilde{B}_1 = \begin{bmatrix} -11 \\ 110 \end{bmatrix},$$

$$A_1 \tilde{A}_1^T = \begin{bmatrix} 13 & 2 \\ 2 & 2 \end{bmatrix}, \quad (A_1 \tilde{A}_1^T)^{-1} = \frac{1}{22}\begin{bmatrix} 2 & -2 \\ -2 & 13 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 2 & 9 \\ 6 & -6 \\ \cdots & \cdots \\ \multicolumn{2}{c}{0_{4\times 2}} \\ \cdots & \cdots \\ -2 & 13 \end{bmatrix}, \quad V^{(1)} = 11\begin{bmatrix} 4 \\ -3 \\ \cdots \\ 0_4 \\ \cdots \\ 6 \end{bmatrix}.$$

Stage 2. $A_2 = [0\ 1\ -2\ -2 : 0_{1\times 3}]$, $B_2 = [-3]$,

$$\Delta_1 = I - L_1 \tilde{A}_1 = \frac{1}{22}\left[\begin{array}{cc:c:c} 9 & -6 & \multirow{2}{*}{$0_{2\times 4}$} & -9 \\ -6 & 4 & & 6 \\ \hdashline \multicolumn{2}{c:}{0_{4\times 2}} & 22\cdot I_{4\times 4} & 0_{4\times 1} \\ \hdashline -9 & 6 & 0_{1\times 4} & 9 \end{array}\right]$$

$$\tilde{A}_2 = A_2 \Delta_1 = \tfrac{1}{11}[-3\ 2\ -22\ -22\ 0\ 0\ 3],$$

$$\tilde{B}_2 = B_2 - A_2 V^{(1)} = [30], \quad A_2 \tilde{A}_2^T = [\tfrac{90}{11}], \quad (A_2 A_2^T)^{-1} = [\tfrac{11}{90}],$$

$$L_2^T = [-\tfrac{1}{30}\ \tfrac{1}{45}\ -\tfrac{11}{45}\ -\tfrac{11}{45}\ 0\ 0\ \tfrac{1}{30}],$$

$$(V^{(2)})^T = (L_2 \tilde{B}_2)^T = [-1\ \tfrac{2}{3}\ -\tfrac{22}{3}\ -\tfrac{22}{3}\ 0\ 0\ 1].$$

Stage 3.

$$A_3 = \begin{bmatrix} 0 & -6 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 7 & 8 & 0 & 1 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 542 \\ -104 \end{bmatrix},$$

$$\Delta_2 = \Delta_1 - L_2 \tilde{A}_2 = \frac{1}{45}\begin{bmatrix} 18 & -12 & -3 & -3 & 0 & 0 & -18 \\ -12 & 8 & 2 & 2 & 0 & 0 & 12 \\ -3 & 2 & 23 & -22 & 0 & 0 & 3 \\ -3 & 2 & -22 & 23 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 45 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 45 & 0 \\ -18 & 12 & 3 & 3 & 0 & 0 & 18 \end{bmatrix},$$

$$\tilde{A}_3 = A_3 \Delta_2 = \frac{1}{15}\begin{bmatrix} -6 & 4 & 1 & 1 & 15 & 0 & 6 \\ -15 & 10 & -5 & 10 & 0 & 15 & 15 \end{bmatrix},$$

$$\tilde{B}_3 = B_3 - A_3(V^{(1)} + V^{(2)}) = \begin{bmatrix} 13 \\ 6 \end{bmatrix},$$

$$A_3\tilde{A}_3^T = \frac{1}{5}\begin{bmatrix} 7 & 5 \\ 5 & 20 \end{bmatrix}, \quad (A_3\tilde{A}_3^T)^{-1} = \frac{1}{23}\begin{bmatrix} 20 & -5 \\ -5 & 7 \end{bmatrix},$$

$$L_3^T = \frac{1}{69}\begin{bmatrix} -9 & 6 & 9 & -6 & 60 & -15 & 9 \\ -15 & 10 & -8 & 13 & -15 & 21 & 15 \end{bmatrix},$$

$$(V^{(3)})^T = (L_3\tilde{B}_3)^T = [-3\ 2\ 1\ 0\ 10\ -1\ 3].$$

Other applications and results will be given in papers [2] and [3].

References

[1] A. Łoś, *Rachunek wyrównawczy*, I (in Polish), Warszawa 1973.

[2] G. Majcher, T. Styrylska, *Wieloetapowe wyrównywanie spostrzeżeń bezpośrednich zawarunkowanych* (in Polish) (in press).

[3] — —, *On a multistage optimization problem* (II) (in press).

[4] T. Styrylska, *Dwuetapowe wyrównywanie spostrzeżeń bezpośrednich zawarunkowanych* (in Polish), Roczniki PTM, Sér. III, Mat. Stos. 4 (1975), p. 29–33.

[5] A. Turowicz, *Teoria macierzy*, skrypt (in Polish), AGH, Kraków 1974.