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Remarks on operator-stable probability measures

In this note we consider Borel probability measures on the Euclidean space R^p. For two probability measures μ and ν, we shall denote by $\mu * \nu$ the convolution of μ and ν. Further, by μ^{*n} we shall denote the n-th power in the sense of convolution. δ_x will denote the probability measure concentrated at the point $x\,(x \in R^p)$. The characteristic function $\hat{\mu}$ of a probability measure μ on R^p is defined by the formula

$$\hat{\mu}(y) = \int_{R^p} e^{i(x,y)}\,\mu(dx) \qquad (y \in R^p),$$

where $(\cdot,\cdot)$ denotes the inner product in R^p. We call a probability measure on R^p *full* if its support is not contained in any $(p-1)$-dimensional hyperplane of R^p. Given a linear operator A on R^p, by $A\mu$ we shall denote the probability measure defined by the formula $A\mu(F) = \mu\big(A^{-1}(F)\big)$ for every Borel subset F of R^p.

Let $\{X_n\}$ be a sequence of independent identically distributed R^p-valued random variables. If there exist sequences $\{A_n\}$ and $\{a_n\}$ of non-singular linear operators on R^p and elements of R^p, respectively, such that the limit distribution μ of normed sums

$$A_n \sum_{j=1}^{n} X_j + a_n \qquad (n = 1, 2, \ldots)$$

exists, then μ is called *operator-stable*. This concept is due to M. Sharpe, who obtained in [3] a characterization of full operator-stable measures. In particular, he proved the following statements:

$(*)$ Every full operator-stable probability measures μ is infinitely divisible, i.e. for every positive integer n there exists a probability measure μ_n with the property $\mu_n^{*n} = \mu$. Hence it follows that for any positive real number t the t-th power of μ in the sense of convolution, in symbols μ^{*t}, is well defined.

(**) A full probability measure μ is operator-stable if and only if there exist a linear operator B on R^p and a collection $\{a_t\colon t > 0\}$ of elements of R^p such that for every positive real number t the equation

$$\mu^{*t} = t^B \mu * \delta_{a_t} \tag{1}$$

holds. Here t^B denotes the operator $e^{B \log t}$. Moreover, the spectrum of B is then contained in the half-plane $\operatorname{Re} z \geqslant \frac{1}{2}$ and all eigenvalues lying on the line $\operatorname{Re} z = \frac{1}{2}$ are simple, i.e. the elementary divisors of B associated with these eigenvalues are of first degree.

(***) Every full operator-stable probability measure μ on R^p can be decomposed into a convolution $\mu = \mu_1 * \mu_2$ of probability measures μ_1 and μ_2 concentrated on subspaces P_1 and P_2 respectively, $R^p = P_1 \oplus P_2$, μ_1 being a full Gaussian measure on P_1 and μ_2 being a full operator-stable probability measure on P_2 without a Gaussian component. Moreover, both subspaces P_1 and P_2 are invariant under B, the real part of all eigenvalues of the restriction of B to P_1 is equal to $\frac{1}{2}$, and the real parts of eigenvalues of the restriction of B to P_2 are greater than $\frac{1}{2}$.

Recently, J. Kucharczak obtained in [1] a representation of the characteristic function of full operator-stable probability measures. Namely, using the extreme point method he proved the following theorem.

THEOREM 1. *A full probability measure μ on R^p is operator-stable if and only if*

$$\hat{\mu}(y) = \exp\Big\{i(a, y) - \tfrac{1}{2}(Qy, y) + \int\limits_{S^{p-1} \cap \operatorname{Ker} Q} \int\limits_0^\infty \Big(e^{i(t^B x, y)} - 1 - \frac{i(t^B x, y)}{1 + \|t^B x\|^2}\Big) \frac{dt}{t^2}\, m(dx)\Big\},$$

where $a \in R^p$, S^{p-1} is the unit sphere in R^p, Q is a non-negative symmetric operator on R^p, the kernel of Q is invariant under B, the real parts of eigenvalues of the restriction of B to $\operatorname{Ker} Q$ *are greater than $\frac{1}{2}$ and m is a finite Borel measure on $S^{p-1} \cap \operatorname{Ker} Q$.*

The aim of this paper is to give a simple proof of Kucharczak representation theorem. By Sharpe decomposition theorem (***) it suffices to establish the representation of the characteristic function for operator-stable measures without a Gaussian component, i.e. to prove the following theorem.

THEOREM 2. *A full probability measure μ on R^p is operator-stable and has no Gaussian component if and only if*

$$\hat{\mu}(y) = \exp\Big\{i(a, y) + \int\limits_{S^{p-1}} \int\limits_0^\infty \Big(e^{i(t^B u, y)} - 1 - \frac{i(t^B u, y)}{1 + \|t^B u\|^2}\Big) \frac{dt}{t^2}\, m(du)\Big\}, \tag{2}$$

where $a \in R^p$, the real parts of eigenvalues of B are greater than $\frac{1}{2}$ and m is a finite Borel measure on S^{p-1}.

Proof. By a simple calculation we can check that each measure μ with the characteristic function of the form (2) satisfies equation (1) for all $t > 0$. Hence, by Sharpe theorems (**) and (***), we get the sufficiency of condition (2).

In order to prove the necessity we assume that μ is a full operator-stable probability measure without a Gaussian component. By (**) and (***), μ satisfies equation (1) for a certain operator B whose eigenvalues have real parts greater than $\frac{1}{2}$. Since, by (*), μ is infinitely divisible, its characteristic function can be written in the Lévy–Khinchine form

$$\hat{\mu}(y) = \exp\left\{i(a, y) + \int_{R^p \setminus \{0\}} \left(e^{i(x,y)} - 1 - \frac{i(x, y)}{1 + \|x\|^2}\right) M(dx)\right\}, \tag{3}$$

where $a \in R^p$ and M is a σ-finite Borel measure on $R^p \setminus \{0\}$ finite outside every neighborhood of 0 with the property $\int_{\|x\| \leq 1} \|x\|^2 M(dx) < \infty$ ([2], p. 181). Moreover, by Proposition 5 in [3],

$$t^B M = tM \tag{4}$$

for every $t > 0$. Since the real parts of eigenvalues of B are greater than $\frac{1}{2}$, each orbit $\{t^B y: t > 0\}$ $(y \neq 0)$ intersects the unit sphere S^{p-1}. Let $\sim$ be a continuous relation in S^{p-1} defined as follows: $x_1 \sim x_2$ iff there exists $t > 0$ such that $x_1 = t^B x_2$. By [2], Theorem 2.4, p. 23, there exists a Borel subset S_0^{p-1} of S^{p-1} such that every element x from $R^p \setminus \{0\}$ has a unique representation $x = t^B u$, where $t > 0$ and $u \in S_0^{p-1}$. Moreover, this representation defines a homeomorphism between $R^p \setminus \{0\}$ and $S_0^{p-1} \times (0, \infty)$. Hence it follows that the σ-field generated by the collection on the sets $\{t^B u: t \in I, u \in E\}$, where I and E are closed intervals on the half-line $(0, \infty)$ and Borel subsets of S_0^{p-1} respectively, consists of all Borel subsets of $R^p \setminus \{0\}$. Put $f(h, E) = M(\{t^B u: t \geq h, u \in E\})$ $(h > 0)$. Taking into account (4) we have the equation $f(h/g, E) = gf(h, E)$. Now setting $h = g$ and $m_0(E) = f(1, E)$ we get $f(h, E) = h^{-1} m_0(E)$ which implies the formula

$$M(\{t^B u: t \in I, u \in E\}) = m(E) \int_I t^{-2}\, dt,$$

where $m(E) = m_0(E \cap S_0^{p-1})$ for any Borel subset E of S_0^{p-1}. This formula can be extended to all Borel subsets F of $R^p \setminus \{0\}$ as follows:

$$M(F) = \int_{S^{p-1}} \int_0^\infty c_F(t^B u) t^{-2}\, dt\, m(du),$$

where c_F denotes the indicator of F. Setting this expression for M into (3), we get the required representation (2) which completes the proof.

References

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