ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXI (1979)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXI (1979)

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Integral representations of linear forms on topological algebras

Abstract. The purpose of the present paper is to give an abstract form of Bochner–Weil–Raikov theorem within the frame of topological algebras theory. So, it is proved that a continuous linear form Φ on a commutative, locally m-convex algebra E, with continuous Gel'fand map, and an approximate identity, admits an integral representation on the spectrum (Gel'fand space) of E, iff Φ satisfies condition (l) (Definition 3.3). An analogous statement is also derived for a tensor product topological algebra, and an application of this latter result yields a form of the preceding theorem for a "generalized group algebra".

1. Introduction. The classical Bochner–Weil–Raikov theorem provides for every positive and extendable linear form on a commutative, Banach algebra with a continuous involution, a (unique) integral representation with respect to the Gel'fand space of the given algebra (cf., for instance, [3], p. 97, Theorem 26.I).

On the other hand, G. Lumer obtains in [4] an abstract form of the last theorem, by replacing the continuous involution by an appropriate finite group of transformations of the given algebra.

Now, Lumer's result can be obtained for a much more general class of topological algebras than the normed ones (they do not also have compact spectra). Hence, an abstract Bochner–Weil–Raikov theorem, for involutive algebras, within the context of locally m-convex algebras, is also derived (Corollary 3.8). Thus, given a Γ-Lumer system (E, Γ) (Definition 3.1), where E is a commutative, locally m-convex algebra, with identity and continuous Gel'fand map, a continuous linear form Φ on E admits an integral representation on the spectrum $\mathfrak{M}(E)$ of E, iff Φ satisfies condition (l) (Definition 3.3). The essential fact here is that, when $\Phi \in E'$ satisfies condition (l), then it is continuous with respect to the "spectral topology" of E. An analogous result for a (Γ, Δ)-Lumer system (E, Γ, Δ) (Definition 4.1), where now E has an approximate identity, is also derived (Theorem 4.6). Finally, a similar representation theorem is obtained for a tensor product topological algebra (Theorem 5.6) and an application of the latter is given for a group algebra of vector-valued functions (generalized group algebra; cf. Corollary 6.2, as well as, Theorem 6.3).

I wish to express my gratitude to my research supervisor Professor Anastasios Mallios for having introduced me into this area of study, and for the very helpful and stimulating discussions we had during the preparation of this paper.

2. Preliminaries. A (complex) topological algebra (topological vector space with a seperately continuous multiplication) is called *spectrally barrelled*, whenever the equicontinuous and the weakly bounded sets of its spectrum coincide (cf. [6]; p. 153).

A *locally m-convex* algebra, is an *algebra* E and a *topological vector space* having a local basis consisting of *m-convex* (multiplicatively convex) subsets of E (cf. [9], [10]).

In this respect, one proves that a locally m-convex algebra E has a local basis consisting of *m-barrels* in E (closed, balanced, absorbing, convex and idempotent subsets of E [9]). Besides, the same topology on E can be defined by a family of submultiplicative semi-norms, say $(p_\alpha)_{\alpha\in A}$ [9]. In the sequel, we shall denote by $(E, (p_\alpha))$ an algebra E thus topologized.

A given topological algebra E is said to be *advertibly complete*, whenever a Cauchy filter $\mathscr{F}$ on E converges, provided that for some element $x\in E$ both of the filter bases $x\circ\mathscr{F}$ and $\mathscr{F}\circ x$ converge to zero in E ($x\circ\mathscr{F} = \{x\circ A\colon A\in\mathscr{F}\}$, with $x\circ y = x+y-xy$ for x, y in E; cf. [11]).

By a *bounded approximate identity* on a topological algebra E we shall mean a bounded net $(e_i)_{i\in I}$ in E such that $\lim_i (e_i x) = \lim_i (xe_i) = x$ for every $x\in E$.

Now, let E be a locally m-convex algebra, and B a family (or a group) of transformations on E (mappings of E into itself, which in case B is a group, are, of course bijections). In this respect, we shall say that a *family* $(p_\alpha)_{\alpha\in A}$ *of submultiplicative semi-norms*, defining the topology of E, is *B-invariant*, if $p_\alpha(\beta(x)) = p_\alpha(x)$ for any $\alpha\in A$, $\beta\in B$, and $x\in E$.

For the notation and terminology further applied, and for basic properties of spectrally barrelled locally m-convex algebras, the reader is referred to [10], [6] and [9].

3. Γ-Lumer systems.

DEFINITION 3.1. A pair (E,Γ) consisting of a locally m-convex algebra $(E,(p_\alpha))$ and a finite group Γ of additive, multiplicative transformations of E (automorphisms of the ring structure of E) such that the family $(p_\alpha)_{\alpha\in A}$ is Γ-invariant, is called a *Γ-Lumer system*.

In case of Banach algebras the elements of Γ are not necessarily additive (cf. [4], p. 136, Theorem). However, they usually are in the applications ([4], p. 140).

Let (E, Γ) be a Γ-Lumer system, and $(E_\alpha = E/\ker(p_\alpha))$, $(\tilde{E}_\alpha = \widetilde{E/\ker(p_\alpha)})$ the projective systems of normed and Banach algebras respectively, corresponding to the locally m-convex algebra $(E, (p_\alpha))$ [10].

Now, by defining, for every $\alpha \in A$,

$$\gamma_\alpha\colon E_\alpha \to E_\alpha\colon x_\alpha = [x]_\alpha = x + \ker(p_\alpha) \to \gamma_\alpha(x_\alpha) = [\gamma(x)]_\alpha, \quad \gamma \in \Gamma,$$

one gets a Γ_α-Lumer system for the normed algebra E_α, $\alpha \in A$, with $\Gamma_\alpha = \{\gamma_\alpha\colon \gamma \in \Gamma\}$. Hence, one also obtains a respective $\tilde{\Gamma}_\alpha$-Lumer system for the Banach algebra $\tilde{E}_\alpha$, $\alpha \in A$, with $\tilde{\Gamma}_\alpha$ the continuous extensions of $\gamma_\alpha \in \Gamma_\alpha$. Thus, the preceding yields the following

LEMMA 3.2. *Let (E, Γ) be a Γ-Lumer system for a given locally m-convex algebra $(E, (p_\alpha))$. Then, for each $\alpha \in A$, $(\tilde{E}_\alpha, \tilde{\Gamma}_\alpha)$ is a $\tilde{\Gamma}_\alpha$-Lumer system for the Banach algebra $\tilde{E}_\alpha$.*

Now, if $(E, (p_\alpha))$ is a given locally m-convex algebra and $\Phi \in E'$ (the topological dual of E), there exists $\alpha \in A$ and $k > 0$ such that

$$|\Phi(x)| \leqslant k \cdot p_\alpha(x), \quad \text{with } x \in E. \tag{3.1}$$

The last relation yields $\Phi_\alpha \in E'_\alpha$ such that $\Phi_\alpha(x_\alpha) \equiv \Phi(x)$, $x_\alpha \in E_\alpha$. Now, we shall actually impose on Φ a stronger "continuity condition" than the preceding one, according to the following

DEFINITION 3.3. Let (E, Γ) be a Γ-Lumer system for a given locally m-convex algebra $(E, (p_\alpha))$ with identity. We shall say that an element $\Phi \in E'$ *satisfies condition* (l), if there exists an index $\alpha \in A$ and $l > 0$ such that

$$|\Phi_\alpha(x_\alpha)| \leqslant l \cdot N_\alpha(x_\alpha), \quad x_\alpha \in \tilde{E}_\alpha, \tag{3.2}$$

for every $\tilde{\Gamma}_\alpha$-invariant norm N_α on $\tilde{E}_\alpha$, equivalent to the given one (where l is the same for all N_α under consideration).

The index $\alpha \in A$ in (3.2) is supposed to be that provided by (3.1). On the other hand, if Φ is a continuous positive linear form on E, then (3.2) is always true with $l = 1$ (cf. proof of Corollary 3.8).

For clarity we restate Lemma 2.1 of [2] (cf. also [4], p. 136, Theorem).

LEMMA 3.4. *Let E be a (complex) Banach algebra with identity, and Γ a finite group of multiplicative, norm-preserving transformations of E. Then for every $x \in E$, such that $r(x) < 1$ (where $r(x)$ is the spectral radius of x), a Γ-invariant norm N is defined on E, equivalent to the original one and such that $N(x) \leqslant 1$. That is, the spectrum $\mathrm{sp}(x)$ of an element $x \in E$ lies in the open unit circle of $\boldsymbol{C}$ (the complexes), iff x belongs to the closed unit ball of E, modulo at most an appropriate norm N on E, equivalent to the given one.*

Note that the above lemma is also true for a Γ-Lumer system (E, Γ), where E is an advertibly complete locally m-convex algebra with identity (cf. also [2], p. 22, Lemma 3.2).

Now, let $(E,(p_\alpha))$ be a commutative, locally m-convex algebra, and $\mathscr{C}_c(\mathfrak{M}(E))$ the algebra of complex-valued functions on the spectrum $\mathfrak{M}(E)$ of E, endowed with the topology of compact convergence in $\mathfrak{M}(E)$. Let also $E^{\wedge} \subseteq \mathscr{C}_c(\mathfrak{M}(E))$ be the Gel'fand transform algebra of E, equipped with the relative topology induced on it by $\mathscr{C}_c(\mathfrak{M}(E))$, where for $\hat{x}\in E^{\wedge}$, one defines $\hat{x}(f) = f(x)$, for any $x\in E$ and $f\in\mathfrak{M}(E)$. Then, a family of sub-multiplicative semi-norms on $E^{\wedge}$ is given by $q_K(\hat{x}) = \sup_{f\in K} |f(x)|$, with $x\in E$ and K a compact subset of $\mathfrak{M}(E)$.

If, moroever, E has an identity and continuous Gel'fand map, then the sets $\mathfrak{M}(E)\cap U_\alpha^0$ ($\alpha\in A$) (where U_α, $\alpha\in A$ is the local basis of E corresponding to the family $(p_\alpha)_{\alpha\in A}$) form a k-covering of $\mathfrak{M}(E)$ (cf. [6], p. 156, Lemma 4.1) in the sense that each $\mathfrak{M}(E)\cap U_\alpha^0$ is compact and for every compact $K\subseteq\mathfrak{M}(E)$ there exists $\alpha\in A$ such that $K\subseteq\mathfrak{M}(E)\cap U_\alpha^0$. On the other hand, $\mathfrak{M}(E_\alpha) = \mathfrak{M}(E)\cap U_\alpha^0 = \mathfrak{M}(\tilde{E}_\alpha)$, $\alpha\in A$, within a homeomorphism (cf. [10], p. 31, Proposition 7.5, c)). Thus $\mathfrak{M}(\tilde{E}_\alpha)$, $\alpha\in A$, are compact subsets of $\mathfrak{M}(E)$ and $q_{\mathfrak{M}(\tilde{E}_\alpha)}(\hat{x}) = \sup_{f\in\mathfrak{M}(\tilde{E}_\alpha)} |f(x)| = r_\alpha(x_\alpha)$, $\alpha\in A$, $\hat{x}\in E^{\wedge}$, where $r_\alpha(x_\alpha)$ denotes the spectral radius of $x_\alpha\in E_\alpha \subsetneq \tilde{E}_\alpha$, $\alpha\in A$.

Lemma 3.5. *Let $(E,(p_\alpha))$ be a commutative, locally m-convex algebra with identity, and (E,Γ) a Γ-Lumer system. Let also $\Phi\in E'$ satisfy condition (l). Then, Φ is continuous with respect to the "spectral topology" of E, i.e. there exists a compact $K\subseteq\mathfrak{M}(E)$ such that*

$$|\Phi(x)| \leqslant l\cdot q_K(\hat{x}) \quad \textit{for every } x\in E.$$

Proof. By hypothesis there exists $\alpha\in A$ such that (3.2) holds true, so that if $y\in E$, let $x = \dfrac{y}{r_\alpha(y_\alpha)+\varepsilon}\in E$, with $\varepsilon>0$, where $y_\alpha = \pi_\alpha(y)$ with $\pi_\alpha\colon E\to E_\alpha$, the canonical quotient map. Let now, $x_\alpha = \pi_\alpha(x) = \dfrac{y_\alpha}{r_\alpha(y_\alpha)+\varepsilon}\in E_\alpha$. Then, $r_\alpha(x_\alpha)<1$, hence, by Lemma 3.4, a $\tilde{\Gamma}_\alpha$-invariant norm N_α is defined on $\tilde{E}_\alpha$, equivalent with the original one, and such that $N_\alpha(x_\alpha)\leqslant 1$. Thus, by (3.2), $|\Phi(x)| \equiv |\Phi_\alpha(x_\alpha)| \leqslant l$, that is $\dfrac{|\Phi(y)|}{r_\alpha(y_\alpha)+\varepsilon}\leqslant l$, for any $y\in E$, $\varepsilon>0$. Therefore, $|\Phi(y)|\leqslant l\cdot r_\alpha(y_\alpha) = l\cdot q_{\mathfrak{M}(\tilde{E}_\alpha)}(\hat{y})$, for every $y\in E$, and this proves the lemma.

The previous lemma (for $l=1$) specializes to Theorem 1 of [4], p. 136, which also shows its validity for any $l>0$.

We are now in position to state the following amendment to Theorems 3.1, 4.1 of [2], which also has a special bearing on the respective result of [4], p. 136, Theorem.

Theorem 3.6. *Let $(E,(p_\alpha))$ be a commutative, locally m-convex algebra with identity and continuous Gel'fand map. Let also (E,Γ) be a Γ-Lumer system, and $\Phi\in E'$. Then the following statements are equivalent:*

(1) *Φ satisfies condition* (l).

(2) *Φ admits (an integral) representation of the form*

$$\Phi(x) = \mu(\hat{x}), \quad x \in E,$$

where μ is a (complex) measure on $\mathfrak{M}(E)$ of positive finite total variation at most l.

Proof. (1)$\Rightarrow$(2). Since $\Phi \in E'$ satisfies condition (l), there exists, by Lemma 3.5, a compact $K \subseteq \mathfrak{M}(E)$ such that $|\Phi(x)| \leqslant l \cdot q_K(\hat{x})$ for every $x \in E$, so that the relation $\hat{\Phi}(\hat{x}) \equiv \Phi(x)$, $\hat{x} \in E^{\wedge}$ is well defined, and $\hat{\Phi} \in (E^{\wedge})'$. Hence $\hat{\Phi}$ is continuously extended, by the Hahn–Banach theorem, to the whole of $\mathscr{C}_c(\mathfrak{M}(E))$, i.e. $\hat{\Phi} \in (\mathscr{C}_c(\mathfrak{M}(E)))'$, so, by [5], Theorem 1.1 (cf. also [1], p. 11, Theorem 5.1) there exists a (complex) measure μ on $\mathfrak{M}(E)$ of positive finite total variation at most l such that

$$\Phi(x) = \hat{\Phi}(\hat{x}) = \mu(\hat{x}) \quad \text{for every } x \in E.$$

(2)$\Rightarrow$(1). By the continuity of μ there exists a compact $K \subseteq \mathfrak{M}(E)$ such that $|\mu(\hat{x})| \leqslant l \cdot q_K(\hat{x})$ for every $\hat{x} \in E^{\wedge}$. Hence, by the continuity of the Gel'fand map of E, there exists $\alpha \in A$ with $K \subseteq \mathfrak{M}(\tilde{E}_\alpha)$, so that $|\Phi(x)| = |\mu(\hat{x})| \leqslant l \cdot r_\alpha(x_\alpha)$, $x \in E$, $x_\alpha \in E_\alpha$ ($\subseteq \tilde{E}_\alpha$) that is, $|\Phi_\alpha(x_\alpha)| \leqslant l \cdot N_\alpha(x_\alpha)$, $x_\alpha \in \tilde{E}_\alpha$, for every $\tilde{\Gamma}_\alpha$-invariant norm N_α on $\tilde{E}_\alpha$ equivalent to the original one, and this finishes the proof.

Theorem 3.6 is the analogon in our case of the abstract Bochner–Weil–Raikov representation theorem, where the algebras involved do not necessarily have compact spectra and the continuous involution is replaced by a finite group of (topological) automorphisms of the respective topological ring structures of the given algebras.

In the sequel, as a corollary of the previous theorem, we obtain the classical Bochner–Weil–Raikov theorem for involutive locally m-convex algebras. Before we continue, we state the following lemma for the proof of which the reader is referred to [4], p. 137.

LEMMA 3.7. *Let E be a Banach algebra with continuous involution $x \to x^*$: $E \to E$, and identity e. Then, every positive linear form Φ on E with $\Phi(e) = 1$, satisfies condition (l), with $l = 1$, and Γ the group of two elements generated by γ, $\gamma(x) = x^*$, $x \in E$.*

COROLLARY 3.8 (Bochner–Weil–Raikov theorem). *Let $(E, (p_\alpha))$ be a commutative, locally m-convex algebra with continuous Gel'fand map, and an involution $x \to x^*$: $E \to E$, such that $(x^*)^{\wedge} = (\hat{x})^{-}$, and $p_\alpha(x^*) = p_\alpha(x)$ for every $x \in E$, and $\alpha \in A$. Let also $\Phi \in E'$. Then, the following statements are equivalent:*

(1) *Φ is positive and extendable.*

(2) *Φ admits (an integral) representation of the form*

$$\Phi(x) = \mu(\hat{x}), \quad x \in E,$$

where μ is a (complex) finite positive measure on $\mathfrak{M}(E)$.

In this respect, we first note that the only use of "extendable" in (1) is that we can get a positive extension of Φ to the algebra $E^+ = E \oplus C$ (cf. also [3], p. 96, Lemma 1). We do not also need the assumption of semi-simplicity for E as the classical case does ([3], p. 97, Theorem 26.I).

Proof of Corollary 3.8. We can assume that E has an identity e such that $\Phi(e) = 1$. Thus, if Γ is the two elements group, generated by γ, $\gamma(x) = x^*$, $x \in E$, (E, Γ) is a Γ-Lumer system. Now, since Φ is continuous, there exists $k > 0$ and $\alpha \in A$ such that $|\Phi(x)| \leqslant k \cdot p_\alpha(x)$, for every $x \in E$, hence $\Phi_\alpha(x_\alpha) \equiv \Phi(x)$, $x_\alpha \in E_\alpha$ defines Φ_α as a continuous positive and extendable linear form on E_α such that $\Phi_\alpha(e_\alpha) = 1$, where $e_\alpha = [e]_\alpha = e + \ker(p_\alpha) \in E_\alpha$ is the identity of E_α. Therefore, Φ_α is extended to a continuous positive linear form on $\tilde{E}_\alpha$, with $\Phi_\alpha(e_\alpha) = 1$; besides $\tilde{E}_\alpha$ is a Banach algebra with an isometric involution, defined by that of E, such that $x_\alpha^* = [x^*]_\alpha$, $x_\alpha \in E_\alpha$. Thus (Lemma 3.7) Φ_α satisfies condition (l), hence the same holds true for Φ, that is (1)$\Rightarrow$(2) by Theorem 3.6. For (2)$\Rightarrow$(1) (cf. [3], p. 97, Theorem 26I).

4. (Γ, Δ)-Lumer systems. In this section we obtain an analogous form of Theorem 3.6, by considering on a suitable locally m-convex algebra E, besides the group Γ, the action of an appropriate family Δ of additive transformations of E. We also notice that in case of a Banach algebra the additivity of the elements of Δ is not necessarily assumed (cf. [4], p. 138, Theorem). Thus, in analogy with Definition 3.1, we have

DEFINITION 4.1. Let $(E, (p_\alpha))$ be a locally m-convex algebra. Then the triad (E, Γ, Δ) will be called a *(Γ, Δ)-Lumer system* if

(1) (E, Γ) is a Γ-Lumer system (Definition 3.1);

(2) Δ is a family of additive transformations of E, such that $\delta(xy) = x\delta(y)$ for any $\delta \in \Delta$, $x, y \in E$, and the family $(p_\alpha)_{\alpha \in A}$, is Δ-invariant.

For simplicity, we shall use in the sequel the term *Lumer system* for a (Γ, Δ)-Lumer system.

Given a Lumer system (E, Γ, Δ) one concludes the continuity of the elements of Γ and Δ by "Γ and Δ-invariance" of the respective family $(p_\alpha)_{\alpha \in A}$. Thus, the elements of Γ are additive multiplicative homeomorphisms of E, and the elements of Δ are additive continuous transformations of E, such that $\delta(xy) = x\delta(y)$ for any $\delta \in \Delta$ and $x, y \in E$.

Now, analogously with Lemma 3.2, we have

LEMMA 4.2. *Let (E, Γ, Δ) be a given Lumer system, for a locally m-convex algebra $(E, (p_\alpha))$. Then for each Banach algebra $\tilde{E}_\alpha$, $\alpha \in A$, a Lumer system $(\tilde{E}_\alpha, \tilde{\Gamma}_\alpha, \tilde{\Delta}_\alpha)$ is defined (where $\tilde{\Delta}_\alpha$ denotes the extentions to $\tilde{E}_\alpha$ of the elements of $\Delta_\alpha = \{\delta_\alpha\colon \delta \in \Delta\}$, with $\delta_\alpha(x_\alpha) = [\delta(x)]_\alpha$, $x_\alpha \in E_\alpha$).*

LEMMA 4.3. *Let E be a Banach algebra with a bounded approximate identity, and (E, Γ, Δ) a Lumer system. Then, for every $x \in E$, with $r(x) < 1$,* *a Γ and Δ-invariant norm N is defined on E, equivalent to the original one, and such that $N(x) \leqslant 1$.*

Proof. cf. [4], p. 138, Theorem.

Scholium. Concerning the preceding Lemma 4.3, the action of Δ on E does not contribute to the proof; instead, the only fact one realizes is that the new norm N defined on E, only by means of Γ [4], p. 138, Theorem, apart from its other properties, is also Δ-invariant. However, a full application of Lemma 4.3 one meets in group algebras (cf. Section 6 below). Thus, if $(E, (p_\alpha))$ is a locally m-convex algebra with approximate identity and (E, Γ, Δ) is a given Lumer system, then if $\Phi \in E'$ satisfies condition (l) in the sense of Definition 3.3, then (3.2) is satisfied for every $\tilde{\Gamma}_\alpha, \tilde{\Delta}_\alpha$-invariant norm N_α on $\tilde{E}_\alpha$, equivalent to the original one. Thus, in the rest of the paper, when we say that $\Phi \in E'$ satisfies condition (l) we shall mean that relation (3.2) is true for every $\tilde{\Gamma}_\alpha, \tilde{\Delta}_\alpha$-invariant norm N_α on $\tilde{E}_\alpha$, equivalent with the given one.

Remark 4.4. In connection with the discussion before Lemma 3.5, we note that in case of a commutative, locally m-convex algebra $(E, (p_\alpha))$ with continuous Gel'fand map, and without identity, we shall consider the extended spectrum $\mathfrak{M}(E)^+ = \mathfrak{M}(E) \cup \{0\}$ of E, a k-covering of which from now the sets $\mathfrak{M}(E)^+ \cap U_\alpha^0 = \mathfrak{M}(\tilde{E}_\alpha)^+$ (cf. [9] and [10], p. 31). Moreover, in that case we have $r_\alpha(x_\alpha) = q_{\mathfrak{M}(\tilde{E}_\alpha)^+}(\hat{x})$, $x_\alpha \in E_\alpha \subseteq \tilde{E}_\alpha$, $\alpha \in A$.

LEMMA 4.5. *Let (E, Γ, Δ) be a given Lumer system, for a commutative, locally m-convex algebra $(E, (p_\alpha))$, with approximate identity. Let also $\Phi \in E'$ such that Φ satisfies condition (l). Then, Φ is continuous with respect to the "spectral topology" of E, in the sense of Lemma 3.5.*

Proof. The assertion comes out by applying the argumentation in the proof of Lemma 3.5, as well as, the previous scholium, Lemma 4.3 and Remark 4.4.

We now state the analogon of Theorem 3.6, for a Lumer system (E, Γ, Δ). That is,

THEOREM 4.6. *Let (E, Γ, Δ) be a given Lumer system, for a commutative, locally m-convex algebra $(E, (p_\alpha))$, with continuous Gel'fand map and a bounded approximate identity. Let also $\Phi \in E'$. Then, the following statements are equivalent:*

(1) *Φ satisfies condition (l).*

(2) *Φ admits (an integral) representation of the form*

$$\Phi(x) = \mu(\hat{x}), \quad x \in E,$$

where μ is a (complex) measure on $\mathfrak{M}(E)^+$ of finite positive total variation at most l.

Proof. The conclusion of the theorem is derived by following the reasoning in the proof of Theorem 3.6 and applying Lemma 4.5.

5. Tensor products. We specialize in the sequel to the case of tensor product algebras, which are topologized in the projective tensorial topology π [7], and seek out integral representations of linear forms on such algebras in the sense of the preceding section. Thus, given two locally m-convex algebras $(E_1, (p_\alpha))$, $(E_2, (q_\beta))$ we denote by $E_1 \underset{\pi}{\otimes} E_2$ the respective tensor product algebra, equipped with the projective tensorial topology π which is defined by the family $r_{\alpha,\beta} = p_\alpha \otimes q_\beta$, $\alpha \in A$, $\beta \in B$, of submultiplicative semi-norms (cf. [7], p. 176, Proposition 3.1). The completion of $E_1 \underset{\pi}{\otimes} E_2$ will be denoted by $E_1 \underset{\pi}{\hat{\otimes}} E_2$.

LEMMA 5.1. *Let $(E_1, \Gamma_1, \Delta_1)$, $(E_2, \Gamma_2, \Delta_2)$ be two Lumer system for the locally m-convex algebras $(E_1, (p_\alpha))$, $(E_2, (q_\beta))$, respectively. Then, a new Lumer system $(E_1 \underset{\pi}{\otimes} E_2, \Gamma_1 \otimes \Gamma_2, \Delta_1 \otimes \Delta_2)$ is defined for the corresponding tensor product locally m-convex algebra $E_1 \underset{\pi}{\otimes} E_2$.*

Proof. If $\Gamma_1 \otimes \Gamma_2 = \{\gamma_1 \otimes \gamma_2 \colon \gamma_i \in \Gamma_i,\ i = 1, 2\}$ with

$$(\gamma_1 \otimes \gamma_2)\Big(\sum_{i=1}^{n} x_i \otimes y_i\Big) = \sum_{i=1}^{n} \gamma_1(x_i) \otimes \gamma_2(y_i),$$

for every

$$z = \sum_{i=1}^{n} x_i \otimes y_i \in E_1 \underset{\pi}{\otimes} E_2,$$

and similarly $\Delta_1 \otimes \Delta_2 = \{\delta_1 \otimes \delta_2 \colon \delta_i \in \Delta_i,\ i = 1, 2\}$, then $(E_1 \underset{\pi}{\otimes} E_2, \Gamma_1 \otimes \Delta_1, \Gamma_2 \otimes \Delta_2)$, is certainly a Lumer system for $E_1 \underset{\pi}{\otimes} E_2$.

LEMMA 5.2. *Let $(E_1 \underset{\pi}{\otimes} E_2, \Gamma_1 \otimes \Gamma_2, \Delta_1 \otimes \Delta_2)$ be a Lumer system for (the tensor product locally m-convex algebra) $E_1 \underset{\pi}{\otimes} E_2$, as in Lemma 5.1. Then, $(E_1 \underset{\pi}{\hat{\otimes}} E_2, \Gamma_1 \hat{\otimes} \Gamma_2, \Delta_1 \hat{\otimes} \Delta_2)$ is a Lumer system for $E_1 \underset{\pi}{\hat{\otimes}} E_2$.*

Proof. By Definitions 3.1, 4.1, the elements of $\Gamma_1 \otimes \Gamma_2$, $\Delta_1 \otimes \Delta_2$ are additive transformations of $E_1 \underset{\pi}{\otimes} E_2$ and continuous at zero, so that they have a unique extension to $E_1 \underset{\pi}{\hat{\otimes}} E_2$. Hence, denoting by $\Gamma_1 \hat{\otimes} \Gamma_2$, $\Delta_1 \hat{\otimes} \Delta_2$ the respective extensions of $\Gamma_1 \otimes \Gamma_2$, $\Delta_1 \otimes \Delta_2$ to $E_1 \underset{\pi}{\hat{\otimes}} E_2$, $(E_1 \underset{\pi}{\hat{\otimes}} E_2, \Gamma_1 \hat{\otimes} \Gamma_2, \Delta_1 \hat{\otimes} \Delta_2)$ is a Lumer system for $E_1 \underset{\pi}{\hat{\otimes}} E_2$.

LEMMA 5.3. *Let $(E, (p_\alpha))$, $(E, (q_\beta))$ be two locally m-convex algebras with bounded approximate identities $(e_i)_{i \in I}$, $(f_j)_{j \in J}$ respectively. Then $(e_i \otimes f_j)$ with $(i, j) \in I \times J$ is also a bounded approximate identity for (the tensor product locally m-convex algebra) $E_1 \underset{\pi}{\otimes} E_2$.*

Proof. By hypothesis, and the definition of the topology in $E_1 \underset{\pi}{\otimes} E_2$, the net $(e_i \otimes f_j)_{(i,j)\in I\times J}$ is bounded in $E_1 \underset{\pi}{\otimes} E_2$. On the other hand, it is readily seen that

$$r_{\alpha,\beta}\Big((e_i \otimes f_j)\,\big(\sum_{k=1}^{n} x_k \otimes y_k\big) - \sum_{k=1}^{n} x_k \otimes y_k\Big) \underset{(i,j)}{\longrightarrow} 0,$$

for any $\alpha \in A$, $\beta \in B$, and this finishes the proof.

For the proof of the following lemma the reader is referred to [9]:

LEMMA 5.4. *Let E be a topological algebra with continuous multiplication, $\tilde{E}$ its completion, and $(e_i)_{i\in I}$ a bounded approximate identity for E. Then $(e_i)_{i\in I}$ is also a bounded approximate identity for $\tilde{E}$.*

COROLLARY 5.5. *Let $(E_1, (p_\alpha))$, $(E_2, (q_\beta))$ be two locally m-convex algebras with bounded approximate identities $(e_i)_{i\in I}$, $(f_j)_{j\in J}$ respectively. Then $(e_i \otimes f_j)_{(i,j)\in I\times J}$ is also a bounded approximate identity for $E_1 \underset{\pi}{\hat{\otimes}} E_2$.*

Proof. The proof immediately results from Lemmata 5.3 and 5.4.

THEOREM 5.6. *Let $(E_1, (p_\alpha))$, $(E_2, (q_\beta))$ be two commutative, spectrally barrelled, locally m-convex algebras, with bounded approximate identities. Let also $(E_1 \underset{\pi}{\hat{\otimes}} E_2, \Gamma_1 \hat{\otimes} \Gamma_2, \Delta_1 \hat{\otimes} \Delta_2)$ be a Lumer system for the completion $E_1 \underset{\pi}{\hat{\otimes}} E_2$ of the tensor product algebra $E_1 \underset{\pi}{\otimes} E_2$, as in Lemma 5.2. Moreover, let $\Phi \in (E_1 \underset{\pi}{\hat{\otimes}} E_2)'$. Then, the following statements are equivalent:*

(1) *Φ satisfies condition (l).*

(2) *Φ admits (an integral) representation of the form*

$$\Phi(z) = \mu(\hat{z}), \qquad z \in E_1 \underset{\pi}{\hat{\otimes}} E_2,$$

where μ is a (complex) measure on $\mathfrak{M}(E_1 \underset{\pi}{\hat{\otimes}} E_2)^+$ of finite positive total variation at most l.

Proof. $E_1 \underset{\pi}{\hat{\otimes}} E_2$ is a commutative, complete, spectrally barrelled, locally m-convex algebra (cf. [6], p. 159, Corollary 4.1; and [7], p. 176, Proposition 3.1). Moreover, $E_1 \underset{\pi}{\hat{\otimes}} E_2$ has a bounded approximate identity by Corollary 5.5. Now, since every spectrally barrelled algebra has continuous Gel'fand map [9], the assertion follows immediately by Theorem 4.6.

6. Generalized group algebras. As an application of the preceding, we specialize below to the case of "generalized group algebras", the latter being expressed as suitable tensor product topological algebras [8].

Let now, G be a locally compact abelian group, and $(E, (p_\alpha))$ a commutative, complete, spectrally barrelled, locally m-convex algebra. Then $L^1_E(G) = L^1(G) \underset{\pi}{\hat{\otimes}} E$ is an algebra of the same type as E (cf. [8], p. 288, Lemma, and [6], p. 159, Corollary 4.1). Now, if E has an involution $x \to x^*$ such that $p_\alpha(x^*) = p_\alpha(x)$, $\alpha \in A$, $x \in E$, and $f^*(t) = \overline{f(-t)}$ defines the usual involution on $L^1(G)$, let Γ, Γ' be the two elements groups generated by $\gamma(f) = f^*$, $f \in L^1(G)$, and $\gamma'(x) = x^*$, $x \in E$ respectively. Then, $(L_E^1(G), \Gamma \hat{\otimes} \Gamma')$ is a $\Gamma \hat{\otimes} \Gamma'$-Lumer system (cf. also Lemmata 5.1, 5.2).

Furthermore, one has the following (cf. [2a], Proposition 3.2):

LEMMA 6.1. *Let $(E, (p_\alpha))$ be a locally m-convex algebra, with a bounded approximate identity, and an involution $x \to x^*$ such that $p_\alpha(x^*) = p_\alpha(x)$, $\alpha \in A$, $x \in E$. Then, each continuous positive linear form Φ on E is extendable.*

Now, we can get the classical from of Bochner's theorem for a generalized group algebra. That is,

COROLLARY 6.2. *Let G be a locally compact abelian group, and $(E, (p_\alpha))$ a commutative, complete, spectrally barrelled, locally m-convex algebra with a locally equicontinuous spectrum, a bounded approximate identity, as well as an involution $x \to x^*$ such that $(x^*)^\wedge = (\hat{x})^-$ and $p_\alpha(x^*) = p_\alpha(x)$ for any $x \in E$, $\alpha \in A$. Let also $\Phi \in (L_E^1(G))'$. Then the following statements are equivalent:*

(1) *Φ is positive.*

(2) *Φ admits (an integral) representation of the form*

$$\Phi(\vec{f}) = \mu((\vec{f})), \qquad \vec{f} \in L_E^1(G),$$

where μ is a finite positive measure on $\mathfrak{M}(L_E^1(G))$.

Proof. $L_E^1(G) = L^1(G) \underset{\pi}{\hat{\otimes}} E$, so that it is a commutative, complete, spectrally barrelled, locally m-convex algebra (cf. [8], p. 288, Lemma and [6], p. 159, Corollary 4.1), hence, it also has continuous Gel'fand map [9]. Furthermore, by hypothesis and the decomposition of the spectrum of $L_E^1(G)$ (cf. [6]), one gets that the latter algebra is self-adjoint as well. Besides, it also has a bounded approximate identity (Corollary 5.5), so that the assertion now follows by the previous Lemma 6.1, and Corollary 3.8.

Now, the following theorem constitutes an abstract form of Bochner's theorem in case of a generalized group algebra $L_E^1(G)$ as before. In this respect, given a Lumer system (E, Γ', Δ') for E, one gets a natural Lumer system $(L_E^1(G), \Gamma \hat{\otimes} \Gamma', \Delta \hat{\otimes} \Delta')$ for $L_E^1(G)$, where Γ is the group generated by the involution of $L^1(G)$ (cf. for instance, discussion before Lemma 6.1), and $\Delta' = \{\delta_x\colon x \in G\}$ with $\delta_x(f)(y) = f(x+y)$, $f \in L^1(G)$, x, $y \in G$ (translations on G) (cf. also Lemmata 5.1, 5.2). Thus, we have herewith a fully applicability of the action of a family such as $\Delta \hat{\otimes} \Delta'$ (cf. also scholium after Lemma 4.3).

THEOREM 6.3.. *Let G be a locally compact abelian group, and E a commutative, complete, spectrally barrelled, locally m-convex algebra, with a bounded approximate identity, Moreover, let $(L_E^1(G), \Gamma \hat{\otimes} \Gamma', \Delta \hat{\otimes} \Delta')$ be the above Lumer system, and let $\Phi \in (L_E^1(G))'$. Then the following statements are equivalent:*

(1) *Φ satisfies condition* (l).

(2) *Φ admits (an integral) representation of the form*

$$\Phi(\vec{f}) = \mu((\vec{f})), \qquad \vec{f} \in L^1_E(G),$$

where μ is a (complex) measure on $\mathfrak{M}(L^1_E(G))^+$ of finite positive total variation at most l.

Proof. The proof is an immediate consequence of Theorem 5.6.

Concerning the preceding family $\Delta \hat{\otimes} \Delta'$ we also note that one might just consider the respective family of translations on G for $L^1_E(G)$.

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