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On Banach algebras with closed set of algebraic elements

Abstract. A necessary and sufficient conditions are given for the set of algebraic elements of a semisimple commutative Banach algebra to be closed.

In this paper we are concerned with commutative Banach algebras over the field of complex numbers. For such an algebra A, $\mathfrak{M}(A)$ stands for the maximal ideal space with the Gelfand topology. If $x \in A$, then $x\hat{}$ denotes its Gelfand transform, $\sigma(x) = \sigma(x; A)$ its spectrum in the algebra A and $\|x\|_s$ its spectral radius.

An element $x \in A$ is called *algebraic* if there exists a monic polynomial p with complex coefficients such that $p(x) = 0$ (*monic* means that the leading coefficient is equal to one).

An element $x \in A$ is algebraic of degree n if $p(x) = 0$ for some monic polynomial p of degree n and $q(x) \neq 0$ for every monic polynomial q of degree less than n.

The set of all algebraic elements of an algebra A will be denoted by $\operatorname{alg} A$.

An element $x \in A$ is *quasi-algebraic* if there exists a sequence of monic polynomials $\{p_n\}$, with degree of p_n equal to $d(n)$ such that $\|p_n(x)\|^{1/d(n)} \to 0$ as $n \to \infty$ or equivalently x is quasi-algebraic if $\operatorname{cap} x = \operatorname{Cap} \sigma(x) = 0$ (see [2], Theorem 2 and Theorem 3).

A set of all quasi-algebraic elements of an algebra A will be denoted by $\operatorname{qalg} A$.

We start with the following lemma.

LEMMA. *An element x of a semisimple commutative Banach algebra A is algebraic if and only if its spectrum consists of finite number of elements.*

Proof. Let x be an algebraic element. Thus there exists a polynomial $p(z) = (z-\lambda_1) \dots (z-\lambda_n)$, $\lambda_1, \dots, \lambda_n \in C$, such that $p(x) = 0$. On the other hand $p(\sigma(x)) = \sigma(p(x)) = \{0\}$. So we have $\sigma(x) \subset \{\lambda_1, \dots, \lambda_n\}$.

Conversely, let us suppose that $\sigma(x) = \{\lambda_1, \dots, \lambda_n\}$. This implies that for the polynomial $p(z) = (z-\lambda_1) \dots (z-\lambda_n)$ we have $p(\sigma(x)) = \{0\}$. It means $\sigma(p(x)) = \{0\}$. So we have $\|p(x)\|_s = 0$. Since the algebra A is semisimple we have $p(x) = 0$.

Now we are going to prove the following fact.

THEOREM. *Let A be a commutative semisimple Banach algebra with unit. Then the following conditions are equivalent*:

(1) *the set of algebraic elements of the algebra A is closed*;

(2) $\operatorname{alg} A = \operatorname{qalg} A$;

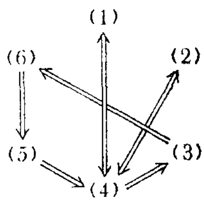
(3) *there exists a positive integer n such that every algebraic element is algebraic of degree not greater than n*;

(4) $\mathfrak{M}(A)$ *has a finite number of components*;

(5) *for every $x \in A$ there exists a positive integer n such that $\sigma(x)$ has n components*;

(6) *there exists a positive integer n such that, for every $x \in A$, $\sigma(x)$ has at most n components.*

Proof. We will carry out the proof according to the following plan:



Let us start with the proof of (3)$\Rightarrow$(6). Let us suppose that condition (6) does not hold. It means that for every positive integer n there exists an element $x_n \in A$ such that $\sigma(x_n)$ has more than n components. Let us take an arbitrary positive integer n. There exists the following decomposition:

$$\sigma(x_n) = E_1 \cup \ldots \cup E_n,$$

where E_i are pairwise disjoint and closed. Then there exist pairwise disjoint and open sets U_i such that $U_i \supset E_i$ for $i = 1, \ldots, n$. So we have $\sigma(x_n) \subset U = U_1 \cup \ldots \cup U_n$. Now we define functions analytic on U by the formulas:

$$f_k(z) = \begin{cases} 1 & \text{for } z \in U_k, \\ 0 & \text{for } z \in U \setminus U_k, \end{cases}$$

for $k = 1, \ldots, n$.

Now, by the functional calculus (see [3], 16.2 Theorem) we get elements $y_k \in A$ for $k = 1, \ldots, n$ such that for every $M \in \mathfrak{M}(A)$ the following equalities hold

$$\hat{y_k}(M) = f_k(\hat{x_n}(M)).$$

This implies that

$$y_k^{\hat{}}(M) = \begin{cases} 1 & \text{for } x_n^{\hat{}}(M) \in \overline{E_k}, \\ 0 & \text{for } x_n^{\hat{}}(M) \in \sigma(x_n)\setminus E_k. \end{cases}$$

Now taking an element $y = \lambda_1 y_1 + \ldots + \lambda_n y_n$, where $\lambda_1, \ldots, \lambda_n$ are pairwise different complex numbers, we have

$$y^{\hat{}}(M) = \lambda_k \quad \text{if } x_n^{\hat{}}(M) \in E_k \ (k = 1, \ldots, n).$$

Thus $\sigma(y) = \{\lambda_1, \ldots, \lambda_n\}$. By the previous lemma the element y is algebraic. It is algebraic of degree n since the numbers λ_i are pairwise different. So we have proved that for every positive integer n there exists an algebraic element of degree n, which means that condition (3) does not hold.

(6) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (4). Let us assume condition (4) is not satisfied. So the set $\mathfrak{M}(A)$ has infinitely many components. We shall construct by induction a sequence of non-empty closed and open sets $\{F_i\}$ such that $F_{i+1} \supsetneqq F_i \subset \mathfrak{M}(A)$ for $i = 1, 2, \ldots$ and every F_i has infinitely many components. Moreover, its corresponding sequence of idempotents $\{f_i\}$ is such that $f_{i+1}f_i = f_{i+1}$ for $i = 1, 2, \ldots$

Let $F_1 = \mathfrak{M}(A)$ and $f_1 = e$ — the unit of the algebra A. Since $\mathfrak{M}(A)$ is disconnected and has infinitely many components then by Šilov idempotent theorem (see [3], 20.2 Theorem), there exist an open and closed subset $F_2 \subsetneqq F_1$, having infinitely many components, and the corresponding idempotent f_2 such that $f_2^{\hat{}}$ is the characteristic function of the set F_2. Proceeding by an induction we obtain a sequence $\{F_i\}$ of closed-open subsets of $\mathfrak{M}(A)$ such that

$$F_1 \supsetneqq F_2 \supsetneqq \ldots \supsetneqq F_n \supsetneqq \ldots$$

and corresponding sequence $\{f_i\}$ of idempotents such that

$$f_{i+1}f_i = f_{i+1}$$

for $i = 1, 2, \ldots$ ($f_i^{\hat{}}$ being the characteristic function of F_i).

Now, we take a new sequence of idempotents $\tilde{f}_n = f_n - f_{n+1}$ for $n = 1, 2, \ldots$ The elements $\tilde{f}_n$ are pairwise orthogonal, i.e., $\tilde{f}_n\tilde{f}_m = 0$ if $n \neq m$. Indeed for $n > m$ we have

$$\begin{aligned} \tilde{f}_n\tilde{f}_m &= (f_n - f_{n+1})(f_m - f_{m+1}) \\ &= f_nf_m - f_{n+1}f_m - f_nf_{m+1} + f_{n+1}f_{m+1} \\ &= f_n - f_{n+1} - f_n + f_{n+1} = 0. \end{aligned}$$

Since $\tilde{f}_n$ are idempotents we have $\|\tilde{f}_n\| \geqslant 1$ for $n = 1, 2, \ldots$ Now, we choose a sequence of numbers $\{\lambda_i\}$ such that

$$1 > |\lambda_n| > |\lambda_{n+1}| > 0 \quad \text{and} \quad |\lambda_n| \leqslant (2^n \|\tilde{f}_n\|)^{-1} \quad \text{for } n = 1, 2, \ldots$$

Then $f = \sum_{n=1}^{\infty} \lambda_n \tilde{f}_n$ belongs to the algebra A. Let us denote

$$\tilde{F}_n = \{M \in \mathfrak{M}(A): \tilde{f}_n^{\wedge}(M) = 1\} \quad \text{for } n = 1, 2, \ldots$$

If $n \neq m$, then $\tilde{F}_n \cap \tilde{F}_m = \emptyset$ because $\tilde{f}_n$ are orthogonal to each other. Thus we have

$$f^{\wedge}(M) = \begin{cases} \lambda_n & \text{for } M \in \tilde{F}_n, \\ 0 & \text{for } M \in \mathfrak{M} \setminus \bigcup_{n=1}^{\infty} \tilde{F}_n. \end{cases}$$

This implies $\sigma(f) = \{0\} \cup \{\lambda_n\}_{n=1}^{\infty}$. So the spectrum $\sigma(f)$ has infinitely many components, i.e., condition (5) is not satisfied.

Let us note that we have simultaneously proved the implications: (1) $\Rightarrow$ (4) (since f is not algebraic but is a limit of the sequence of the algebraic elements $\{\sum_{i=1}^{n} \lambda_i \tilde{f}_i\}$) and (2) $\Rightarrow$ (4) (since the capacity of any countable set is zero, the element f is quasi-algebraic but it is not algebraic).

(4) $\Rightarrow$ (3). Let us suppose $\mathfrak{M}(A) = F_1 \cup \ldots \cup F_n$, where F_i are closed and pairwise disjoint. We also assume that each F_i is connected. By Šilov Theorem (see [3], 20.4 Corollary) there exist closed ideals $I_1, \ldots, I_n$ such that

$$A = I_1 \oplus \ldots \oplus I_n$$

and, moreover, $\mathfrak{M}(I_i) = F_i$ for $i = 1, \ldots, n$.

Let $x \in A$ be an arbitrary algebraic element. There exists a monic polynomial p such that $p(x) = 0$. Since $x = (x_1, \ldots, x_n)$, where $x_i \in I_i$ for $i = 1, \ldots, n$ and $p(x) = (p(x_1), \ldots, p(x_n)) = 0$ we obtain $p(x_i) = 0$ for every i. So every element x_i is algebraic. Since $\mathfrak{M}(I_i) = F_i$ is connected the set $\sigma(x_i; I_i) = \{x_i^{\wedge}(M): M \in F_i\}$ is also connected. Thus it must contain only one point; otherwise it would have to contain a continuum and consequently it would have positive capacity (see [1], p. 296), and the element x_i would not be algebraic (see [2], Theorem 2 and Theorem 3). We have then $\sigma(x_i; I_i) = \{\lambda_i\}$ for $i = 1, \ldots, n$. Since the algebra I_i is semisimple, we have $x_i = \lambda_i e_i$ (e_i denotes the unit of the algebra I_i). Consider the polynomial $q(z) = (z - \lambda_1) \ldots (z - \lambda_n)$. Then $q(x) = (q(x_1), \ldots \ldots, q(x_n)) = 0$. Therefore x is algebraic of degree n at most.

(4) $\Rightarrow$ (2). Let us suppose that the algebra A has a decomposition as above

$$A = I_1 \oplus \ldots \oplus I_n,$$

where $\mathfrak{M}(I_i) = F_i$ and F_i is connected for $i = 1, \ldots, n$. It is sufficient to show that $\operatorname{qalg} A \subset \operatorname{alg} A$.

If $x \in \operatorname{qalg} A$, then $x = (x_1, \ldots, x_n)$ and every $x_i \in \operatorname{qalg} I_i$. But $\sigma(x_i; I_i)$ is connected, so it contains only one point; otherwise the element x_i would not be quasi-algebraic (see the proof of implication (4) $\Rightarrow$ (3)). Therefore for $i = 1, \ldots, n$ we have $\sigma(x_i; I_i) = \{\lambda_i\}$. Now we proceed in the same way as in the proof of implication (4) $\Rightarrow$ (3).

Finally, we prove (4) $\Rightarrow$ (1). Let as above $A = I_1 \oplus \ldots \oplus I_n$ with $\mathfrak{M}(I_i) = F_i$ and F_i connected for $i = 1, \ldots, n$. We have to show that $\overline{\operatorname{alg} A} \subset \operatorname{alg} A$. Let $x \in \overline{\operatorname{alg} A}$. There exists a sequence of algebraic elements $\{x_k\}$ such that $\|x - x_k\| \to 0$ as $k \to \infty$. We have $x_k = (x_1^{(k)}, \ldots, x_n^{(k)})$ and $x_i^{(k)} \in \operatorname{alg} I_i$ for $i = 1, \ldots, n$ and for all k. Moreover, $x_i^{(k)} \to x_i$ as $k \to \infty$. As above (see the proof of (4) $\Rightarrow$ (3)) there exists $\{\lambda_i^{(k)}\}$ such that $\sigma(x_i^{(k)}; I_i) = \{\lambda_i^{(k)}\}$. Since the spectrum in a commutative Banach algebra is a continuous function (see [3], 17.12 Theorem) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sigma(x_i; I_i) \subset \sigma(x_i^{(k)}; I_i) + \{z: |z| < \varepsilon\}$$

for $\|x_i - x_i^{(k)}\| < \delta$. Thus there exists an integer N such that for all $k > N$ we have

$$\sigma(x_i; I_i) \subset \lambda_i^{(k)} + \{z: |z| < \varepsilon\}.$$

Hence $|\alpha - \lambda_i^{(k)}| < \varepsilon$ for $k > N$ and for every $\alpha \in \sigma(x_i; I_i)$. This implies that each point in $\sigma(x_i; I_i)$ is the accumulation point of the sequence $\{\lambda_i^{(k)}\}_{k=1}^{\infty}$ $(i = 1, \ldots, n)$, moreover, the sequence itself is convergent. Thus $\sigma(x_i; I_i)$ is a singleton $\{\lambda_i\}$. From now on the proof is exactly the same as that of (4) $\Rightarrow$ (3). So the proof of the theorem is completed.

Remark. Conditions (4), (5) and (6) are equivalent without the assumption of semisimplicity. But this assumption is essential if we want all these conditions to be equivalent. It is seen in the following example.

EXAMPLE. Let $A_0 = L_1(0, 1)$ denote the Banach algebra of complex-valued functions absolutely summable on the interval $(0, 1)$, with the multiplication defined as convolution. Let $A = A_0 \oplus \{\lambda e\}$ be the algebra obtained by adjunction of a unit to A. Then we have $\mathfrak{M}(A) = \{A_0\}$ and $\operatorname{rad} A = A_0$ (see [3], 12.10 Example). Spectrum of an arbitrary element of the algebra A contains only one point, namely $\sigma(x + \lambda e) = \{\lambda\}$. So conditions (4), (5) and (6) of the theorem are satisfied. But for every integer $n > 1$ there exists a nilpotent element of degree n. It is sufficient to consider the function

$$(*) \qquad x(t) = \begin{cases} 0 & \text{for } 0 < t < 1/n, \\ 1 & \text{for } 1/n \leqslant t < 1. \end{cases}$$

It is, obviously, the algebraic element of degree n. Moreover, such elements are dense in $L_1(0, 1)$. The element $x = x(t) = 1$ for $t \in (0, 1)$ is quasi-nilpotent but it is not algebraic, since we have $x^n(t) = t^{n-1}/(n-1)!$. What is more this element is a limit of nilpotent elements of type (*). Thus we see that conditions (1), (2) and (3) are not satisfied.

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