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Some kinds of the unicoherence

Abstract. It is proved that a continuum X is strongly unicoherent (for the definition see below) if and only if every subcontinuum of X with a non-empty interior is unicoherent. This result gives a positive answer to a problem asked in [2].

In this paper a continuum is a compact connected metric space. A continuum X is said to be *unicoherent* provided that the intersection of any two subcontinua, whose union is X, is connected. The continuum X is called *hereditarily unicoherent* if every subcontinuum of X is unicoherent, or, what is equivalent, if the intersection of any two subcontinua of X is connected.

The concept of strongly unicoherent continua was introduced in [1]. We say that a unicoherent continuum X is *strongly unicoherent* provided that for any pair of proper subcontinua K and L such that $X = K \cup L$, each of K and L is unicoherent. D. E. Bennett asked the following problem in [2], p. 3: is every subcontinuum of a strongly unicoherent continuum X with a non-empty interior a unicoherent continuum? The answer is given by the following

THEOREM. *A continuum X is strongly unicoherent if and only if every subcontinuum of X with a non-empty interior is unicoherent.*

Proof. A sufficient condition is obvious. Assume now that X is strongly unicoherent. Let Q be a proper subcontinuum of X with a non-empty interior.

(1) If there is a non-empty open set V contained in Q, which does not separate X, then Q is unicoherent.

Indeed, let V be a non-empty open set contained in Q, which does not separate X. Then $X \setminus V$ and Q are proper subcontinua of X and $X = (X \setminus V) \cup Q$. Thus $X \setminus V$ and Q are unicoherent by the strong unicoherence of X.

(2) If V is a non-empty open set contained in Q and Q is not unicoherent, then there are at least two components A' and B' of $X \setminus V$, each of which is not contained in Q.

In fact, since $\emptyset \neq X\setminus Q \subset X\setminus V$, we conclude that there is a component A' of $X\setminus V$ which is not contained in Q. If any component of $X\setminus V$ other than A' is contained in Q, then $X = A'\cup Q$. Thus Q is unicoherent by the strong unicoherence of X.

Suppose, on the contrary, that Q is not unicoherent. Then there are continua Q_1, Q_2 and closed non-empty disjoint sets P and R such that $Q = Q_1\cup Q_2$ and $Q_1\cap Q_2 = P\cup R$. Since Q has a non-empty interior, we infer that either Q_1 or Q_2 has a non-empty interior.

Assume that $\mathrm{Int}Q_1 \neq \emptyset$. Consider two cases.

(a) $(\mathrm{Int}Q_1)\setminus Q_2 \neq \emptyset$. Then there is a non-empty open set V such that $V \subset Q_1\setminus Q_2$. It follows from (1) and (2) (cf. also [3], § 46, IV, p. 142) that there are closed non-empty disjoint sets A and B such that

(3) $X\setminus V = A\cup B$, $A\setminus Q \neq \emptyset$ and $B\setminus Q \neq \emptyset$.

Since $Q_2 \subset X\setminus V$ and Q_2 is connected, we can assume that $Q_2 \subset B$. Obviously

(4) $A\cap Q_2 = \emptyset$.

Moreover,

(5) sets $A\cup Q$ and $A\cup Q_1$ are proper subcontinua of X.

Indeed, since any component of A intersects $\overline{V}$ (see [3], § 47, III, Theorem 1, p. 172) and since $\overline{V} \subset Q_1 \subset Q$, we infer that the sets $A\cup Q$ and $A\cup Q_1$ are continua. Suppose that $A\cup Q = X$. Then $(A\cup Q_1)\cup Q_2 = X$. Hence the set $(A\cup Q_1)\cap Q_2$ is connected by the unicoherence of X. But $(A\cup Q_1)\cap Q_2 = Q_1\cap Q_2 = P\cup R$ by (4), a contradiction.

(6) The set $B\cup Q$ is a proper subcontinuum of X.

Indeed, since any component of B intersects $\overline{V}$ and since $\overline{V} \subset Q$, we conclude that the set $B\cup Q$ is a continuum. Suppose that $B\cup Q = X$. Then $A \subset Q$, because $A\cap B = \emptyset$. But $A\setminus Q \neq \emptyset$ by (3), a contradiction.

We have $X = (X\setminus V)\cup V = (A\cup B)\cup Q = (A\cup Q)\cup(B\cup Q)$. Thus, according to the strong unicoherence of X, we infer that the continuum $A\cup Q$ is unicoherent by (5) and (6). But $A\cup Q = (A\cup Q_1)\cup Q_2$. Therefore $(A\cup Q_1)\cap Q_2$ is connected by (5), a contradiction, because $(A\cup Q_1)\cap Q_2 = Q_1\cap Q_2 = P\cup R$ by (4). This completes the proof of case (a).

(b) $\mathrm{Int}Q_1 \subset P\cup R$. Take a continuum I irreducible between P and R in Q_2. Then I is irreducible between every pair of points p, r, where $p \in P\cap I$ and $r \in R\cap I$ (see [3], § 48, IX, Theorem 2, p. 222).

If I is an indecomposable continuum, then there exists a composant C of I (for the definition of a composant see [3], § 48, VI, p. 208) such that $C\cap\big((P\cap I)\cup(R\cap I)\big) = \emptyset$. Since C is dense in I (see [3], § 48, VI, Theorem 2, p. 209), we infer that $I\cap\mathrm{Int}Q_1 = \emptyset$, thus $(\mathrm{Int}Q_1)\setminus I \neq \emptyset$. Taking $Q_1\cup I$ instead of Q and I instead of Q_2, one can obtain a contradiction as in case (a).

If I is decomposable, then there are continua I_1 and I_2 such that $I = I_1 \cup I_2$ and $I_1 \cap P = \emptyset$ and $I_2 \cap R = \emptyset$. Since $\mathrm{Int} Q_1 \subset P \cup R$, we can assume that there is a non-empty open set U contained in R. Then $\big(\mathrm{Int}(Q_1 \cup \cup I_1)\big) \setminus I_2 \neq \emptyset$. Taking $Q_1 \cup I$ instead of Q, $Q_1 \cup I_1$ instead of Q_1 and I_2 instead of Q_2, one can obtain a contradiction as in case (a). The proof of Theorem is complete.

I have introduced the concept of weakly hereditarily unicoherent continua in [4]: a continuum X is *weakly hereditarily unicoherent* in case the intersection of any two subcontinua of X with non-empty interiors is connected.

Above theorem implies that

COROLLARY 1. *Any strongly unicoherent continuum is weakly hereditarily unicoherent.*

In fact, let A and B be continua with non-empty interiors contained in the strongly unicoherent continuum X. If $A \cap B \neq \emptyset$, then $A \cup B$ is a subcontinuum of X with a non-empty interior. Therefore $A \cup B$ is unicoherent by Theorem. This implies that the set $A \cap B$ is connected.

Recall that a *dendroid* is an arcwise connected hereditarily unicoherent continuum. From Theorem 2 of [4] and from Corollary 1 we obtain

COROLLARY 2. *Let a continuum X be arcwise connected. The following conditions are equivalent*:

(i) *X is a dendroid,*

(ii) *X is strongly unicoherent,*

(iii) *X is weakly hereditarily unicoherent.*

The converse implication of Corollary 1 is not true in general (Example below). But we have

PROPOSITION. *Let a continuum X be hereditarily decomposable. If X is weakly hereditarily unicoherent, then X is strongly unicoherent.*

Proof. Let Q be a subcontinuum of X with a non-empty interior. Suppose, on the contrary, that Q is not unicoherent (cf. Theorem). Then there are continua Q_1 and Q_2 and closed non-empty disjoint sets P and R such that $Q = Q_1 \cup Q_2$ and $Q_1 \cap Q_2 = P \cup R$. Since Q has a non-empty interior, we conclude that either Q_1 or Q_2 has a non-empty interior. We can assume that $\mathrm{Int} Q_1 \neq \emptyset$. Take a continuum I irreducible between P and R in Q_2. Since X is hereditarily decomposable, we infer that there are continua I_1 and I_2 such that $I = I_1 \cup I_2$ and $I_1 \cap P = \emptyset$ and $I_2 \cap R = \emptyset$. Sets $Q_1 \cup I_1$ and $Q_1 \cup I_2$ are subcontinua of X with non-empty interiors. Since X is weakly hereditarily unicoherent, we have that the set $(Q_1 \cup \cup I_1) \cap (Q_1 \cup I_2) = Q_1 \cup (I_1 \cap I_2)$ is connected. Thus the set $Q_1 \cap I_1 \cap I_2$ is non-empty, but $Q_1 \cap I_1 \cap I_2 \subset P \cap R = \emptyset$, a contradiction.

Example. Let B denote the Brouwer's indecomposable continuum (see [3], § 48, V, Example 1, p. 204) and let p and q be points of B lying in different composants. Put $B_0 = (B\times\{0\})\cup(\{p,q\}\times[0,1])$ and $B_1 = B\times\{1\}$. The continuum X consists of the sets B_0 and B_1 and of two disjoint lines, one of which aproximates B_0, the other $B_0\cup B_1$. It is easy to see that X is weakly hereditarily unicoherent but it is not strongly unicoherent if we contract intervals $\{p\}\times[0,1]$ and $\{q\}\times[0,1]$ to points.

References

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