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Connectedness of the hyperspace of closed connected subsets

Abstract. Let X be a connected Hausdorff space and $C(X)$ be its collection of closed connected subsets with the Vietoris topology. An example is given to show that $C(X)$ is not necessarily connected. A sufficient condition is given to obtain connected $C(X)$. This condition implies that the "explosion point" example of Knaster and Kuratowski has connected $C(X)$.

It is known that if X is a connected T_1 space, then 2^X (the hyperspace of non-empty closed subsets of X) is connected. It is also known that if X is a continuum (= compact connected T_2), then $C(X)$ is a continuum.

Let X be a space and $C(X)$ be the collection of all non-empty closed connected subsets of X. The Vietoris topology on $C(X)$ is generated by sets of the form $\langle V_1, \ldots, V_n\rangle$, where each V_i is open in X and $A \in \langle V_1, \ldots$ $\ldots, V_n\rangle$ iff $A \in C(X)$, $A \cap V_i \neq \emptyset$ for each i and $A \subseteq \bigcup\{V_i \mid i = 1, \ldots, n\}$. Denote the closure of A by A^*. We will discuss when $C(X)$ is connected if X is connected. A counterexample will be given and some sufficient conditions are provided to obtain connected $C(X)$.

A space X has property (1) provided there exists $p \in X$ such that if $p \in V$ and V is open in X, then there is a non-degenerate $K \in C(X)$ such that $K \subseteq V$.

THEOREM 1. *If X is a non-degenerate connected T_2 space and $C(X)$ is connected, then X has property* (1).

Proof. Suppose for each $p \in X$, there exists an open set V_p containing p such that $\langle V_p\rangle = \langle V_p\rangle \cap \hat{X}$, where $\hat{X}$ is the set of all singletons in X. Then $\hat{X} = \bigcup\{\langle V_p\rangle \mid p \in X\}$ which is open in $C(X)$. But $\hat{X}$ is closed since X is T_2. This contradicts $C(X)$ being connected.

EXAMPLE A. Let X be Bing's countable connected Hausdorff space in [1]. Then X does not have property (1); hence $C(X)$ is not connected.

EXAMPLE B. Let p be a point on the x-axis in Bing's example and adjoin an arc I to X with one endpoint of I equal to p. Then $\hat{X} \cup C(I)$

is closed and open in $C(X \cup I)$. Furthermore, $X \cup I$ has property (1) at each point of I.

A space X has property (2) provided that if K is a non-degenerate proper closed connected subset of X and $K \subseteq V$ open in X, then there exists $H \in C(X)$ such that $K \subseteq H \subseteq V$ and $K \neq H$.

THEOREM 2. *If X is connected T_2 and has properties* (1) *and* (2), *then $C(X)$ is connected.*

Proof. Let $C(X)$ be separated by $\mathscr{U}$ and $\mathscr{V}$. Since $\hat{X}$ is connected, one can assume $\hat{X} \subseteq \mathscr{V}$. Let p be the point given by property (1). Then $\{p\} \in \mathscr{V}$ implies that there is an open set V such that $\{p\} \in \langle V \rangle \subseteq \mathscr{V}$. Choose a non-degenerate $K \in C(X)$ and $K \subseteq V$. Let $\mathscr{C} = \{H \in \mathscr{V} \mid K \subseteq H\}$ and $A \leqslant B$ iff $A \subseteq B$. Choose a maximal chain $\mathscr{M}$ in $\mathscr{C}$ and put $M = (\bigcup \mathscr{M})^* \in C(X)$.

Let $M \in \langle V_1, \ldots, V_n \rangle$. Then for each i, there exists $M_i \in \mathscr{M}$ and $M_i \cap V_i \neq \emptyset$. Since $\{M_i\}$ is a chain, then there is a largest M_j which meet each V_i, i.e., $M \in \mathscr{V}^* = \mathscr{V}$. Property (2) and the openness of $\mathscr{V}$ imply $M = X$.

Let $P \in \mathscr{U}$. Since $\hat{X} \subseteq \mathscr{V}$, then P is non-degenerate. Do the same construction for P in $\mathscr{U}$. One obtains $X \in \mathscr{U}$ which is a contradiction.

COROLLARY 1. *If X is T_2 and $X = \bigcup K_\alpha$ and $\bigcap K_\alpha \neq \emptyset$, where each K_α is a continuum, then $C(X)$ is connected.*

Proof. Let $p \in \bigcap K_\alpha$ and assume X is non-degenerate. Then $p \in K_\beta$ for some non-degenerate continuum K_β. We will show that X has properties (1) and (2). Let $p \in V$ open in X. Choose U open in K_β such that $p \in U \subseteq U^* \subseteq V \cap K_\beta$. Then the component of p in U^* is a non-degenerate continuum contained in V.

Let C be a non-degenerate proper closed connected subset of X and $C \subseteq U$ open in X. Assume $p \in C$. Since $C \neq X$, there exists K_α which is not contained in C. Let $C \cap K_\alpha \subseteq V \subseteq V^* \subseteq U \cap K_\alpha$, where V is open in K_α. The component D of p in V^* meets the boundary of V relative to K_α. Hence $C \cup D$ is the required closed connected subset contained in U.

Suppose $p \notin C$. Choose any $q \in C \cap K_\alpha$ and do the same construction as in the previous paragraph.

COROLLARY 2. *If X is locally connected T_3 and connected, then $C(X)$ is connected.*

Proof. To prove properties (1) and (2), let $C \in C(X)$, and $C \neq X$ and $C \subseteq W$ open in X. Since C is not open, there exists $q \in C$ which is not an interior point of C. Choose a connected open set U such that $q \in U \subseteq U^* \subseteq W$. Then $C \cup U^*$ is the required closed connected set.

EXAMPLE C. The converse of Theorem 2 is false. If n is a positive integer, let $L_n = \{(t, t/n) \mid 0 \leqslant t \leqslant 1\}$ and $L = \bigcup L_n$. Put $U = \{(x, 0) \mid$

$0 \leqslant x \leqslant 1$ and $x \neq 1/2\}$ and $X = L \cup U$. Since $M = \{K \in C(X) \mid (0, 0) \in K\}$ is connected (same proof as Theorem 2) and each $C(L_n)$ meets M at $\{(0, 0)\}$, then $M \cup (\bigcup C(L_n))$ is connected. It suffices to show it is dense in $C(X)$.

Let $K \in C(X)$ and $(0, 0) \notin K$. Then $K \subseteq L_n$ or $K \subseteq U$. Assume $K \subseteq U$. Then $K = \bigcup K_j$, where $\{K_j\}$ is an increasing sequence of compact connected subsets of U. If $K \in \langle V_1, \ldots, V_n \rangle$, then some $K_j \in \langle V_1, \ldots, V_n \rangle$. There exists $A \in C(L_n)$ such that $A \in \langle V_1, \ldots, V_n \rangle$. Note that property (2) fails at $K = \{(x, 0) \mid \frac{1}{2} < x \leqslant 1\}$.

EXAMPLE D. The "explosion point" example of Knaster and Kuratowski (see [3], p. 22] will be shown to have properties (1) and (2). Let $C \times \{0\}$ be the canonical Cantor set on the x-axis and Y the union of all line segments $L(x)$ from $(x, 0)$ to $a = (\frac{1}{2}, \frac{1}{2})$. Let $C = P \cup Q$, where P consists of endpoints and Q its complement in C. Then the explosion point example X is the set of all $(z, t) \in L(x)$, where t is rational iff $x \in P$ and t is irrational iff $x \in Q$. Note that A^* and $B(A)$ mean closure and boundary of A in Y (not in X).

It is easy to see that property (1) is satisfied at a (and nowhere else). The following two lemmas show X has property (2).

LEMMA 1. *If $H \in 2^X$, $a \in H$, $H \subseteq U$ which is open in Y, then $F = \{x \in C \mid L(x) \cap H^* \cap B(U) = \emptyset\}$ is of first category in C.*

Proof. Let $\{r_i \mid i = 1, 2, \ldots\}$ be all the rationals in $[0, \frac{1}{2})$ and $G_i = \{(b, r_i) \in H^* \cap B(U) \mid (b, r_i) \in L(x)$ for some $x \in Q\}$.

Let $x \in F \cap Q$ and $g \in L(x) \cap H^* \cap B(U)$. Since $g \notin U$, then $g \notin H$. But $H = H^* \cap X$ yields $g \notin X$. The y-coordinate of g is rational which gives us $g \in G_i$ for some i.

Note that $G_i^* \subseteq H^* \cap B(U) \cap \{(b, r_i) \mid (b, r_i) \in Y\}$. But if $q \in G^*$, then $q \notin X$ since $H = X \cap H^*$. So $q \in L(x)$ for some $x \in Q$. Let F_i be the image of the (stereographic) projection of G_i^* down the line segments into the Cantor set. F_i is nowhere dense since $F_i \cap P = \emptyset$. Moreover, $F \subseteq (\bigcup F_i) \cup P$.

LEMMA 2. *If $H \in C(X)$ and H is non-degenerate and proper, $H \subseteq U$ open in Y, then there exists $L \in C(X)$ and $H \subseteq L \subseteq U$ and $H \neq L$.*

Proof. We know $a \in H$ and by the previous lemma, one can get $x \in C$ such that $L(x) \cap H^* \cap B(U) = \emptyset$. Note that $L(x) \cap H^*$ is an arc. In fact, we can choose x such that $\{a\} \neq L(x) \cap H^* \neq L(x)$. Since $a \in L(x) \cap H^*$ which is connected and $(L(x) \cap H^*) \cap B(U) = \emptyset$, then $L(x) \cap \cap H^* \subseteq U$. Choose $(e, t) \in (L(x) \cap X) \setminus H^*$ such that $\{(f, s) \in L(x) \mid s \geqslant t\} \subseteq U$. Choose $M \in C(X)$ such that $(e, t) \in M$ and $M \subseteq U$. Now $L = H \cup M$ is the appropriate connected set in X since $a \in M$ and $(e, t) \in M \setminus H$.

Remark. In contrast to hyperspaces of continua: (a) there are maximal chains in $C(X)$ which are not connected, (b) $\{a\}$ is a cutpoint of $C(X)$.

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