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On some two functional equations in the theory of geometric objects

In the present paper all solutions of the functional equation

$$g(x\cdot y) = F(x)\cdot g(y)+g(x) \tag{0.1}$$

are given, where x, y are non-singular 2×2 real matrices, i.e., $x, y \in \mathrm{GL}(2, R)\cdot g$ is an unknown function whose values are 3×1 real matrices, and F is a given function of the form

$$F(x) = \begin{bmatrix} 1 & a_1(\Delta) & a_2(\Delta) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{0.2}$$

or

$$F(x) = (\operatorname{sgn}\Delta)\begin{bmatrix} 1 & a_1(\Delta) & a_2(\Delta) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{0.3}$$

where $\Delta = \det x$, a_1 and a_2 are arbitrary lineary independent solutions of the functional equation

$$a(\xi\eta) = a(\xi)+a(\eta) \quad (\text{for all } \xi\eta \neq 0). \tag{0.4}$$

We do not make any assumptions concerning the regularity of the function g.

Equation (0.1) appears in the theory of geometric objects when we want to find the solutions of the system of functional equations

$$F(x\cdot y) = F(x)\cdot F(y), \qquad g(x\cdot y) = F(x)\cdot g(y)+g(x)$$

([2], p. 152) in order to determine the geometric objects of type [3], [2], [1] with linear non-homogeneous transformation rule.

The main results of the paper are Theorem 0.1 and Theorem 0.2.

THEOREM 0.1. *Any solution of* (0.1) *defined on* $\mathrm{GL}(2, R)$ *in the case when the function* $F(x)$ *is defined by formula* (0.2), *where the functions*

α_1, α_2 *are lineary independent solutions of equation* (0.4), *is of the form*

$$g(x) = \begin{bmatrix} \alpha_0(\Delta)+\frac{1}{2}\varepsilon_1\ \alpha_1^2(\Delta)+\frac{1}{2}\bar{\varepsilon}_2\ \alpha_2^2(\Delta)+\bar{\varepsilon}_1\ \alpha_1(\Delta)\alpha_2(\Delta) \\ \varepsilon_1\ \alpha_1(\Delta)+\bar{\varepsilon}_1\ \alpha_2(\Delta) \\ \bar{\varepsilon}_1\ \alpha_1(\Delta)+\bar{\varepsilon}_2\ \alpha_2(\Delta) \end{bmatrix}. \tag{0.5}$$

In formula (0.5) $\varepsilon_1, \bar{\varepsilon}_1, \bar{\varepsilon}_2$ are some real constants, $\Delta = \det x$, α_0 denotes a function satisfying equation (0.4).

THEOREM 0.2. *Any solution of* (0.1) *defined on* GL$(2, R)$ *in the case when the function* $F(x)$ *is defined by formula* (0.3), *where the functions* α_1, α_2 *are lineary independent solutions of equation* (0.4), *is of the form*

$$g(x) = [F(x)-E]\cdot q. \tag{0.6}$$

In formula (0.6) E denotes the unit 3×3 matrix, i.e., $[\delta_j^i] = E$ $(i, j = 1, 2, 3)$ and q is a 3×1 matrix whose entries are real parameters, F is defined by (0.3).

Equation (0.4) as well as its solutions are well known (cf. [1]). From (0.4) we have

$$\alpha_k(1) = \alpha_k(-1) = 0 \quad \text{for } k = 0, 1, 2, 3, \tag{0.7}$$

$$\alpha_k(\xi) = \alpha_k(-\xi) \quad \text{for all } \xi \neq 0 \text{ and } k = 0, 1, 2, 3. \tag{0.8}$$

Let us observe that from the above-mentioned properties of the solution of (0.4) follows in particular that, if the functions α_1, α_2 are lineary independent on $\boldsymbol{R}-\{0\}$, then they are also lineary independent if we confine ourselves to $\boldsymbol{R}_+$ only. Thus, in the sequel of the present paper we shall use the condition, that

(0.9) the functions α_1 and α_2 in formulae (0.2) and (0.3) are lineary independent on $\boldsymbol{R}_+$.

Remark 0.1. The case when the functions α_1 and α_2 occurring in formulae (0.2) and (0.3) are lineary dependent on $\boldsymbol{R}-\{0\}$ but $\alpha_1 \not\equiv 0$ or $\alpha_2 \not\equiv 0$, can easily be reduced to such a case when equation (0.1) contains $F(x)$ defined by the formulae:

$$F(x) = Q\cdot \begin{bmatrix} 1 & \alpha(\Delta) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot Q^{-1} \tag{0.10}$$

or

$$F(x) = T\cdot(\operatorname{sgn}\Delta) \begin{bmatrix} 1 & \alpha(\Delta) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot T^{-1} \tag{0.11}$$

according to the form of the function F ((0.2) or (0.3)), where Q, T are some constant matrices $\in \mathrm{GL}(3, R)$, $\Delta = \det x$ and

$$a = \begin{cases} a_1, \\ \text{or} \\ a_2. \end{cases}$$

The general solution of (0.1) for all $x, y \in \mathrm{GL}(2, R)$ in this case has been determined and given in my paper [4].

Now

$$g(x) = Q \begin{bmatrix} a_0(\Delta) + \omega a^2(\Delta) \\ 2\omega a(\Delta) \\ a_3(\Delta) \end{bmatrix}, \tag{0.12}$$

when $F(x)$ is defined by (0.10) and

$$g(x) = [F(x) - E]\cdot q, \tag{0.13}$$

when $F(x)$ is of the form (0.11).

a_0, a_3 in formula (0.12) are arbitrary solutions of equation (0.4).

1. The auxiliary lemmas. In the sequel of the present paper we shall apply the following lemmas:

LEMMA 1.1 (cf. [6]). *The general solution of the functional equation*

$$\gamma(x\cdot y) = \gamma(y)\varphi(\Delta) + \gamma(x) \tag{1.1}$$

for all $x, y \in \mathrm{GL}(2, R)$, *where* φ *is an arbitrary not vanishing identically solution of the equation*

$$\varphi(\xi\eta) = \varphi(\xi)\varphi(\eta) \quad \text{for all } \xi\eta \neq 0, \tag{1.1*}$$

is given by the formulae:

$$\gamma(x) = \lambda[\varphi(\Delta) - 1] \quad \text{if } \varphi \not\equiv 1 \tag{1.2}$$

and

$$\gamma(x) = \ln|\Phi_0(\Delta)| \quad \text{if } \varphi \equiv 1. \tag{1.3}$$

$\Delta = \det x$, Φ_0 *is an arbitrary multiplicative function non-identically zero for* $\xi \neq 0$, λ *is a real parameter.*

LEMMA 1.2. *If a function* ϱ *satisfies for all* $x, y \in \mathrm{GL}(2, R)$ *the functional equation*

$$\varrho(x\cdot y) = (\operatorname{sgn}\Delta)\varrho(y) + \varrho(x) \tag{1.4}$$

and

$$\varrho(j) = \varrho\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0, \tag{1.5}$$

where $\Delta = \det x$, *then*

(1.6) $$\varrho(x) = 0 \quad \textit{for every } x \in \mathrm{GL}(2, R).$$

In fact, if we put $\varphi(\Delta) = \operatorname{sgn}\Delta$ in equation (1.1) of Lemma 1.1 and if we consider (1.2), then we obtain the general solution of equation (1.4) defined by the formula

(1.7) $$\varrho(x) = \lambda[\operatorname{sgn}\Delta - 1].$$

In particular, substituting in (1.7) $x = j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we get $\varrho(j) = -2\lambda$. From (1.5) follows that $\lambda = 0$.

Thus, taking into account (1.7) we obtain $\varrho(x) = 0$ for every $x \in \mathrm{GL}(2, R)$.

Lemma 1.3 (cf. [5], p. 64, 65). *The general solution of the system of equations*

(1.8) $$\omega_1(x\cdot y) = \omega_1(y) + \alpha(\Delta_x)\omega_2(y) + \omega_1(x),$$

(1.9) $$\omega_2(x\cdot y) = \omega_2(x) + \omega_2(y)$$

for all $x, y \in \mathrm{GL}(2, R)$, *where* a *is an arbitrary not vanishing solution of equation* (0.4), *is given by the formulae*:

(1.10) $$\omega_1(x) = \ln|\Phi(\Delta)| + \omega\alpha^2(\Delta),$$

(1.11) $$\omega_2(x) = 2\omega\alpha(\Delta),$$

where ω *is an arbitrary constant,* Φ *is an arbitrary non-zero multiplicative function,* $\Delta = \det x = \Delta_x$.

Lemma 1.4 ([5], p. 65, 66). *The general solution of the system of the equations*

(1.12) $$\omega_1(x\cdot y) = (\operatorname{sgn}\Delta_x)\omega_1(y) + (\operatorname{sgn}\Delta_x)\alpha(\Delta_x)\omega_2(y) + \omega_1(x),$$

(1.13) $$\omega_2(x\cdot y) = (\operatorname{sgn}\Delta_x)\omega_2(y) + \omega_2(x)$$

for all $x, y \in \mathrm{GL}(2, R)$, *where* a *is an arbitrary not vanishing solution of equation* (0.4), *is given by the formulae*:

(1.14) $$\omega_1(x) = \omega\alpha(\Delta)\operatorname{sgn}\Delta - \delta(\operatorname{sgn}\Delta - 1),$$

(1.15) $$\omega_2(x) = \omega(\operatorname{sgn}\Delta - 1),$$

where $\det x = \Delta$, ω, δ *are arbitrary constants* ($\delta = \frac{1}{2}\omega_1(j)$).

2. Proof of Theorem 0.1. A straightforward verification shows that the function defined by (0.2) satisfies the equation $F(x\cdot y) = F(x)\cdot F(y)$ (cf. [3]) and any function of the form (0.5) satisfies (0.1). Thus, it remains to prove that the function g satisfying equation (0.1) for all $x, y \in \mathrm{GL}(2, R)$ must be of the form (0.5).

Let

$$(2.1)\qquad g(x) = \begin{bmatrix}\gamma_1(x)\\ \gamma_2(x)\\ \gamma_3(x)\end{bmatrix} = [\gamma_k(x)], \quad k = 1, 2, 3,$$

be an arbitrary solution of equation (0.1), where $F(x)$ is defined by (0.2). Let us notice that the matrix equation (0.1) is equivalent to system of three equations:

$$(2.2)\qquad \gamma_1(x\cdot y) = \gamma_1(y)+\alpha_1(\Delta)\gamma_2(y)+\alpha_2(\Delta)\gamma_3(y)+\gamma_1(x),$$

$$(2.3)\qquad \gamma_2(x\cdot y) = \gamma_2(y)+\gamma_2(x),$$

$$(2.4)\qquad \gamma_3(x\cdot y) = \gamma_3(y)+\gamma_3(x),$$

where $\Delta = \det x$ and the functions α_1, α_2 are arbitrary solutions of the functional equation (0.4) fulfilling condition (0.9).

Applying Lemma 1.1 we obtain

$$(2.5)\qquad \gamma_2(x) = \ln|\Phi_2(\Delta)|,$$

$$(2.6)\qquad \gamma_3(x) = \ln|\Phi_3(\Delta)|.$$

Φ_2 and Φ_3 in formulae (2.5), (2.6) are multiplicative functions non-identically zero on $\boldsymbol{R}\setminus\{0\}$.

From condition (0.9) follows in particular that $\alpha_1 \not\equiv 0$ and $\alpha_2 \not\equiv 0$ on $\boldsymbol{R}_+$. Since $\alpha_1 \not\equiv 0$ on $\boldsymbol{R}_+$, then there exists a number $\beta > 0$ such that $\alpha_1(\beta) \neq 0$.

Writing

$$(2.7)\qquad x_0 = \begin{bmatrix}\sqrt{\beta} & 0\\ 0 & \sqrt{\beta}\end{bmatrix}$$

we have

$$x_0\cdot x = x\cdot x_0 \quad \text{and} \quad \gamma_1(x_0\cdot x) = \gamma_1(x\cdot x_0)$$

for every matrix $x \in \mathrm{GL}(2, \boldsymbol{R})$. From (2.2) we obtain

$$\gamma_1(x_0)+\alpha_1(\Delta)\gamma_2(x_0)+\alpha_2(\Delta)\gamma_3(x_0)+\gamma_1(x) = \gamma_1(x)+\alpha_1(\beta)\gamma_2(x)+\alpha_2(\beta)\gamma_3(x)+\gamma_1(x_0),$$

i.e.,

$$\alpha_1(\beta)\gamma_2(x) = \alpha_1(\Delta)\gamma_2(x_0)+\alpha_2(\Delta)\gamma_3(x_0)-\alpha_2(\beta)\gamma_3(x).$$

Consequently, from $\alpha_1(\beta) \neq 0$ we obtain

$$(2.8)\qquad \gamma_2(x) = 2\omega_1\alpha_1(\Delta)+2\omega_2\alpha_2(\Delta)-2\lambda\gamma_3(x),$$

where

$$(2.9)\qquad 2\omega_1 = \frac{\gamma_2(x_0)}{\alpha_1(\beta)}, \quad 2\omega_2 = \frac{\gamma_3(x_0)}{\alpha_1(\beta)}, \quad 2\lambda = \frac{\alpha_2(\beta)}{\alpha_1(\beta)},$$

$\Delta = \det x, \quad \det x_0 = \beta.$

Analogously taking into account the condition $a_2 \not\equiv 0$ on $\boldsymbol{R}_+$ it follows that there exists $\tau > 0$ such that $a_2(\tau) \neq 0$.

Writing

$$x_1 = \begin{bmatrix} \sqrt{\tau} & 0 \\ 0 & \sqrt{\tau} \end{bmatrix} \tag{2.10}$$

we have

$$x_1 \cdot x = x \cdot x_1 \quad \text{and} \quad \gamma_1(x_1 \cdot x) = \gamma_1(x \cdot x_1)$$

for every matrix $x \in \mathrm{GL}(2, R)$. In the present case from (2.2) we obtain

$$\gamma_1(x_1) + a_1(\Delta)\gamma_2(x_1) + a_2(\Delta)\gamma_3(x_1) + \gamma_1(x) = \gamma_1(x) + a_1(\tau)\gamma_2(x) + a_2(\tau)\gamma_3(x) + \gamma_1(x_1),$$

i.e.,

$$a_2(\tau)\gamma_3(x) = a_1(\Delta)\gamma_2(x_1) + a_2(\Delta)\gamma_3(x_1) - a_1(\tau)\gamma_2(x).$$

Finally

$$\gamma_3(x) = 2\bar{\omega}_1 a_1(\Delta) + 2\bar{\omega}_2 a_2(\Delta) - 2\bar{\lambda}\gamma_2(x), \tag{2.11}$$

where

$$2\bar{\omega}_1 = \frac{\gamma_2(x_1)}{a_2(\tau)}, \quad 2\bar{\omega}_2 = \frac{\gamma_3(x_1)}{a_2(\tau)}, \quad 2\bar{\lambda} = \frac{a_1(\tau)}{a_2(\tau)}, \tag{2.12}$$

$$\Delta = \det x, \quad \det x_1 = \tau.$$

From (2.8) and (2.11) follows that the functions γ_2 and γ_3 are the solutions of the system of equations:

$$\begin{aligned} \gamma_2(x) + 2\lambda\gamma_3(x) &= 2\omega_1 a_1(\Delta) + 2\omega_2 a_2(\Delta), \\ 2\bar{\lambda}\gamma_2(x) + \gamma_3(x) &= 2\bar{\omega}_1 a_1(\Delta) + 2\bar{\omega}_2 a_2(\Delta). \end{aligned} \tag{2.13}$$

The determinant of system (2.13)

$$W(x) = 1 - 4\lambda\bar{\lambda} = 1 - \frac{a_1(\tau)}{a_2(\tau)} \cdot \frac{a_2(\beta)}{a_1(\beta)}. \tag{2.14}$$

Now we have two possibilities: either

$$1 - 4\lambda\bar{\lambda} \neq 0, \tag{2.15}$$

i.e., the case when there exist the numbers $\beta > 0$ and $\tau > 0$ such that $a_1(\beta) \neq 0$, $a_2(\tau) \neq 0$ and

$$1 - 4\lambda\bar{\lambda} = 1 - \frac{a_1(\tau)}{a_2(\tau)} \cdot \frac{a_2(\beta)}{a_1(\beta)} \neq 0$$

or the opposite case to (2.15), i.e., when do not exist the numbers $\beta > 0$ and $\tau > 0$ such that $a_1(\beta) \neq 0$, $a_2(\tau) \neq 0$ and (2.15) is satisfied, it means that for all numbers $\xi > 0$, $\eta > 0$ if $\alpha_1(\xi) \neq 0$, $\alpha_2(\eta) \neq 0$, then

$$(2.16) \qquad 1-4\lambda\bar{\lambda} = 1-\frac{\alpha_1(\eta)}{\alpha_2(\eta)}\cdot\frac{\alpha_2(\xi)}{\alpha_1(\xi)} = 0.$$

In particular substituting $\eta = \tau$ into (2.16) we obtain for every $\xi > 0$

$$1-\frac{\alpha_1(\tau)}{\alpha_2(\tau)}\cdot\frac{\alpha_2(\xi)}{\alpha_1(\xi)} = 0,$$

i.e.,

$$(2.17) \qquad \alpha_2(\tau)\alpha_1(\xi)-\alpha_1(\tau)\alpha_2(\xi) = 0,$$

where $\alpha_2(\tau) \neq 0$.

From (2.17) follows that the functions α_1 and α_2 are lineary dependent on $\boldsymbol{R}_+$ which contradicts condition (0.9).

Thus, in the sequel it remains to consider only case (2.15), i.e., when $W(x) = 1-4\lambda\bar{\lambda} \neq 0$. Taking into account the system of the functional equations (2.13) and case (2.15) we obtain

$$(2.18) \qquad \gamma_2(x) = \varepsilon_1\alpha_1(\Delta)+\varepsilon_2\alpha_2(\Delta),$$

$$(2.19) \qquad \gamma_3(x) = \bar{\varepsilon}_1\alpha_1(\Delta)+\bar{\varepsilon}_2\alpha_2(\Delta),$$

where

$$(2.20) \qquad \begin{aligned} \varepsilon_i &= \frac{2(\omega_i-2\lambda\bar{\omega}_i)}{1-4\lambda\bar{\lambda}} \quad \text{for } i = 1, 2, \\ \bar{\varepsilon}_i &= \frac{2(\bar{\omega}_i-2\lambda\omega_i)}{1-4\lambda\bar{\lambda}} \quad \text{for } i = 1, 2. \end{aligned}$$

Evidently, the functions γ_2 and γ_3 are the solutions of equation (0.4).

Applying (2.18) and (2.19) in formula (2.2) we have for all $x, y \in \in \mathrm{GL}(2, \boldsymbol{R})$

$$(2.21) \qquad \begin{aligned} \gamma_1(x\cdot y) &= \gamma_1(\Delta_y)+\gamma_1(\Delta_x)+ \\ &+\alpha_1(\Delta_x)[\varepsilon\alpha_1(\Delta_y)+\varepsilon_2\alpha_2(\Delta_y)]+\alpha_2(\Delta_x)[\bar{\varepsilon}_1\alpha_1(\Delta_y)+\bar{\varepsilon}_2\alpha_2(\Delta_y)]. \end{aligned}$$

Since for every $x \in \mathrm{GL}(2, R)$ we have $x\cdot x_1 = x_1\cdot x$, where $x_1 = \begin{bmatrix} \sqrt{\tau} & 0 \\ 0 & \sqrt{\tau} \end{bmatrix}$, τ is a positive number such that $\alpha_2(\tau) \neq 0$, then $g(x\cdot x_1) = g(x_1\cdot x)$. From this follows in particular that

$$\gamma_1(x\cdot x_1) = \gamma_1(x_1\cdot x).$$

In the sequel putting in formula (2.21) $y = x_1$ and taking into account the above relation we obtain

$$\gamma_1(x)+\gamma_1(x_1)+\alpha_1(\Delta_x)[\varepsilon_1\alpha_1(\tau)+\varepsilon_2\alpha_2(\tau)]+\alpha_2(\Delta_x)[\bar{\varepsilon}_1\alpha_1(\tau)+\bar{\varepsilon}_2\alpha_2(\tau)]$$
$$=\gamma_1(x_1)+\gamma_1(x)+\alpha_1(\tau)[\varepsilon_1\alpha_1(\Delta_x)+\varepsilon_2\alpha_2(\Delta_x)]+$$
$$+\alpha_2(\tau)[\bar{\varepsilon}_1\alpha_1(\Delta_x)+\bar{\varepsilon}_2\alpha_2(\Delta_x)].$$

Introducing $\Delta:=\Delta_x$ we have for every $x\in \mathrm{GL}(2,R)$

$$(\varepsilon_2-\bar{\varepsilon}_1)[\alpha_1(\Delta)\alpha_2(\tau)-\alpha_1(\tau)\alpha_2(\Delta)]=0. \tag{2.22}$$

Taking into account condition (0.9) and considering that $\alpha_2(\tau)\neq 0$ or $\alpha_1^2(\tau)+\alpha_2^2(\tau)>0$ it follows that $\alpha_2(\tau)\alpha_1(\Delta)-\alpha_1(\tau)\alpha_2(\Delta)$ does not vanish identically on $\boldsymbol{R}_+$. From (2.22) we obtain $\varepsilon_2-\bar{\varepsilon}_1=0$. Thus, we have

$$\varepsilon_2=\bar{\varepsilon}_1. \tag{2.23}$$

Applying (2.23) to (2.18), (2.19) and (2.21) we obtain

$$\gamma_2(x)=\varepsilon_1\alpha_1(\Delta)+\bar{\varepsilon}_1\alpha_2(\Delta), \tag{2.24}$$

$$\gamma_3(x)=\bar{\varepsilon}_1\alpha_1(\Delta)+\bar{\varepsilon}_2\alpha_2(\Delta), \tag{2.25}$$

$$\gamma_1(x\cdot y)=\gamma_1(y)+\gamma_1(x)+ \tag{2.26}$$
$$+\alpha_1(\Delta_x)[\varepsilon_1\alpha_1(\Delta_y)+\bar{\varepsilon}_1\alpha_2(\Delta_y)]+\alpha_2(\Delta_x)[\bar{\varepsilon}_1\alpha_1(\Delta_y)+\bar{\varepsilon}_2\alpha_2(\Delta_y)].$$

Now let us introduce the function γ_0 defined by the formula

$$\gamma_0(x):=\gamma_1(x)-\tfrac{1}{2}\varepsilon_1\alpha_1^2(\Delta)-\tfrac{1}{2}\bar{\varepsilon}_2\alpha_2^2(\Delta)-\bar{\varepsilon}_1\alpha_1(\Delta)\alpha_2(\Delta). \tag{2.27}$$

It follows from (2.26) that the function γ_0 evidently satisfies the equation

$$\gamma_0(x\cdot y)=\gamma_0(x)+\gamma_0(y). \tag{2.28}$$

Consequently, by Lemma 1.1 and formula (2.28) we have

$$\gamma_0(x)=\ln|\Phi_0(\Delta)|,$$

where Φ_0 is a multiplicative function not vanishing indentically on $\boldsymbol{R}\setminus\{0\}$. From $\Phi_0\not\equiv 0$ and from properties of equation (1.1*) follows that $\Phi_0(\xi)\neq 0$ for every $\xi\neq 0$.

Let us notice that $\ln|\Phi_0|$ is a solution of equation (0.4), and let a_0 denote the function $\ln|\Phi_0|$.

Now for every $x\in \mathrm{GL}(2,R)$

$$\gamma_0(x)=a_0(\Delta), \tag{2.29}$$

where a_0 is a solution of (0.4).

Applying (2.29) to formula (2.27) we obtain

$$\gamma_1(x)=a_0(\Delta)+\tfrac{1}{2}\varepsilon_1\alpha_1^2(\Delta)+\tfrac{1}{2}\bar{\varepsilon}_2\alpha_2^2(\Delta)+\bar{\varepsilon}_1\alpha_1(\Delta)\alpha_2(\Delta). \tag{2.30}$$

Thus, applying the results of the above consideration to $g(x)$ we have in the present case

$$(2.31)\qquad g(x) = \begin{bmatrix} \alpha_0(\Delta)+\frac{1}{2}\varepsilon_1\alpha_1^2(\Delta)+\frac{1}{2}\bar{\varepsilon}_2\alpha_2^2(\Delta)+\bar{\varepsilon}_1\alpha_1(\Delta)\alpha_2(\Delta) \\ \varepsilon_1\alpha_1(\Delta)+\bar{\varepsilon}_1\alpha_2(\Delta) \\ \varepsilon_1\alpha_1(\Delta)+\bar{\varepsilon}_2\alpha_2(\Delta) \end{bmatrix}.$$

We have considered all the possible cases and so the proof of Theorem 0.1 has been completed.

3. Proof of Theorem 0.2. A straightforward verification shows that the function defined by (0.3) satisfies the equation $F(x\cdot y) = F(x)\cdot F(y)$ (cf. [3]) and any function of form (0.6) satisfies (0.1). In fact, $g(x\cdot y) = [F(x\cdot y)-E]\cdot q = [F(x)\cdot F(y)-F(x)+F(x)-E]\cdot q = \{F(x)\cdot[F(y)-E]+ +[F(x)-E]\}\cdot q = F(x)\cdot g(y)+g(x)$.

It remains to prove that the function $g(x)$ satisfying equation (0.1) for all $x, y \in \mathrm{GL}(2, R)$ must have form (0.6).

Let

$$(3.1)\qquad g(x) = [\gamma_k(x)] \quad (k = 1, 2, 3)$$

satisfy equation (0.1), where $F(x)$ is of form (0.3). Now, equation (0.1) is equivalent to the system of three equations:

$$(3.2)\qquad \gamma_1(x\cdot y) = (\operatorname{sgn}\Delta)\gamma_1(y)+(\operatorname{sgn}\Delta)\alpha_1(\Delta)\gamma_2(y)+ +(\operatorname{sgn}\Delta)\alpha_2(\Delta)\gamma_3(y)+\gamma_1(x),$$

$$(3.3)\qquad \gamma_2(x\cdot y) = (\operatorname{sgn}\Delta)\gamma_2(y)+\gamma_2(x),$$

$$(3.4)\qquad \gamma_3(x\cdot y) = (\operatorname{sgn}\Delta)\gamma_3(y)+\gamma_3(x),$$

where $\Delta = \det x$ and the functions α_1, α_2 are arbitrary solutions of the functional equation (0.4) fulfilling condition (0.9).

Since the function $\operatorname{sgn}\xi$ is not identically 1 on $\boldsymbol{R}-\{0\}$ the solution of equation (1.1*) thus, according to Lemma 1.1, the general solution of the functional equation (3.3) and (3.4) is given by the corresponding formula

$$(3.5)\qquad \gamma_2(x) = \gamma(\operatorname{sgn}\Delta-1),$$

$$(3.6)\qquad \gamma_3(x) = \bar{\gamma}(\operatorname{sgn}\Delta-1),$$

where γ, $\bar{\gamma}$ are some constants.

Applying (3.5) and (3.6) in equation (3.2) we obtain for all $x, y \in \mathrm{GL}(2, R)$

$$(3.7)\qquad \gamma_1(x\cdot y) = (\operatorname{sgn}\Delta_x)\gamma_1(y)+\gamma_1(x)+ +(\operatorname{sgn}\Delta_x)\gamma[\operatorname{sgn}\Delta_y-1][\gamma\alpha_1(\Delta_x)+\bar{\gamma}\alpha_2(\Delta_x)]$$

or

$$(3.8)\qquad \gamma_1(x\cdot y) = (\operatorname{sgn}\Delta_x)\gamma_1(y)+\gamma_1(x)+\alpha^*(\Delta_x)(\operatorname{sgn}\Delta_x)(\operatorname{sgn}\Delta_y-1),$$

where

$$\alpha^*(\xi) = \gamma\alpha_1(\xi)+\bar{\gamma}\alpha_2(\xi) \tag{3.9}$$

from (0.9) is not vanishing identically solution of equation (0.4). Taking into account Lemma 1.4, from (3.5), (3.6) [cf. (3.3), (3.4)] we obtain

$$\gamma_1(x) = \alpha^*(\Delta)\operatorname{sgn}\Delta+\delta(\operatorname{sgn}\Delta-1) \tag{3.10}$$

or

$$\gamma_1(x) = [\gamma\alpha_1(\Delta)+\bar{\gamma}\alpha_2(\Delta)]\operatorname{sgn}\Delta+\delta(\operatorname{sgn}\Delta-1), \tag{3.11}$$

where $\gamma, \bar{\gamma}, \delta = -\frac{1}{2}\gamma_1(j)$ are some constants.

Now let us put

$$\varrho(x) = \gamma_1(x)-\alpha^*(\Delta)\operatorname{sgn}\Delta-\delta(\operatorname{sgn}\Delta-1). \tag{3.12}$$

The function ϱ evidently satisfies the equation

$$\varrho(x\cdot y) = (\operatorname{sgn}\Delta_x)\varrho(y)+\varrho(x). \tag{3.13}$$

We notice, that $\varrho(j) = 0$. Substituting in formula (3.10) $x = j$ and applying (3.12), (0.7) from $\delta = -\frac{1}{2}\gamma_1(j)$ we obtain

$$\varrho(j) = \gamma_1(j)-(-2)\delta = \gamma_1(j)-\gamma_1(j) = 0.$$

Thus, $\varrho(j) = 0$ and ϱ is the solution of equation (3.13) (i.e., (1.4)). So in the present case according to Lemma 1.2 $\varrho(x) = 0$ for every $x \in GL(2, R)$. It follows from (3.12) that

$$\begin{aligned} \gamma_1(x) &= \varrho(x)+\alpha^*(\Delta)\operatorname{sgn}\Delta+\delta(\operatorname{sgn}\Delta-1), \\ \gamma_1(x) &= (\operatorname{sgn}\Delta)\alpha^*(\Delta)+\delta(\operatorname{sgn}\Delta-1) \\ &= (\operatorname{sgn}\Delta)[\gamma\alpha_1(\Delta)+\bar{\gamma}\alpha_2(\Delta)]+\delta(\operatorname{sgn}\Delta-1), \end{aligned} \tag{3.14}$$

where $\delta, \gamma, \bar{\gamma}$ are some constants.

Thus, in the present case from (3.14), (3.5) and (3.6) we obtain

$$g(x) = \begin{bmatrix} (\operatorname{sgn}\Delta)\alpha^*(\Delta)+\delta(\operatorname{sgn}\Delta-1) \\ \gamma(\operatorname{sgn}\Delta-1) \\ \bar{\gamma}(\operatorname{sgn}\Delta-1) \end{bmatrix} \tag{3.15}$$

$$= \begin{bmatrix} (\operatorname{sgn}\Delta)[\gamma\alpha_1(\Delta)+\bar{\gamma}\alpha_2(\Delta)]+\delta(\operatorname{sgn}\Delta-1) \\ \gamma(\operatorname{sgn}\Delta-1) \\ \bar{\gamma}(\operatorname{sgn}\Delta-1) \end{bmatrix},$$

which can be writen in the form

$$g(x) = \begin{bmatrix} \operatorname{sgn}\Delta - 1 & a_1(\Delta)\operatorname{sgn}\Delta & a_2(\Delta)\operatorname{sgn}\Delta \\ 0 & \operatorname{sgn}\Delta - 1 & 0 \\ 0 & 0 & \operatorname{sgn}\Delta - 1 \end{bmatrix} \cdot \begin{bmatrix} \delta \\ \gamma \\ \bar{\gamma} \end{bmatrix}$$

$$= [F(x) - E]\cdot q, \quad \text{where } q = \begin{bmatrix} \delta \\ \gamma \\ \bar{\gamma} \end{bmatrix}.$$

So the proof of Theorem 0.2 is accomplished.

References

[1] J. Aczél, *Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Basel und Stuttgart 1961.

[2] — und S. Gołąb, *Funktionalgleichungen der Theorie der geometrischen Objekte*, Warszawa 1960.

[3] Z. Kareńska, *O równaniu funkcyjnym $F(x\cdot y) = F(x)\cdot F(y)$*, Zeszyty Naukowe Politechniki Krakowskiej No. 6 (1972), p. 3–269.

[4] — *On a system of functional equations occurring in the theory of geometric objects*, Ann. Polon. Math. 37.1 (1979) (to appear).

[5] M. Kucharzewski and M. Kuczma, *On a system of functional equations occurring in the theory of geometric objects*, ibidem 14 (1963), p. 59–67.

[6] M. Kuczma, *On lineary differential geometric objects of the first class with one component*, Publ. Math. Debrecen 6 (1959), p. 72–78.