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On the osculating k-plane of a curve in an n-dimensional Euclidean space

In considering the general problem to find possibly weak analytic conditions for the existence of osculating k-dimensional surface M of a p-dimensional surface L in an n-dimensional Euclidean space, I restricted myself to the case where $p = 1$ and M is a k-sphere or a k-plane. The results for the case of a k-sphere are in [1]. In this note I present some results of this kind for M being a k-plane.

Let $\boldsymbol{L}(t) = \sum_{j=1}^{n} x^j(t)\boldsymbol{k}_j$ be the radius vector of a curve L in E^n, where $x^j\colon t \to x^j(t)$ are C^{k-1} functions, $\boldsymbol{k}_j$ are the unit vectors and a coordinate system in E^n is chosen so that at some point $P_0(t_0)$ of L the derivatives $\dot{\boldsymbol{L}}(t_0), \ddot{\boldsymbol{L}}(t_0), \ldots, \overset{(k-1)}{\boldsymbol{L}}(t_0)$ are linearly independent. Then, by a proper choice of the vectors $\boldsymbol{k}_j$ we may achieve that the determinant

$$\begin{vmatrix} \overset{(k-1)}{x}{}^{n-k+2}(t_0) & \ldots & \overset{(k-1)}{x}{}^{n}(t_0) \\ \cdots & \cdots & \cdots \\ \dot{x}^{n-k+2}(t_0) & \ldots & \dot{x}^{n}(t_0) \end{vmatrix} \overset{\text{df}}{=} V$$

does not vanish at $P_0(t_0)$. In the sequel we assume also that P_0 is not a point of inflection of order $k-1$ of L. It means that exists such neighbourhood of P_0, that k points of it, chosen on the curve L, do not lie in any $(k-1)$-dimensional plane.

A k-plane in E^n is uniquely determined by any $k-h+1$ of its points and h directions, where h is any integer not exceeding $\frac{1}{2}(k+1)$; the points should not lie in any $(k-h-1)$-dimensional plane. In the sequel we shall consider only k-planes determined either by $k+1$ points and 0 directions or by k points and 1 direction; the points will always belong to the given curve L and the direction will be tangent to L at one of chosen points.

Let the points $P_1(t_1), P_2(t_2), \ldots, P_{k-h+1}(t_{k-h+1})$ of L be chosen so that P_r lies between P_{r-1} and P_{r-2} for each $r \in \{3, 4, \ldots, k-h+1\}$, and any tangent line let be tangent at P_k, whenever it is considered; the sequence of these points will be called *regular* and denoted $\mathrm{RSP}(k-h+1)$. By ${}^h\pi^k(P_s)$ we mean the k-plane through the points P_s ($s \in \{1, 2, \ldots, k+1\}$) when $h = 0$, and the k-plane through the points P_s ($s \in \{1, 2, \ldots, k\}$) and through the tangent line at P_k when $h = 1$.

DEFINITION. Let $\mathrm{RSP}(k-h+1)$ converge to P_0 along L; the limit k-plane of the sequence $\{{}^h\pi^k(P_s^a)\}$, whenever it exists, will be called the *osculating k-plane* of the curve L at P_0 and will be denoted by ${}^h_r\pi^k$ ($r \in \{1, 2, 3\}$, $h \in \{0, 1\}$):

$$ {}^h_r\pi^k = \lim_{\mathrm{RSP}(k-h+1)\to P_0} {}^h\pi^k(P_s^a), $$

whereby we write

(a) ${}^h_1\pi^k$ if P_0 belongs to none of the arcs $\breve{P_1^aP_2^a}$,

(b) ${}^h_2\pi^k$ if P_0 belongs to each of the arcs $\breve{P_1^aP_2^a}$ and is different from P_1^a and P_2^a for each $a \in \boldsymbol{N}$,

(c) ${}^h_3\pi^k$ if P_0 is one of P_1^a or P_2^a in each sequence of the family $\mathrm{RSP}^a(k-h+1)$, $a \in \boldsymbol{N}$.

Choose the points $P_1, P_2, \ldots, P_{k-h+1}$ on L in some neighbourhood of $P_0 \in L$ and let P_{k-h+1} be the origin of each vector $\boldsymbol{P_iP_{k-h+1}}$ ($i \in \{1, 2, \ldots$ $\ldots, k-h\}$). Then a k-plane generated by these vectors, and by tangent line at P_k (when $h = 1$), is described by the system of $n-k$ equations

$$ \sum (x^s - x^s_{k-h+1})\,{}^hX^{n-k+1\ldots n} = 0 \tag{1} $$

for each $s \in \{1, 2, \ldots, n-k\}$, where we write x^j_p instead of $x^j(t_p)$,

(2) $$ {}^hX^{r+1\ldots nsn-k+1\ldots r-1} = $$

$$ \begin{vmatrix} \dfrac{x_1^{r+1}-x_{k-h+1}^{r+1}}{t_1-t_{k-h+1}} & \ldots & \dfrac{x_{k-1}^{r+1}-x_{k-h+1}^{r+1}}{t_{k-1}-t_{k-h+1}} & (1-h)\dfrac{x_k^{r+1}-x_{k+1}^{r+1}}{t_k-t_{k+1}}+h\dot{x}_k^{r+1} \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{x_1^{n}-x_{k-h+1}^{n}}{t_1-t_{k-h+1}} & \ldots & \dfrac{x_{k-1}^{n}-x_{k-h+1}^{n}}{t_{k-1}-t_{k-h+1}} & (1-h)\dfrac{x_k^{n}-x_{k+1}^{n}}{t_k-t_{k+1}}+h\dot{x}_k^{n} \\ \dfrac{x_1^{s}-x_{k-h+1}^{s}}{t_1-t_{k-h+1}} & \ldots & \dfrac{x_{k-1}^{s}-x_{k-h+1}^{s}}{t_{k-1}-t_{k-h+1}} & (1-h)\dfrac{x_k^{s}-x_{k+1}^{s}}{t_k-t_{k+1}}+h\dot{x}_k^{s} \\ \dfrac{x_1^{n-k+1}-x_{k-h+1}^{n-k+1}}{t_1-t_{k-h+1}} & \ldots & \dfrac{x_{k-1}^{n-k+1}-x_{k-h+1}^{n-k+1}}{t_{k-1}-t_{k-h+1}} & (1-h)\dfrac{x_k^{n-k+1}-x_{k+1}^{n-k+1}}{t_k-t_{k+1}}+h\dot{x}_k^{n} \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{x_1^{r-1}-x_{k-h+1}^{r-1}}{t_1-t_{k-h+1}} & \ldots & \dfrac{x_{k-1}^{r-1}-x_{k-h+1}^{r-1}}{t_{k-1}-t_{k-h+1}} & (1-h)\dfrac{x_k^{r-1}-x_{k+1}^{r-1}}{t_k-t_{k+1}}+h\dot{x}_k^{r-1} \end{vmatrix} $$

for $r \in \{n-k+1, \ldots, n\}$ and $\sum$ denotes the summation over all cyclic permutations of the sequence of superscripts $s, n-k+1, n-k+2, \ldots, n$.

Each equation of (1) describes any hyperplane H_s. A normal unit vector of such hyperplane is

$$(3)\qquad \boldsymbol{H}_s = \Bigg\langle \underbrace{\underbrace{0, \ldots, 0}_{s-1}, \frac{{}^{h}X^{n-k+1\ldots n}}{\sqrt{\sum_s({}^{h}X^{i_1\ldots i_k})^2}}, 0, \ldots, 0}_{n-k}, \frac{{}^{h}X^{n-k+2\ldots ns}}{\sqrt{\sum_s({}^{h}X^{i_1\ldots i_k})^2}}, \ldots, \frac{{}^{h}X^{sn-k+1\ldots n-1}}{\sqrt{\sum_s({}^{h}X^{i_1\ldots i_k})^2}}\Bigg\rangle,$$

where $\sum_s$ denotes the summation over all sequences of superscripts $i_1, \ldots, i_k$, which are elements of the set $\{s, n-k+1, \ldots, n\}$. By this we mean that the reare formed all cyclic permutations of the sequence of $s, n-k+1, \ldots, n$, in each of that permutation the first term is omited and the rest of them is denoted by $i_1, \ldots, i_k$.

If the points $P_1, P_2, \ldots, P_{k-h+1}$ converge to P_0 along L, all components of $\boldsymbol{H}_s$ become indeterminate forms of the type $0/0$. To avoid of it we must transform each component in (3). Do it as follows:

1° Write ${}_1^0X^j[t_{k-h+1}t_i]$ in (3) instead of $(x_i^j - x_{k-h+1}^j)/(t_i - t_{k-h+1})$ and the k-th column of ${}^{h}X^{r+1\ldots nsn-k+1\ldots r-1}$ subtract from the i-th column for each $i \in \{1, 2, \ldots, k-1\}$; obtained difference divide by $t_i - t_k$. Then, in the i-th column of the newly obtained determinant, we have the expression of the form

$$\frac{{}_1^0X^j[t_{k+1}t_i] - {}_1^0X^j[t_{k+1}t_k]}{t_i - t_k} \stackrel{\text{df}}{=} {}_2^0X^j[t_{k+1}t_kt_i] \qquad (\text{when } h = 0)$$

or

$$\frac{{}_1^0X^j[t_kt_i] - \dot{x}_k^j}{t_i - t_k} \stackrel{\text{df}}{=} {}_1^1X^j[t_kt_i] \qquad (\text{when } h = 1) \text{ for each } j \in \{1, 2, \ldots, n\}.$$

This expression is called a *finite difference of order* 2 *and of kind* 0 or is called a *finite difference of order* 1 *and of kind* 1, accordingly.

2° In the newly obtained determinant subtract the $(k-1)$-th column from its i-th column for each $i \in \{1, 2, \ldots, k-2\}$ and obtained difference divide by $t_i - t_{k-1}$. Then, in the i-th column of the obtained determinant, we have the finite difference of order $3-h$ and of kind h:

$$\frac{{}_{2-h}^{\;h}X^j[t_{k-h+1}\ldots t_i] - {}_{2-h}^{\;h}X^j[t_{k-h+1}\ldots t_{k-1}]}{t_i - t_{k-1}} \stackrel{\text{df}}{=} {}_{3-h}^{\;h}X^j[t_{k-h+1}\ldots t_{k-1}t_i]$$

for $h \in \{0, 1\}$, and so on. In the last step of such transformation subtract the second column from the first one of the last obtained determinant and obtained difference divide by $t_1 - t_2$. Then in the first column we obtain the finite difference of order $k-h$ and of kind h:

$$\frac{{}_{k-h-1}^{\quad h}X^j[t_{k-h+1}\ldots t_3t_1] - {}_{k-h-1}^{\quad h}X^j[t_{k-h+1}\ldots t_3t_2]}{t_1 - t_2} \stackrel{\text{df}}{=} {}_{k-h}^{\;h}X^j[t_{k-h+1}\ldots t_3t_2t_1].$$

For such finite difference we have

LEMMA 1. *Let $x^j\colon t\to x^j(t)$ be a C^{k+h-1} function ($h\in\{0,1\}$) in an interval $\langle a,b\rangle$ and let $t_1,\ldots,t_{k-h-1}$ be a regular sequence of points in (a,b). Then for every point $t_s\in(t_m,t_{k-h-1})$ ($m\in\{-1,0\}$, letting $t_{-1}=a$, $t_0=b$) with $t_s\neq t_p$ ($p\in\{1,2,\ldots,k-h-2\}$) there exists a point $u\in(t_m,t_{1-m})$ such that*

$$ {}^{h}_{q}X^j[t_{k-h-1}\ldots t_{k-q-h-1}]=\frac{1}{(q+h)!}\overset{(q+h)}{x}{}^j(u)\qquad(q\leqslant k-h-2). $$

(It is Theorem 1 in [2].)

LEMMA 2. *Let $x^j\colon t\to x^j(t)$ be a C^{k-1} function in an interval $\langle a,b\rangle$; let a finite derivative $\overset{(k)}{x}{}^j(t_0)=B^j_k$ exist at a point $t_0\in(a,b)$ and $a,b,t_1,t_2,\ldots$ $\ldots,t_{k-h-1}$ ($h\in\{0,1\}$) be a regular sequence of points in the interval $\langle a,b\rangle$. Then there exists the finite limit*

$$ \lim_{\substack{\mathrm{RSP}(k-h+1)\to t_0\\ t_0\in(a,b)}} {}^{h}_{k-h}X^j[t_{k-h-1}\ldots t_1ba]=\frac{1}{k!}B^j_k, $$

at the point t_0. (It is Theorem 2 in [2].)

LEMMA 3. *Let $x^j\colon t\to x^j(t)$ be a C^{k-1} function in an interval $\langle A,B\rangle$ $\supset\langle a,b\rangle$ and let exist the finite limit*

$$ \lim_{a,b\to t_0}\frac{\overset{(k-1)}{x}{}^j(a)-\overset{(k-1)}{x}{}^j(b)}{a-b}=C_k $$

at such point $t_0\in\langle A,B\rangle$ that $t_0\notin\langle a,b\rangle$; moreover, let the points $a,b,t_1,t_2,\ldots$ $\ldots,t_{k-h-1}$ chosen in $\langle a,b\rangle$ form a regular sequence of points ($h\in\{0,1\}$). Then there exists the finite limit

$$ \lim_{\substack{\mathrm{RSP}(k-h+1)\to t_0\\ t_0\notin\langle a,b\rangle}} {}^{h}_{k-h}X^j[t_{k-h-1}\ldots t_1ba]=\frac{1}{k!}C^j_k $$

at the point t_0. (It is Theorem 3 in [2].)

Recall here one more notion.

DEFINITION. *If $x^j\colon t\to x^j(t)$* is the C^{k-1} function in a closed interval $\langle a,b\rangle$, then by the right-hand side (left-hand side) k-th derivative of the function x^j at a (b) we mean

$$ \overset{(k+)}{x}{}^j(a)=\lim_{\substack{u\to0\\u>0}}\frac{\overset{(k-1)}{x}{}^j(a+u)-\overset{(k-1)}{x}{}^j(a)}{u} \tag{4} $$

$$ \left(\overset{(k-)}{x}{}^j(b)=\lim_{\substack{u\to0\\u<0}}\frac{\overset{(k-1)}{x}{}^j(b+u)-\overset{(k-1)}{x}{}^j(b)}{u}\right). $$

Then we have

LEMMA 4. *Let x^j: $t \to x^j(t)$ be a C^{k-1} function in a closed interval $\langle a, b\rangle$; let the finite right-hand side (left-hand side) derivative $\overset{(k+)}{x}{}^j(a) = D^j_{k+}$ ($\overset{(k-)}{x}{}^j(b) = D^j_{k-}$) exist and let $a, b, t_1, t_2, \ldots, t_{k-h-1}$ ($h \in \{0, 1\}$) be a regular sequence of points in the interval $\langle a, b\rangle$. Then there exists the finite limit*

$$\lim_{\mathrm{RSP}(k-h+1)\to a} {}^h_{k-h}X^j[t_{k-h-1} \ldots t_1 ba] = \frac{1}{k!} D^j_{k+}$$

$$\left(\lim_{\mathrm{RSP}(k-h+1)\to b} {}^h_{k-h}X^j[t_{k-h-1} \ldots t_1 ba] = \frac{1}{k!} D^j_{k-}\right)$$

at the point a (or b).

A proof of this Lemma is identical with the one of Theorem 2 in [2] if we write the limit of (4) instead of

$$\lim_{t\to t_0} \frac{\overset{(q)}{x}{}^j(t) - \overset{(q)}{x}{}^j(t_0)}{t - t_0}$$

using there.

After the indicated above transformations of (2) we use Lemma 1 to the finite differences in a term for the components of vector $\boldsymbol{H}_s$ and write $\frac{1}{(q+h)!} \overset{(q+h)}{x}{}^j(\zeta^r_i)$ instead of ${}^h_qX^j[t_{k-h+1} \ldots t_{k-h-q+1}]$ for each $q \in \{1, 2, \ldots$ $\ldots, k-h-1\}$. Any determinant in one of components of vector $\boldsymbol{H}_s$ has now the form (5) (see p. 320).

Assume now that P_0 belongs to the interior of each arc $\widetilde{P^a_1 P^a_2}$ for $a \in \boldsymbol{N}$ and let the derivative $\overset{(k)}{\boldsymbol{L}}(t_0) = \sum_{j=1}^{n} \overset{(k)}{x}{}^j \cdot \boldsymbol{k}_j$ exist at P_0 and be linearly independent with $\dot{\boldsymbol{L}}(t_0)$, $\ddot{\boldsymbol{L}}(t_0)$, $\ldots$, $\overset{(k-1)}{\boldsymbol{L}}(t_0)$. Then we may choose a coordinate system in E^n so that the derivatives $\overset{(k)}{x}{}^j(t_0) = B^j_k$ are non-zero for each $j \in \{1, 2, \ldots, n-k+1\}$. Converge with $\mathrm{RSP}(k-h+1)$ to P_0 and apply Lemma 2. Then, since L is of C^{k-1}-class, we have the determinant

$$\prod_{r=1}^{k-h+1} \frac{1}{r!} \begin{vmatrix} B^{r+1}_k & \overset{(k-1)}{x}{}^{r+1}(t_0) & \ldots & \ddot{x}^{r+1}(t_0) & (1-h)\dot{x}^{r+1}(t_0) & +h\dot{x}^{r+1}(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B^n_k & \overset{(k-1)}{x}{}^{n}(t_0) & \ldots & \ddot{x}^{n}(t_0) & (1-h)\dot{x}^{n}(t_0) & +h\dot{x}^{n}(t_0) \\ B^s_k & \overset{(k-1)}{x}{}^{s}(t_0) & \ldots & \ddot{x}^{s}(t_0) & (1-h)\dot{x}^{s}(t_0) & +h\dot{x}^{s}(t_0) \\ B^{n-k+1}_k & \overset{(k-1)}{x}{}^{n-k+1}(t_0) & \ldots & \ddot{x}^{n-k+1}(t_0) & (1-h)\dot{x}^{n-k+1}(t_0) & +h\dot{x}^{n-k+1}(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B^{r-1}_k & \overset{(k-1)}{x}{}^{r-1}(t_0) & \ldots & \ddot{x}^{r-1}(t_0) & (1-h)\dot{x}^{r-1}(t_0) & +h\dot{x}^{r-1}(t_0) \end{vmatrix}$$

$$\prod_{r=1}^{k-h-1}\frac{1}{r!}\begin{vmatrix} {}_{k-h}^{\;\;h}X^{r+1}[t_{k-h+1}\ldots t_1] & \overset{(k-1)}{x}{}^{r+1}(\zeta_1^{r+1}) & \ldots & \ddot{x}^{r+1}(\zeta_{k-2}^{r+1}) & (1-h)\dot{x}^{r+1}(\zeta_{k-1}^{r+1}) & +h\dot{x}_k^{r+1} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ {}_{k-h}^{\;\;h}X^{n}[t_{k-h+1}\ldots t_1] & \overset{(k-1)}{x}{}^{n}(\zeta_1^{n}) & \ldots & \ddot{x}^{n}(\zeta_{k-2}^{n}) & (1-h)\dot{x}^{n}(\zeta_{k-1}^{n}) & +h\dot{x}_k^{n} \\ {}_{k-h}^{\;\;h}X^{s}[t_{k-h+1}\ldots t_1] & \overset{(k-1)}{x}{}^{s}(\zeta_1^{s}) & \ldots & \ddot{x}^{s}(\zeta_{k-2}^{s}) & (1-h)\dot{x}^{s}(\zeta_{k-1}^{s}) & +h\dot{x}_k^{s} \\ {}_{k-h}^{\;\;h}X^{n-k+1}[t_{k-h+1}\ldots t_1] & \overset{(k-1)}{x}{}^{n-k+1}(\zeta_1^{n-k+1}) & \ldots & \ddot{x}^{n-k+1}(\zeta_{k-2}^{n-k+1}) & (1-h)\dot{x}^{n-k+1}(\zeta_{k-1}^{n-k+1}) & +h\dot{x}_k^{n-k+1} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ {}_{k-h}^{\;\;h}X^{r-1}[t_{k-h+1}\ldots t_1] & \overset{(k-1)}{x}{}^{r-1}(\zeta_1^{r-1}) & \ldots & \ddot{x}^{r-1}(\zeta_{k-2}^{r-1}) & (1-h)\dot{x}^{r-1}(\zeta_{k-1}^{r-1}) & +h\dot{x}_k^{r-1} \end{vmatrix} \tag{5}$$

instead of that in (5). Among all determinants which have appeared in (3), the determinant

$$\begin{vmatrix} B_k^j & \overset{(k-1)}{x}{}^j(t_0) & \dots & \ddot{x}^j(t_0) & (1-h)\dot{x}^j(t_0) & +h\dot{x}^j(t_0) \\ B_k^{n-k+2} & \overset{(k-1)}{x}{}^{n-k+2}(t_0) & \dots & \ddot{x}^{n-k+2}(t_0) & (1-h)\dot{x}^{n-k+2}(t_0) & +h\dot{x}^{n-k+2}(t_0) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ B_k^n & \overset{(k-1)}{x}{}^n(t_0) & \dots & \ddot{x}^n(t_0) & (1-h)\dot{x}^n(t_0) & +h\dot{x}^n(t_0) \end{vmatrix}$$

is non-zero, according to the remark about numbers B_k^j $(j \in \{1, 2, \dots \dots, n-k+1\})$ and about the determinant V. This determinant appears in the denominator of each component of vectors $\boldsymbol{H}_s$ and appears as the numerator of the one such component. That means that there exists the limit non-zero vector of a sequence of normal unit vectors $\boldsymbol{H}_s$ of hyperplane for each $s \in \{1, 2, \dots, n-k\}$. It follows that there exists the limit k-plane of a sequence of k-planes described by (1). This limit k-plane, by its definition, is the osculating k-plane ${}_2^h\pi^k$. Then we have

THEOREM 1. *Let* $\boldsymbol{L}(t) = \sum_{j=1}^{n} x^j(t)\cdot \boldsymbol{k}_j$ *be the radius vector of a curve* L *in* E^n, *where* $x^j\colon t \to x^j(t)$ *are* C^{k-1}*-functions. Moreover, let* P_0 *be not a poin of inflection of order* $k-1$ *of* L. *If there exist the derivatives* $\dot{\boldsymbol{L}}(t_0)$, $\ddot{\boldsymbol{L}}(t_0)$, $\dots$ $\dots$, $\overset{(k)}{\boldsymbol{L}}(t_0)$ *of* $\boldsymbol{L}(t)$ *and if they are linearly independent, then there exists the osculating k-plane* ${}_2^h\pi^k$ $(h \in \{0, 1\})$ *of* L *at the point* P_0.

Also it is true

THEOREM 2. *Under the same hypothesis on the curve* L *and the point* P_0 *as in Theorem* 1, *the osculating k-plane* ${}_1^h\pi^k$ $(h \in \{0, 1\})$ *of* L *at the point* P_0 *exists provided there exist vectors* $\dot{\boldsymbol{L}}(t_0), \dots, \overset{(k-1)}{\boldsymbol{L}}(t_0)$ *and vector* $\boldsymbol{C}$ *such that* $\boldsymbol{C} = \sum_{j=1}^{n} C_k^j \cdot \boldsymbol{k}_j$, *where*

$$C_k^j = \lim_{a,b\to t_0} \frac{\overset{(k-1)}{x}{}^j(a) - \overset{(k-1)}{x}{}^j(b)}{a-b},$$

and all these vectors are linearly independent.

The proof of this Theorem runs in the same way as the one of Theorem 1, but here is exploited Lemma 3 instead of Lemma 2.

Using Lemma 4 instead of Lemma 2 in the proof of Theorem 1 we can prove

THEOREM 3. *Under the same hypothesis on the curve* L *and the point* P_0 *as in Theorem* 1, *the osculating k-plane* ${}_3^h\pi^k$ $(h \in \{0, 1\})$ *of* L *at the point* P_0 *exists provided there exist vectors* $\dot{\boldsymbol{L}}(t_0), \dots, \overset{(k-1)}{\boldsymbol{L}}(t_0)$ *and vector* $\boldsymbol{D}$ *such that*

$D = \sum_{j=1}^{n} D_k^j \cdot k_j$, *where* D_k^j *is a one-side k-th derivative of* x^j *at the point* P_0, *and all these vectors are linearly independent.*

References

[1] S. Fudali, *On the osculating k-sphere of a curve in an n-dimensional Euclidean space*, Comm. Math. 18 (1974), p. 11–20,

[2] — *Несколько замечаний о конечных разностях*, Prace Naukowe Instytutu Matematyki i Fizyki Teoretycznej Politechniki Wrocławskiej, Studia i Materiały 8 (1973), p. 75–88.

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