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On solutions of some linear difference equations

In my paper[1] (Theorem 2) we have considered asymptotic properties (for $x\to\infty$) of integrals of the differential equation

$$y^{(n)}-\sum_{v=0}^{n-1} c_v(x)y^{(v)} = 0.$$

In this paper we shall prove an analogous theorem (Theorem 2) concerning asymptotic properties of solutions of the difference equation

$$\Delta^n y(x-n)-\sum_{v=0}^{n-1} a_v(x)\Delta^v y(x-v) = 0, \tag{1}$$

where $\Delta^v y(x-v) = \sum_{s=0}^{v}(-1)^s\binom{v}{s} y(x-s)$. The v-th difference $\Delta^v y(x)$ of the function $y(x)$ is an analogue of the derivative $y^{(v)}(x)$. The asymptotic relations (11) in Theorem 2 and the relations occurring in Theorem 2 in my paper[1] are similar if we notice that we have $\Delta P_k(x-1)/P_k(x) = \varepsilon_k a_0^{1/n}(x)$ and $\varphi'(x)/\varphi(x) = \varepsilon_k c_0^{1/n}(x)$, where $\varphi(x) = \exp\varepsilon_k \int_{x_0}^{x} c_0^{1/n}(t)\,dt$. The methods of the proofs for both theorems are similar. We write here $\Delta y(x) = \Delta^1 y(x)$.

From Theorem 2 some oscillation theorem follows (Theorem 3). We obtain Theorem 2 from the

THEOREM 1. *Let us suppose that the functions $b_{mj}(x)$, $m, j = 1, \ldots, n$ $(n\geqslant 2)$ are complex valued and locally bounded for $x\geqslant x_0$; moreover, suppose $b_{mm}(x)\neq 0$ for $x\geqslant x_0$, $m = 1, \ldots, n$,*

$$\sum_{s=0}^{\infty}|b_{11}(\xi+s)-1| < \infty \quad \textit{uniformly for } x_0\leqslant \xi\leqslant x_0+1, \tag{2}$$

and for every m $(1\leqslant m\leqslant n)$ one of the two following hypotheses is satisfied:

(1) Z. Polniakowski, *On some linear differential equations*, Comm. Math. 19 (1976), p. 349–367.

(3a) *The series* $\sum_{s=0}^{\infty}\big(|b_{mm}(\xi+s)|-1\big)$ and $\sum_{s=0}^{\infty} b_{mj}(\xi+s)$ *are convergent absolutely and uniformly for* $x_0 \leqslant \xi \leqslant x_0+1$, $j = 1, \ldots, n$ *and* $j \neq m$, *or*

(3b) $\sum_{s=0}^{\infty}\big||b_{mm}(\xi+s)|-1\big| = \infty$ *uniformly for* $x_0 \leqslant \xi \leqslant x_0+1$, $|b_{mm}(x)| > 1$ *for* $x \geqslant x_0$ *or* $|b_{mm}(x)| < 1$ *for these* x *and* $\lim_{x\to\infty} b_{mj}(x)/\big(|b_{mm}(x)|-1\big) = 0$, $j = 1, \ldots, n$ *and* $j \neq m$.

Then the system of equations

$$u_m(x-1) = \sum_{j=1}^{n} b_{mj}(x)u_j(x), \qquad m = 1, \ldots, n, \tag{4}$$

has for $x \geqslant x_0+1$ *a solution* $\bar{u}_1(x), \ldots, \bar{u}_n(x)$ *such that* $\lim_{x\to\infty}\bar{u}_1(x) = 1$ *and* $\lim_{x\to\infty}\bar{u}_m(x) = 0$ *for* $m = 2, \ldots, n$.

THEOREM 2. *Suppose the functions* $a_v(x)$, $v = 0, \ldots, n-1$, *are defined for* $x \geqslant n$ *and real valued. Furthermore, let us suppose that*

(5) $a_0(x) \neq 0$ *and* $a_0(x)$ *has constant sign for* $x > n$,

(6) $a_0(x-1)/a_0(x)$ *tends monotonically to* 1 *as* $x\to\infty$,

(7) $a_0(x)$ *tends monotonically to* a *as* $x\to\infty$, *where* $0 < |a| \leqslant \infty$,

$$\sum_{s=n}^{\infty}\big(\Delta a_0(\xi+s)\big)^2 a_0^{-2}(\xi+s) < \infty \qquad \textit{uniformly for } n \leqslant \xi \leqslant n+1, \tag{8}$$

$$\lim_{x\to\infty} a_v(x)a_0^{(v+1-n)/n}(x) = 0 \qquad \textit{for } v = 1, \ldots, n-1, \tag{9}$$

$$\sum_{s=n}^{\infty}|a_v(\xi+s)a_0^{(v-n)/n}(\xi+s)| < \infty \qquad \textit{uniformly for } n \leqslant \xi \leqslant n+1 \textit{ and for } v = 1, \ldots, n-1. \tag{10}$$

Then the difference equation (1) *has for* $x \geqslant n$ *solutions* $y_k(x)$, $k = 1, \ldots, n-1$, *such that we have for* $x\to\infty$

$$\begin{aligned} y_k(x) &\sim a_0^{(1-n)/2n}(x)P_k(x), \\ \Delta^m y_k(x-m) &\sim \varepsilon_k^m a_0^{m/n}(x)y_k(x), \qquad m = 1, \ldots, n-1, \end{aligned} \tag{11}$$

where $\varepsilon_k = e^{2k\pi i/n}$ *and* $P_k(x) = 1\Big/\prod_{s=0}^{[x]-n}\big(1-\varepsilon_k a_0^{1/n}(x-s)\big)$ *for* $x \geqslant n$.

If, in addition,

$$a_0(x) \neq 1 \quad \text{for } x \geqslant n \quad \text{and} \quad \lim_{x\to\infty} a_0(x) \neq 1, \tag{12}$$

then the difference equation (1) has for $x \geqslant n$ a solution $y_n(x)$ satisfying relations (11) for $k = n$.

The functions $y_1(x), \ldots, y_n(x)$ are linearly independent. If $|a| < \infty$ and $\cos(2k+\theta)\pi/n > \frac{1}{2}|a^{1/n}|$, then $|\Delta^m y_k(x)| \to \infty$ as $x \to \infty$; if $\cos(2k+\theta)\pi/n < \frac{1}{2}|a^{1/n}|$, then $\Delta^m y_k(x) \to 0$ as $x \to \infty$, $m = 0, \ldots, n-1$. (We set here $\theta = 0$ if $a_0(x) > 0$ and $\theta = 1$ if $a_0(x) < 0$ for $x > n$.)

If $\lim_{x\to\infty} |a_0(x)| = \infty$, then $\Delta^m y_k(x) \to 0$ as $x \to \infty$ for $k = 1, \ldots, n$ and $m = 0, 1, \ldots, n-1$.

Let us notice that the functions $a_v(x) = \pm x^{p_v}$, where $p_0 \geqslant 0$ and $p_v < \min\big((n-v-1)p_0/n,\ (n-v)p_0/n-1\big)$ for $v = 1, \ldots, n-1$, satisfy the hypotheses of Theorem 2.

THEOREM 3. *Let us suppose that assumptions* (5)–(10) *and* (12) *of Theorem 2 are satisfied. Moreover, suppose that* $a_0(x)$ *is continuous for* $x \geqslant n$ *and* $a_0(n) = 0$. *If* $\operatorname{im} \varepsilon_k a_0^{1/n}(x) \neq 0$ *for* $x > n$, *then the difference equation* (1) *has two real solutions* $y_k^*(x)$ *and* $y_k^{**}(x)$ *such that for* $x \geqslant n$

$$\Delta^m y_k^*(x-m) = \big(\cos\big(B_k(x)+\lambda_{km}\big)+\delta_{km}(x)\big)\cdot|a_0^{(2m+1-n)/2n}(x)P_k(x)|,$$
$$\Delta^m y_k^{**}(x-m) = \big(\sin\big(B_k(x)+\lambda_{km}\big)+\eta_{km}(x)\big)\cdot|a_0^{(2m+1-n)/2n}(x)P_k(x)|,$$

where $\lambda_{km} = (2k+\theta)m\pi/n+\theta(1-n)\pi/2n$ *and* $\lim_{x\to\infty}\delta_{km}(x) = \lim_{x\to\infty}\eta_{km}(x) = 0$ *for* $m = 0, 1, \ldots, n-1$. *The function* $B_k(x)$ *is real valued and continuous for* $x \geqslant n$, *has a constant sign for sufficiently large* x *and* $\lim_{x\to\infty}|B_k(x)| = \infty$. *The functions* $\Delta^m y_k^*(x)$ *and* $\Delta^m y_k^{**}(x)$ *change the sign as* $x \to \infty$ *infinitely many times.*

Proof of Theorem 1. We shall apply the theorem of Cauchy (difference analogue of de l'Hôspital rule) in the following formulation: Suppose we have for $x \geqslant x_0$:

$$\text{(i)}\ \lim_{x\to\infty}|g(x)| = \infty \quad \text{and} \quad \sum_{v=1}^{[x-x_0]} |\Delta g(x-v)| \leqslant K|g(x)|$$

or

$$\text{(ii)}\ g(x) \neq 0,\ \lim_{x\to\infty} g(x) = 0 \quad \text{and} \quad \sum_{v=0}^{\infty} |\Delta g(x+v)| \leqslant K|g(x)|.$$

In case (i) we assume that there exists a sequence $\{x_n\}$ tending to ∞ such that the function $f(x)$ is bounded in every interval $\langle x_n, x_n+1\rangle$, $n = 1, 2, \ldots$ In case (ii) we assume that $\lim_{x\to\infty} f(x) = 0$. If $\Delta f(x) = s(x) \times \Delta g(x)$, then $\overline{\lim_{x\to\infty}}|f(x)/g(x)| \leqslant K \overline{\lim_{x\to\infty}}|s(x)|$.

We set, for $x \geqslant x_0$, $g_1(x) = 1/\prod_{v=1}^{\infty} b_{11}(x+v)$, $g_m(x) = \prod_{s=0}^{[x-x_0]} b_{mm}(x-s) = \prod_{v=0}^{[x-x_0]} b_{mm}(\xi+v)$, where $\xi = x-[x-x_0]$, for $m = 2, \ldots, n$.

By (2) and the inequality $|\ln(1+z)| \leqslant 2|z|$, true for $|z| \leqslant 1/2$, we get $|\ln g_1(x+s)| \leqslant 2\sum_{v=s+1}^{\infty}|b_{11}(x+v)-1|$ for $x_0 \leqslant x \leqslant x_0+1$ and sufficiently large s.

We obtain from this that $g_1(x+s)\to 1$ as $s\to\infty$ uniformly for $x_0\leqslant x\leqslant x_0+$ $+1$ and consequently $\lim_{x\to\infty} g_1(x) = 1$.

By (3a) and (3b) we have the following three cases for $2\leqslant m\leqslant n$:

(13) In case (3a) we have $\lim_{s\to\infty}|g_m(\xi+s)| = \prod_{v=0}^{\infty}|b_{mm}(\xi+v)| = G_m(\xi)$ uniformly for $x_0\leqslant\xi\leqslant x_0+1$, where $0<a_1\leqslant G_m(\xi)\leqslant a_2$ with some constants a_1 and a_2.

(14) In case (3b) and $|b_{mm}x)|(>1$ we have $|g_m(\xi+s)|\uparrow\infty$ a ss$\to\infty$, uniformly in $\langle x_0, x_0+1\rangle$ (by the inequality $|g_m(x)|\geqslant\sum_{v=0}^{[x-x_0]}\big(|b_{mm}(\xi+v)|-1\big)$ and $\lim_{x\to\infty}|g_m(x)| = \infty$.

(15) In case (3b) and $|b_{mm}(x)|<1$ we have $|g_m(\xi+s)|\downarrow 0$ as $s\to\infty$, uniformly in $\langle x_0, x_0+1\rangle$ (by the inequality $\ln x\leqslant x-1$ $(x>0)$) and $\lim_{x\to\infty} g_m(x) = 0$.

Let us notice that in case (13), i.e., (3a), the function $g_m(x)$ is bounded for $x\geqslant x_0$. Namely, there exists an $N\geqslant 0$ such that $\sum_{v=N+1}^{\infty}\big|\ln|b_{mm}(\xi+v)|\big|$ $\leqslant 1$ for $x_0\leqslant\xi\leqslant x_0+1$. Then

$$\big|\ln|g_m(x)|\big|\leqslant\sum_{v=0}^{\infty}\big|\ln|b_{mm}(\xi+v)|\big|\leqslant\sum_{v=0}^{N}\big|\ln|b_{mm}(\xi+v)|\big|+1.$$

Moreover, we obtain that $\overline{\lim_{x\to\infty}}\big|\ln|g_m(x)|\big| = \sup_{\langle x_0,x_0+1\rangle}|\ln G_m(\xi)|<\infty$.

In the m-th equation in (4) we represent the function $u_m(x)$, $1\leqslant m\leqslant n$, in the form $u_m(x) = c_m(x)/g_m(x)$, where the function $c_m(x)$ may be evaluated. We obtain from (4) for $x\geqslant x_0+1$

$$c_m(x-1)/g_m(x-1) = b_{mm}(x)c_m(x)/g_m(x)+\sum_{\substack{j=1\\ j\neq m}}^{n} b_{mj}(x)u_j(x),$$

$$c_m(x-1)-c_m(x) = g_m(x-1)\sum_{\substack{j=1\\ j\neq m}}^{n} b_{mj}(x)u_j(x).$$

In the formal way we obtain from this in cases $m = 1$, (13) and (15)

$$c_m(x) = \gamma_m(x)+\sum_{s=0}^{\infty} g_m(x+s)\sum_{\substack{j=1\\ j\neq m}}^{n} b_{mj}(x+s+1)u_j(x+s+1)\qquad (x\geqslant x_0)$$

and in case (14)

$$c_m(x) = \gamma_m(x) - \sum_{s=1}^{[x-x_0]} g_m(x-s) \sum_{\substack{j=1\\ j\neq m}}^{n} b_{mj}(x-s+1)u_j(x-s+1) \qquad (x \geqslant x_0+1),$$

where $\gamma_m(x)$ is some periodic function with the period $w = 1$. (We have $\gamma_m(x) = -\lim_{s\to\infty} c_m(x+s)$ in cases $m = 1$, (13) and (15), and $\gamma_m(x) = c_m(x-[x-x_0])$ in case (14).) The obtained result may be written in the following form:

$$(16) \qquad u_m(x) = \gamma_m(x)/g_m(x) + \sum_{j=1}^{n} J_{mj}^{[0]}(u_j), \qquad m = 1, \ldots, n,\ x \geqslant x_0+1,$$

where

(17)

$$J_{mj}^{[0]}(u) = \begin{cases} 0 & \text{if } m = j, \\ \big(1/g_m(x)\big) \sum_{s=0}^{\infty} g_m(x+s) b_{mj}(x+s+1) u(x+s+1) & \\ & \text{if } m \neq j, \text{ in cases } m = 1, \text{ (13) and (15)}, \\ -\big(1/g_m(x)\big) \sum_{s=1}^{[x-x_0]} g_m(x-s) b_{mj}(x-s+1) u(x-s+1) & \\ & \text{if } m \neq j, \text{ in case (14)}. \end{cases}$$

It is easy to show that in cases $m = 1$, (13) and (15) the series $\sum_{s=0}^{\infty} g_m(\xi+s) b_{mj}(\xi+s+1)$ are uniformly convergent for $x_0 \leqslant \xi \leqslant x_0+1$, $j = 1, \ldots, n$ and $j \neq m$. For $m = 1$ and in case (13) this follows from hypothesis (3a). In case (15) there follows, for a given $\varepsilon > 0$, the existence of the point $x_1 \geqslant x_0+1$ such that we have $|g_m(x)| \leqslant \varepsilon$ and, by (3b), $|b_{mj}(x)| \leqslant 1-|b_{mm}(x)|$ for $x \geqslant x_1$. We obtain for $x \geqslant x_1$

$$\sum_{s=0}^{\infty} |g_m(x+s) b_{mj}(x+s+1)| \leqslant \sum_{s=0}^{\infty} |g_m(x+s)| \big(1-|b_{mm}(x+s+1)|\big)$$
$$= -\sum_{s=0}^{\infty} \Delta |g_m(x+s)| = |g_m(x)| \leqslant \varepsilon.$$

Now, we shall prove that

$$(18) \qquad \lim_{x\to\infty} J_{mj}^{[0]}(1) = 0 \qquad \text{for } m, j = 1, \ldots, n.$$

This is easy to see in cases $m = 1$ and (13). In cases (14) and (15), applying theorem of Cauchy (for $g(x) = |g_m(x)|$ and $K = 1$), we get

$$\lim_{x\to\infty} |J_{mj}^{[0]}(1)| = \lim_{x\to\infty} |g_m(x-1) b_{mj}(x)| / \big(|g_m(x-1)| - |g_m(x)|\big)$$
$$= \lim_{x\to\infty} |b_{mj}(x)| / \big(1-|b_{mm}(x)|\big) = 0,$$

by hypothesis (3b).

There exists $x_2 \geqslant x_0+1$ such that $|g_1(x)| \geqslant 1/2$ and $\left|J_{mj}^{[0]}(1/g_1(x))\right| \leqslant 1/2n$ for $m, j = 1, \ldots, n$ and $x \geqslant x_2$, by (18).

We set $J_{mj}^{[k]}(u) = \sum_{s=1}^{n} J_{ms}^{[0]}\left(J_{sj}^{[k-1]}(u)\right)$ for $k = 1, 2, \ldots$ and $x \geqslant x_0+1$. It is easy to prove by induction that $\left|J_{m1}^{[k]}\left(1/g_1(x)\right)\right| \leqslant 2^{-k}$ for $m = 1, \ldots, n$, $k = 1, 2, \ldots$, and $x \geqslant x_2$. Setting for $x \geqslant x_2$

$$\bar{u}_1(x) = 1/g_1(x) + \sum_{k=1}^{\infty} J_{11}^{[k]}\left(1/g_1(x)\right),$$

$$\bar{u}_m(x) = \sum_{k=0}^{\infty} J_{m1}^{[k]}\left(1/g_1(x)\right) \quad \text{for } m = 2, \ldots, n,$$

we obtain that $|\bar{u}_1(x)| \leqslant 3$ and $|\bar{u}_m(x)| \leqslant 2$ for $m = 2, \ldots, n$ and $x \geqslant x_2$. The functions $u_m(x) = \bar{u}_m(x)$ satisfy for $x \geqslant x_2$ the system of equations (16) for $\gamma_1(x) = 1$ and $\gamma_m(x) = 0$, $m = 2, \ldots, n$. Namely we have for $m = 1$

$$\begin{aligned} 1/g_1(x) + \sum_{j=2}^{n} J_{1j}^{[0]}(\bar{u}_j) &= 1/g_1(x) + \sum_{j=2}^{n} J_{1j}^{[0]}\Big(\sum_{k=0}^{\infty} J_{j1}^{[k]}\left(1/g_1(x)\right)\Big) \\ &= 1/g_1(x) + \sum_{k=0}^{\infty}\sum_{j=2}^{n} J_{1j}^{[0]}\left(J_{j1}^{[k]}\left(1/g_1(x)\right)\right) \\ &= 1/g_1(x) + \sum_{k=0}^{\infty} J_{11}^{[k+1]}\left(1/g_1(x)\right) = \bar{u}_1(x) \end{aligned}$$

and similarly for $m = 2, \ldots, n$.

By (18) we obtain $\lim_{x\to\infty} J_{mj}^{[0]}(\bar{u}_j) = 0$ for $m, j = 1, \ldots, n$, since the functions $\bar{u}_j(x)$ are bounded for $x \geqslant x_2$. From (16) for $\gamma_1(x) = 1$ and $\gamma_m(x) = 0$ $(m = 2, \ldots, n)$ we get $\lim_{x\to\infty} \bar{u}_1(x) = 1$ and $\lim_{x\to\infty} \bar{u}_m(x) = 0$ for $m = 2, \ldots, n$. By (4) the functions $\bar{u}_m(x)$, $m = 1, \ldots, n$, may be extended to the point x_0.

LEMMA 1. *If the function $a_0(x)$ satisfies assumptions* (5)–(7) *of Theorem* 2, *then the equation*

$$(19) \qquad \psi(x)\psi(x-1)\ldots\psi(x-n+1) = a_0(x) \qquad (n \geqslant 2 \text{ and } x > n)$$

has n solutions $\psi(x) = \psi_k(x)$, $k = 1, \ldots, n$, such that $\psi_k(x) \sim \varepsilon_k a_0^{1/n}(x)$ as $x \to \infty$ $(\varepsilon_k = e^{2k\pi i/n})$. Moreover, we have $\psi_k(x)/\psi_k(x-1) \geqslant 1$ for $x > 2$ or $\psi_k(x)/\psi_k(x-1) \leqslant 1$ for these x.

Proof. From (19) we obtain $\psi(x) \neq 0$ for $x > 1$ and

$$\psi(x)/\psi(x-n) = a_0(x)/a_0(x-1) \qquad \text{for } x > n+1.$$

If $\sigma(x) = \psi(x)/\psi(x-1)$ for $x > 2$, then

$$(20) \qquad \sigma(x)\sigma(x-1)\ldots\sigma(x-n+1) = a_0(x)/a_0(x-1) \qquad \text{for } x > n+1.$$

Setting $\ln\sigma(x) = \tau(x)$ for $x > 2$ and $\ln\big(a_0(x)/a_0(x-1)\big) = f(x)$ for $x > n+1$, we obtain from (20) the equation

$$\tau(x)+\tau(x-1)+\ldots+\tau(x-n+1) = f(x) \quad \text{for } x > n+1, \tag{21}$$

where by assumption the function $f(x)$ tends monotonically to 0 as $x\to\infty$. Then $\tau(x-n)-\tau(x) = f(x-1)-f(x)$ for $x > n+2$, and

$$\begin{aligned}\tau(x) = \tau_0(x) &= \sum_{s=1}^{\infty}\big\{\tau_0\big(x+(s-1)n\big)-\tau_0(x+sn)\big\} \\ &= \sum_{s=1}^{\infty}\{f(x+sn-1)-f(x+sn)\} \quad \text{for } x > 2.\end{aligned}$$

We obtain from this

$$\tau_0(x) = \lambda(x)f(x+n-1) \quad \text{for } x > 2, \tag{22}$$

where $0 \leqslant \lambda(x) \leqslant 1$. It follows that $\lim_{x\to\infty}\tau_0(x) = 0$ and $\operatorname{sgn}\tau_0(x) = \operatorname{sgn} f(x)$ for $x > 2$.

For $x > 2$ we set $\sigma_0(x) = \exp\tau_0(x)$ and

$$\psi_0(x) = \gamma(x)\prod_{s=0}^{[x]-3}\sigma_0(x-s) \quad \text{for } x \geqslant 3, \tag{23}$$

where $\gamma(x)$ is some periodic function with the period $w = 1$, which will be defined below. From (23) and (20) we obtain for $x \geqslant n+2$

$$\begin{aligned}&\psi_0(x)\psi_0(x-1)\ldots\psi_0(x-n+1) \\ &= \gamma^n(x)\prod_{s=0}^{[x]-3}\sigma_0(x-s)\prod_{s=0}^{[x]-4}\sigma_0(x-s-1)\ldots\prod_{s=0}^{[x]-n-2}\sigma_0(x-s-n+1) \\ &= \gamma^n(x)\prod_{s=3}^{n+1}\sigma_0^{n-s+2}(x-[x]+s)\prod_{s=0}^{[x]-n-2}a_0(x-s)/a_0(x-s-1) \\ &= \gamma^n(x)\prod_{s=3}^{n+1}\sigma_0^{n-s+2}(x-[x]+s)\,a_0(x)/a_0(x-[x]+n+1).\end{aligned}$$

Setting $\gamma^n(x) = a_0(x-[x]+n+1)\Big/\prod_{s=3}^{n+1}\sigma_0^{n-s+2}(x-[x]+s)$, we obtain that the function $\psi_0(x)$ satisfies for $x \geqslant n+2$ equation (19). We get from (23) $\psi_0(x)/\psi_0(x-1) = \sigma_0(x) = \exp\tau_0(x)\to 1$ as $x\to\infty$ and by (19)

$$\begin{aligned}a_0(x) = \psi_0(x)\psi_0(x-1)\ldots\psi_0(x-n+1) &= \psi_0^n(x)\{\psi_0(x-1)/\psi_0(x)\}\ldots \\ &\ldots\{\psi_0(x-n+1)/\psi_0(x)\} \sim \psi_0^n(x) \quad \text{as } x\to\infty.\end{aligned}$$

Then $|\psi_0(x)| \sim |a_0^{1/n}(x)|$ as $x\to\infty$. Instead of $a_0(x)$ the function $|\psi_0(x)|$ satisfies (19) for $|a_0(x)|$.

We set $\theta = 0$ if $a_0(x) > 0$ and $\theta = 1$ if $a_0(x) < 0$ for $x > n$. Then the functions $\psi_k(x) = e^{(2k+\theta)\pi i/n}|\psi_0(x)|$, $k = 1, \ldots, n$, satisfy (19) and they have the desired asymptotic properties. From (22) there follows that $\tau_0(x) \geqslant 0$ for $x > 2$ or $\tau_0(x) \leqslant 0$ for $x > 2$, and $\sigma_0(x) = \psi_0(x)/\psi_0(x-1) \geqslant 1$ or $\psi_0(x)/\psi_0(x-1) \leqslant 1$ for these x. The same equalities satisfy the functions $\psi_k(x)$, $k = 1, \ldots, n$.

LEMMA 2. *Suppose the functions $F_v(x)$ are defined and different from 0 for $x \geqslant x_0$, $\sum_{s=0}^{\infty} |\Delta F_v(\xi+s)/F_v(\xi+s)|^2 < \infty$ for $v = 1, \ldots, p$, uniformly for $x_0 \leqslant \xi \leqslant x_0+1$. Furthermore, suppose the series $\sum_{s=0}^{\infty} |F(\xi+s)|$ is uniformly convergent in $\langle x_0, x_0+1\rangle$. If the function $\varphi(x)$ satisfies for $x \geqslant x_0$ the equality*

$$\Delta\varphi(x)/\varphi(x) = \sum_{v=1}^{p} r_v \Delta F_v(x)/F_v(x) + F(x) \tag{24}$$

and is defined and different from 0 in the interval $\langle x_0, x_0+1\rangle$, then we have $\varphi(x) \sim \gamma_1(x) \prod_{v=1}^{p} F_v^{r_v}(x)$ as $x \to \infty$. Here $\gamma_1(x)$ is some periodic function with the period $w = 1$, defined and different from 0 in the interval $\langle 0, 1)$.

Proof. Let us notice that from the inequality $|\ln(1+z)-z| \leqslant |z|^2$, true for $|z| \leqslant 1/2$, there follows that the convergence of the series $\sum_{s=0}^{\infty} |a_s|^2$ implies the convergence of the series $\sum_{s=0}^{\infty} |\ln(1+a_s)-a_s|$.

By hypothesis we obtain that $\sum_{s=0}^{\infty} |G_v(\xi+s)| < \infty$ for $v = 1, \ldots, p$, uniformly in $\langle x_0, x_0+1\rangle$, where

$$G_v(x) = \ln\big(1+\Delta F_v(x)/F_v(x)\big) - \Delta F_v(x)/F_v(x). \tag{25}$$

On the other hand,

$$\sum_{s=0}^{a-1} \ln\big(1+\Delta F_v(\xi+s)/F_v(\xi+s)\big) = \sum_{s=0}^{a-1} \ln\big(F_v(\xi+s+1)/F_v(\xi+s)\big)$$
$$= \ln F_v(x) - \ln F_v(\xi),$$

where $\xi = x-a$ and $a = [x-x_0]$. By (25) we obtain that

$$\sum_{s=0}^{a-1} \Delta F_v(\xi+s)/F_v(\xi+s) = \ln F_v(x) - \ln F_v(\xi) - \sum_{s=0}^{a-1} G_v(\xi+s). \tag{26}$$

By (24), the inequality $|ab| \leqslant \frac{1}{2}(a^2+b^2)$, and by hypothesis we get $\sum_{=0}^{\infty} |\Delta\varphi(\xi+s)/\varphi(\xi+s)|^2 < \infty$ uniformly in $\langle x_0, x_0+1\rangle$, and, as above,

$$\ln\varphi(x) - \ln\varphi(\xi) = \sum_{s=0}^{a-1} \Delta\varphi(\xi+s)/\varphi(\xi+s) + \sum_{s=0}^{a-1} \Phi(\xi+s), \tag{27}$$

the series $\sum_{s=0}^{\infty} |\Phi(\xi+s)|$ being uniformly convergent in $\langle x_0, x_0+1\rangle$. By (27), (24) and (26) we get

$$\ln\varphi(x)-\ln\varphi(\xi) = \sum_{s=0}^{a-1}\sum_{v=1}^{p} r_v \Delta F_v(\xi+s)/F_v(\xi+s)+\sum_{s=0}^{a-1}\big(F(\xi+s)+\Phi(\xi+s)\big)$$

$$= \sum_{v=1}^{p} r_v\big(\ln F_v(x)-\ln F_v(\xi)\big)-\sum_{v=1}^{p} r_v\sum_{s=0}^{a-1} G_v(\xi+s)+\sum_{s=0}^{a-1}\big(F(\xi+s)+\Phi(\xi+s)\big).$$

We complete the proof of Lemma 2 putting

$$\gamma_1(x) = K(\xi)\varphi(\xi)\prod_{v=1}^{p} F_v^{-r_v}(\xi),$$

where

$$K(\xi) = \exp\Big\{-\sum_{v=1}^{p} r_v\sum_{s=0}^{\infty} G_v(\xi+s)+\sum_{s=0}^{\infty}\big(F(\xi+s)+\Phi(\xi+s)\big)\Big\}.$$

LEMMA 3. *Suppose that the function $a_0(x)$ satisfies assumptions* (5)–(8) *of Theorem* 2 *and $\psi_k(x)$ is defined as in Lemma* 1. *We put for $x \geqslant n+1$*

$$A(x) = (1/n)\sum_{s=1}^{n} 1/\psi_k(x-s+1)$$

and

$$1/\varphi_1(x) = \prod_{s=0}^{[x]-n-1} \psi_k(x-s)A(x-s).$$

Then $\varphi_1(x) \sim \gamma_2(x)a_0^{(1-n)/2n}(x)$ as $x\to\infty$, where $\gamma_2(x)$ is some periodic function with the period $w = 1$, defined and different from 0 *in the interval* $\langle 0, 1)$.

Proof. Let us notice that if $\eta_v = \sum_{s=1}^{v}\lambda_s$ for $v = 1, \ldots, n-1$, and $\sum_{s=1}^{n}\lambda_s = 0$, then for every sequence $\{c_v\}$ we have the identity

$$\sum_{v=1}^{n}\lambda_v c_v = \sum_{v=1}^{n-1}\eta_v(c_v-c_{v+1}), \tag{28}$$

which may be obtained by the transformation of Abel.

From the definitions we get for $x \geqslant n+1$

$$\Delta\varphi_1(x)/\varphi_1(x) = 1/\psi_k(x+1)A(x+1)-1$$

$$= \big(1/nA(x+1)\big)\Big(n/\psi_k(x+1)-\sum_{v=1}^{n} 1/\psi_k(x-v+2)\Big)$$

$$= \big(1/nA(x+1)\big)\sum_{v=1}^{n}\lambda_v/\psi_k(x-v+2),$$

where $\lambda_1 = n-1$ and $\lambda_v = -1$ for $v = 2, \ldots, n$. Applying (28) we get

$$\Delta\varphi_1(x)/\varphi_1(x) = \big(1/nA(x+1)\big)\sum_{v=1}^{n-1}(n-v)\big(1/\psi_k(x-v+2)-1/\psi_k(x-v+1)\big)$$
$$= \big(-1/nA(x+1)\big)\sum_{v=1}^{n-1}(n-v)f_v(x)/\psi_k(x-v+2),$$

where $f_v(x) = \Delta\psi_k(x-v+1)/\psi_k(x-v+1)$. Setting $\lambda_s^* = 1/n$ for $s = 1, \ldots, n$ and $s \neq v$, $\lambda_v^* = 1/n-1$, we get for $x \geqslant n+1$

$$\Delta\varphi_1(x)/\varphi_1(x) = -\sum_{v=1}^{n-1}\big((n-v)/n\big)f_v(x)+F(x), \tag{29}$$

where

$$F(x) = \big(1/nA(x+1)\big)\sum_{v=1}^{n-1}(n-v)f_v(x)\Big(-1/\psi_k(x-v+2)+(1/n)\sum_{s=1}^{n}1/\psi_k(x-s+2)\Big)$$
$$= \big(1/nA(x+1)\big)\sum_{v=1}^{n-1}(n-v)f_v(x)\sum_{s=1}^{n}\lambda_s^*/\psi_k(x-s+2)$$
$$= \big(1/nA(x+1)\big)\sum_{v=1}^{n-1}(n-v)f_v(x)\sum_{s=1}^{n-1}\eta_s^*\big(1/\psi_k(x-s+2)-1/\psi_k(x-s+1)\big)$$
$$= -\sum_{v=1}^{n-1}\sum_{s=1}^{n-1}(n-v)\eta_s^* f_v(x)f_s(x)/nA(x+1)\psi_k(x-s+2),$$

where $\eta_s^* = \sum_{j=1}^{s}\lambda_j^*$.

By Lemma 1 and (19) we get for $v = 1, \ldots, n$

$$|\Delta\psi_k(x-v+1)| \leqslant |\Delta\psi_k(x)+\Delta\psi_k(x-1)+\ldots+\Delta\psi_k(x-n+1)|$$
$$= |\psi_k(x+1)-\psi_k(x-n+1)| = |\psi_k(x+1)\,\Delta a_0(x)/a_0(x+1)|.$$

By (6) there exists a constant $M > 0$ such that for large x and $v = 1, \ldots, n$ there is

$$|f_v(x)| = |\Delta\psi_k(x-v+1)/\psi_k(x-v+1)| \leqslant M|\Delta a_0(x)/a_0(x)|,$$

since $\psi_k(x+1)/\psi_k(x-v+1)\to 1$ as $x\to\infty$. We have also $\lim_{x\to\infty}A(x)\psi_k(x-s+1) = 1$. By (8) we obtain the series $\sum_{s=0}^{\infty}|f_v^2(x+s)|$ and $\sum_{s=0}^{\infty}|F(x+s)|$ are uniformly convergent for $n+1 \leqslant x \leqslant n+2$.

We have $\varphi_1(x) \neq 0$ for $x \geqslant n+1$. By (29) and Lemma 2 for $F_v(x) = \psi_k(x-v+1)$ and $r_v = (v-n)/n$ $(v = 1, \ldots, n-1)$ we obtain

$$\varphi_1(x) \sim \gamma_1(x)\prod_{v=1}^{n-1}\psi_k^{(v-n)/n}(x-v+1) \sim \gamma_1(x)\psi_k^{(1-n)/2}(x) \sim \gamma_2(x)a_0^{(1-n)/2n}(x)$$

as $x\to\infty$.

LEMMA 4. *Suppose $a_0(x)$ satisfies assumptions* (5)–(8) *and* (12) *of Theorem* 2, *and $A(x)$ is defined as in Lemma* 3. *We set*

$$P(x) = 1 \Big/ \prod_{s=0}^{[x]-n-1} \big(1-1/A(x-s)\big) \quad \textit{for } x \geqslant n+1$$

and

$$P_k(x) = 1 \Big/ \prod_{s=0}^{[x]-n} \big(1-\varepsilon_k a_0^{1/n}(x-s)\big) \quad \textit{for } x \geqslant n,$$

cf. (11). *Then $P_k(x) \sim \gamma_3(x)P(x)$ as $x\to\infty$, where $\gamma_3(x)$ is some periodic function with the period $w = 1$, defined and different from* 0 *in the interval* $\langle 0, 1)$.

Proof. Since $A(x) \sim 1/\psi_k(x) \sim \varepsilon_k^{-1}a_0^{-1/n}(x)$ as $x\to\infty$ and $\lim\limits_{x\to\infty} a_0(x) \neq 1$ we may assume that $A(x) \neq 1$ for $x \geqslant n+1$.

We set for $x \geqslant n+1$

$$\begin{aligned} P_k(x)/P(x) &= \prod_{v=0}^{[x]-n-1} \big(1+R(x-v)\big)/\big(1-\varepsilon_k a_0^{1/n}(x-[x]+n)\big) \\ &= \prod_{s=n+1}^{[x]} \big(1+R(x-[x]+s)\big)/\big(1-\varepsilon_k a_0^{1/n}(x-[x]+n)\big). \end{aligned}$$

Then

$$\begin{aligned} R(x) &= \big(1-1/A(x)\big)/\big(1-\varepsilon_k a_0^{1/n}(x)\big)-1 \\ &= \big(\varepsilon_k a_0^{1/n}(x)-1/A(x)\big)/\big(1-\varepsilon_k a_0^{1/n}(x)\big) \\ &= \big(A(x)-\varepsilon_k^{-1}a_0^{-1/n}(x)\big)/A(x)\big(\varepsilon_k^{-1}a_0^{-1/n}(x)-1\big). \end{aligned}$$

We shall prove that the series $\sum\limits_{s=0}^{\infty} |R(x+s)|$ is uniformly convergent in the interval $\langle n+1, n+2\rangle$. We consider the functions $f_1(x_1, \ldots, x_n) = \sum\limits_{v=1}^{n} x_v^n/n$ and $f_2(x_1, \ldots, x_n) = x_1\ldots x_n$. We have

$$\begin{aligned} f_1(a+h_1, \ldots, a+h_n) &= f_1(a, \ldots, a)+\sum_{v=1}^{n} h_v \frac{\partial}{\partial x_v} f_1(a, \ldots, a)+ \\ &\quad +\frac{1}{2}\sum_{v=1}^{n}\sum_{s=1}^{n} h_v h_s \frac{\partial^2}{\partial x_v \partial x_s} f_1(a+\theta h_1, \ldots, a+\theta h_n) \\ &= a^n+a^{n-1}\sum_{v=1}^{n} h_v+\big((n-1)/2\big)\sum_{v=1}^{n} h_v^2(a+\theta h_v)^{n-2}, \end{aligned}$$

where $0<\theta<1$, and similarly

$$f_2(a+h_1, \ldots, a+h_n) = a^n + a^{n-1} \sum_{v=1}^{n} h_v + \tfrac{1}{2}(a+\theta_1 h_1) \ldots (a+\theta_1 h_n) \times$$
$$\times \sum_{v=1}^{n} \sum_{\substack{s=1 \\ s\neq v}}^{n} h_v h_s / (a+\theta_1 h_v)(a+\theta_1 h_s),$$

where $0 < \theta_1 < 1$. Then

$$(30) \qquad f_1(a+h_1, \ldots, a+h_n) - f_2(a+h_1, \ldots, a+h_n)$$
$$= \big((n-1)/2\big) \sum_{v=1}^{n} h_v^2 (a+\theta h_v)^{n-2} -$$
$$- \tfrac{1}{2}(a+\theta_1 h_1) \ldots (a+\theta_1 h_n) \sum_{v=1}^{n} \sum_{\substack{s=1 \\ s\neq v}}^{n} h_v h_s / (a+\theta_1 h_v)(a+\theta_1 h_s).$$

Setting $a = |\psi_k^{-1/n}(x)|$ and $h_v = |\psi_k^{-1/n}(x-v+1)| - |\psi_k^{-1/n}(x)|$ for $v = 1, \ldots, n$, by (19) we get

$$|A(x) - \varepsilon_k^{-1} a_0^{-1/n}(x)| = f_1(a+h_1, \ldots, a+h_n) - f_2(a+h_1, \ldots, a+h_n).$$

We have $|a+\theta h_v| \leqslant |\psi_k^{-1/n}(x)|$, $v = 1, \ldots, n$, in the case $|\psi_k(x)| \leqslant |\psi_k(x-1)|$, and $|a+\theta h_v| \leqslant |\psi_k^{-1/n}(x-v+1)| \leqslant |\psi_k^{-1/n}(x-n)|$ in the case $|\psi_k(x)| \geqslant |\psi_k(x-1)|$, cf. Lemma 1. Moreover, $|h_v| \leqslant |\psi_k^{-1/n}(x-n) - \psi_k^{-1/n}(x)|$ for $v = 1, \ldots, n$ and $x > n+1$. By (30) we get

$$|A(x) - \varepsilon_k^{-1} a_0^{-1/n}(x)| \leqslant n(n-1) |\psi_k^{(2-n)/n}(x)| \cdot |\psi_k^{-1/n}(x-n) - \psi_k^{-1/n}(x)|^2$$

in the case $|\psi_k(x)| \leqslant |\psi_k(x-1)|$ and

$$|A(x) - \varepsilon_k^{-1} a_0^{-1/n}(x)| \leqslant n(n-1) |\psi_k^{(2-n)/n}(x-n)| \cdot |\psi_k^{-1/n}(x-n) - \psi_k^{-1/n}(x)|^2$$

in the case $|\psi_k(x)| \geqslant |\psi_k(x-1)|$.

We obtain

$$\psi_k^{-1/n}(x-n) - \psi_k^{-1/n}(x) = \big(\psi_k^{-1}(x-n) - \psi_k^{-1}(x)\big) / \big(\psi_k^{(1-n)/n}(x-n) +$$
$$+ \psi_k^{(2-n)/n}(x-n)\psi_k^{-1/n}(x) + \ldots + \psi_k^{(1-n)/n}(x)\big) \sim \big(\psi_k^{-1}(x-n) -$$
$$- \psi_k^{-1}(x)\big) / n\psi_k^{(1-n)/n}(x) \sim \big(\psi_k(x) - \psi_k(x-n)\big) / n\psi_k^{(n+1)/n}(x)$$

as $x \to \infty$. By (19) we get

$$\psi_k(x) - \psi_k(x-n) = \psi_k(x)\,\Delta a_0(x-1)/a_0(x)$$

and

$$|\psi_k(x) - \psi_k(x-n)| \sim |a_0^{(1-n)/n}(x)\,\Delta a_0(x-1)| \quad \text{as } x \to \infty.$$

We deduce from this that

$$|\psi_k^{-1/n}(x-n) - \psi_k^{-1/n}(x)| \sim |a_0^{(1-n)/n}(x)\,\Delta a_0(x-1)/n\psi_k^{(n+1)/n}(x)|$$
$$\sim (1/n)|a_0^{-1-1/n^2}(x)\,\Delta a_0(x-1)| \quad \text{as } x \to \infty.$$

There exists $\varepsilon > 0$ such that we have $|\varepsilon_k^{-1} a_0^{-1/n}(x) - 1| \geqslant \varepsilon$ for sufficiently large x. (In the case $k = n$ this follows from (12).) We obtain for large x

$$|R(x)| \leqslant |A(x) - \varepsilon_k^{-1} a_0^{-1/n}(x)|/\varepsilon |A(x)| \leqslant \big(n(n-1)/\varepsilon\big) M(x),$$

where

$$\begin{aligned} M(x) &= |\psi_k^{(2-n)/n}(x)| \cdot |\psi_k^{-1/n}(x-n) - \psi_k^{-1/n}(x)|^2 / |A(x)| \\ &\sim n^{-2} |a_0^{(2-n)/n^2}(x)/A(x)| \cdot |a_0^{-2-2/n^2}(x)| \cdot |\Delta a_0(x-1)|^2 \\ &\sim n^{-2} a_0^{-2}(x-1)\big(\Delta a_0(x-1)\big)^2 \quad \text{as } x \to \infty, \end{aligned}$$

in the case $|\psi_k(x)| \leqslant |\psi_k(x-1)|$, and similarly in the case $|\psi_k(x)| \geqslant |\psi_k(x-1)|$.

By (8) the series $\sum_{s=0}^{\infty} M(x+s)$ and $\sum_{s=0}^{\infty} |R(x+s)|$ are uniformly convergent in the interval $\langle n+1, n+2\rangle$. We complete the proof of Lemma 4 setting

$$\begin{aligned} \gamma_3(x) &= \lim_{x\to\infty} \prod_{v=0}^{[x]-n-1} \big(1 + R(x-v)\big) / \big(1 - \varepsilon_k a_0^{1/n}(x - [x] + n)\big) \\ &= \prod_{s=n+1}^{\infty} \big(1 + R(x - [x] + s)\big) / \big(1 - \varepsilon_k a_0^{1/n}(x - [x] + n)\big), \end{aligned}$$

since

$$\prod_{s=[x]+1}^{\infty} \big(1 + R(x - [x] + s)\big) = \prod_{s=1}^{\infty} \big(1 + R(x+s)\big) \to 1 \quad \text{as } x \to \infty.$$

Proof of Theorem 2. We write the difference equation (1) in the form of the following system of equations

$$(31) \qquad \begin{aligned} \Delta z_v(x-1) &= z_{v+1}(x) \quad \text{for } v = 1, \ldots, n-1, \\ \Delta z_n(x-1) &= \sum_{j=0}^{n-1} a_j(x) z_{j+1}(x), \end{aligned}$$

where $z_v(x) = \Delta^{v-1} y(x - v + 1)$ for $v = 1, \ldots, n$.

For a given index k ($1 \leqslant k \leqslant n$) we define the function $\psi_k(x)$ as in Lemma 1. Moreover, for $x \geqslant n+1$ we put (cf. Lemmas 3 and 4):

$$\begin{aligned} A(x) &= (1/n) \sum_{s=1}^{n} 1/\psi_k(x - s + 1), \\ 1/\varphi_1(x) &= \prod_{s=0}^{[x]-n-1} \psi_k(x-s) A(x-s), \\ 1/P(x) &= \prod_{s=0}^{[x]-n-1} \big(1 - 1/A(x-s)\big), \\ \varphi_v(x) &= \varphi_1(x) \prod_{s=0}^{v-2} \psi_k(x-s) \quad \text{for } v = 2, \ldots, n. \end{aligned}$$

We substitute into (31)

$$z_v(x) = \varphi_v(x)P(x)w_v(x) \quad \text{for } v = 1, \ldots, n \text{ and } x \geqslant n+1,$$

and we obtain the system

$$(32)\quad \begin{aligned} &\varphi_v(x)P(x)w_v(x) - \varphi_v(x-1)P(x-1)w_v(x-1) = \varphi_{v+1}(x)P(x)w_{v+1}(x) \\ &\qquad\qquad (v = 1, \ldots, n-1), \\ &\varphi_n(x)P(x)w_n(x) - \varphi_n(x-1)P(x-1)w_n(x-1) \\ &\qquad = P(x)\sum_{j=0}^{n-1} a_j(x)\varphi_{j+1}(x)w_{j+1}(x). \end{aligned}$$

Dividing the v-th equation in (32) by $\varphi_{v+1}(x)P(x)$ $(v = 1, \ldots, n-1)$ and the n-th equation by $a_0(x)\varphi_1(x)P(x)$ we get

$$(33)\quad \begin{aligned} &\frac{\varphi_v(x-1)P(x-1)}{\varphi_{v+1}(x)P(x)} w_v(x-1) = \frac{1}{\psi_k(x-v+1)} w_v(x) - w_{v+1}(x) \\ &\qquad\qquad (v = 1, \ldots, n-1), \\ &\frac{\varphi_n(x-1)P(x-1)}{a_0(x)\varphi_1(x)P(x)} w_n(x-1) \\ &\qquad = \frac{1}{\psi_k(x-n+1)} w_n(x) - \sum_{j=0}^{n-1} \frac{a_j(x)\varphi_{j+1}(x)}{a_0(x)\varphi_1(x)} w_{j+1}(x). \end{aligned}$$

Since $\varphi_v(x-1)/\varphi_{v+1}(x) = \varphi_n(x-1)/a_0(x)\varphi_1(x) = A(x)$ for $v = 1, \ldots$ $\ldots, n-1$, we obtain from (33)

$$(34)\quad \begin{aligned} &\big(A(x)-1\big)w_v(x-1) = \big(1/\psi_k(x-v+1)\big)w_v(x) - w_{v+1}(x) \\ &\qquad\qquad (v = 1, \ldots, n-1), \\ &\big(A(x)-1\big)w_n(x-1) = -w_1(x) + \big(1/\psi_k(x-n+1)\big)w_n(x) - \\ &\qquad - \sum_{j=1}^{n-1} \Big(a_j(x) \Big/ \prod_{s=j}^{n-1} \psi_k(x-s)\Big) w_{j+1}(x). \end{aligned}$$

For a given index m $(1 \leqslant m \leqslant n)$ we multiply the v-th equation in (34) by ε_{m-1}^{v-1} for $v = 1, \ldots, n-1$, and the last equation in (34) by ε_{m-1}^{n-1}. Adding the obtained n equations and setting $\sum_{v=1}^{n} \varepsilon_{m-1}^{v-1} w_v(x) = nu_m(x)$ we get

$$\begin{aligned} n\big(A(x)-1\big)u_m(x-1) &= \sum_{v=1}^{n} \big(\varepsilon_{m-1}^{v-1}/\psi_k(x-v+1)\big)w_v(x) - \sum_{v=0}^{n-1} \varepsilon_{m-1}^{v-1} w_{v+1}(x) - \\ &\quad - \varepsilon_{m-1}^{n-1} \sum_{v=1}^{n-1} \Big(a_v(x) \Big/ \prod_{s=v}^{n-1} \psi\ (x-s)\Big) w_{v+1}(x) \\ = \sum_{v=1}^{n} \big(\varepsilon_{m-1}^{v-1}/\psi_k(x-v+1) - \varepsilon_{m-1}^{v-2}\big) w_v(x) &- \varepsilon_{m-1}^{n-1} \sum_{v=1}^{n-1} \Big(a_v(x) \Big/ \prod_{s=v}^{n-1} \psi_k(x-s)\Big) w_{v+1}(x) \\ &= S_1 - S_2. \end{aligned}$$

Since $w_v(x) = \sum_{j=1}^{n} \varepsilon_{v-1}^{n-j+1} u_j(x)$ we get

$$S_1 = \sum_{v=1}^{n} \left(\varepsilon_{m-1}^{v-1}/\psi_k(x-v+1) - \varepsilon_{m-1}^{v-2}\right) \sum_{j=1}^{n} \varepsilon_{v-1}^{n-j+1} u_j(x)$$

$$= \sum_{j=1}^{n} u_j(x) \sum_{v=1}^{n} \left(\varepsilon_{m-1}^{v-1}/\psi_k(x-v+1) - \varepsilon_{m-1}^{v-2}\right) \varepsilon_{v-1}^{n-j+1}$$

$$= \sum_{j=1}^{n} u_j(x) \sum_{v=1}^{n} \varepsilon_{v-1}^{n+m-j} \left(1/\psi_k(x-v+1) - \varepsilon_{m-1}^{n-1}\right)$$

$$= \sum_{\substack{j=1\\ j\neq m}}^{n} u_j(x) \sum_{v=1}^{n} \varepsilon_{v-1}^{m-j}/\psi_k(x-v+1) + n\left(A(x) - \varepsilon_{m-1}^{n-1}\right) u_m(x),$$

$$S_2 = \varepsilon_{m-1}^{n-1} \sum_{v=1}^{n-1} \left(a_v(x) \Big/ \prod_{s=v}^{n-1} \psi_k(x-s)\right) \sum_{j=1}^{n} \varepsilon_v^{n-j+1} u_j(x)$$

$$= \varepsilon_{m-1}^{n-1} \sum_{j=1}^{n} u_j(x) \sum_{v=1}^{n-1} \varepsilon_v^{n-j+1} a_v(x) \Big/ \prod_{s=v}^{n-1} \psi_k(x-s) = \sum_{j=1}^{n} \varphi_{mj}(x) u_j(x),$$

where $\varphi_{mj}(x) = \varepsilon_{m-1}^{n-1} \sum_{v=1}^{n-1} \varepsilon_v^{n-j+1} a_v(x) / \sum_{s=v}^{n-1} \psi_k(x-s)$.

By Lemma 1 and (6) we obtain $A(x) \sim \varepsilon_k^{-1} a_0^{-1/n}(x)$ as $x \to \infty$. Then there exist $x_0 \geqslant n+1$ and $\varepsilon > 0$ such that $|A(x)-1| \geqslant \varepsilon$ for $x \geqslant x_0$. (In the case $k = n$ we apply (12).) Moreover, we have $A(x) \neq 0$ for $x \geqslant n+1$.

We obtain the system of equations

$$u_m(x-1) = \sum_{j=1}^{n} b_{mj}(x) u_j(x), \qquad m = 1, \ldots, n \text{ and } x \geqslant x_0,$$

where

(35) $b_{mj}(x)$

$$= \begin{cases} \sum_{v=1}^{n} \varepsilon_{v-1}^{m-j}/\psi_k(x-v+1) n\left(A(x)-1\right) - \varphi_{mj}(x)/n\left(A(x)-1\right) & \text{if } m \neq j, \\ \left(A(x) - \varepsilon_{m-1}^{n-1}\right)/\left(A(x)-1\right) - \varphi_{mm}(x)/n\left(A(x)-1\right) & \text{if } m = j. \end{cases}$$

We shall prove that the functions $b_{mj}(x)$ defined by (35) satisfy hypotheses of Theorem 1. By (10) and Lemma 1 there follows that $\sum_{s=0}^{\infty} |\varphi_{mj}(x+s)| < \infty$, uniformly for $n+1 \leqslant x \leqslant n+2$ and $m, j = 1, \ldots, n$. Then hypothesis (2) is satisfied. From (35) and (28) we obtain for $m, j = 1, \ldots, n$, $m \neq j$, $c_v = 1/\psi_k(x-v+1)$ and $x \geqslant x_0$ that

$$b_{mj}(x) = \sum_{v=1}^{n-1} \eta_v \left(1/\psi_k(x-v+1) - 1/\psi_k(x-v)\right)/n\left(A(x)-1\right) - \varphi_{mj}(x)/n\left(A(x)-1\right),$$

$$\text{where } \eta_v = \sum_{s=1}^{v} \varepsilon_{s-1}^{m-j}.$$

By (7) we obtain that

$$\sum_{s=0}^{\infty} |1/\psi_k(x+s)-1/\psi_k(x+s+1)| < \infty \quad \text{uniformly for } n+1 \leqslant x \leqslant n+2.$$

We infer from this that second hypothesis in (3a) is satisfied for $m, j = 1, \ldots, n$ and $j \neq m$.

We have for $m = 1, \ldots, n$

$$\begin{aligned}
&|b_{mm}(x)|-1 \\
&= \left|\frac{A(x)+\varphi_{mm}(x)/n-\varepsilon_{m-1}^{n-1}}{A(x)-1}\right|-1 = \frac{\big||A(x)|+\varphi_m(x)-a_m\big|-\big||A(x)|-\varepsilon_k\big|}{\big||A(x)|-\varepsilon_k\big|} \\
&= \left(\big||A(x)|+\varphi_m(x)-a_m\big|^2-\big||A(x)|-\varepsilon_k\big|^2\right)/R(x) \\
&= \left(2|A(x)|\left(\mathrm{re}\,\varphi_m(x)+\mathrm{re}\,\varepsilon_k-\mathrm{re}\,a_m\right)+|\varphi_m(x)|^2-2\,\mathrm{re}\left(\varphi_m(x)/a_m\right)\right)/R(x),
\end{aligned}$$

where $a_m = \varepsilon_{m-1}^{n-1}\varepsilon_k$, $\varphi_m(x) = \varepsilon_k\varphi_{mm}(x)/n$ and

$$R(x) = \big||A(x)|-\varepsilon_k\big|\left\{\big||A(x)|+\varphi_m(x)-a_m\big|+\big||A(x)-\varepsilon_k\big|\right\}.$$

By (9) we obtain $\varphi_{mj}(x) = o\big(A(x)\big)$ and $\varphi_{mj}(x) = o(1)$ as $x \to \infty$ $(m, j = 1, \ldots, n)$, since by (7) the function $A(x)$ remains bounded for $x \to \infty$. Moreover, by (7) and (12) we get $\lim\limits_{x\to\infty} R(x) = c$, where $0 < c < \infty$.

Let us suppose that for some m we have $\mathrm{re}\,a_m \neq \mathrm{re}\,\varepsilon_k$. Then we get $|b_{mm}(x)|-1 \sim 2|A(x)|\;(\mathrm{re}\,\varepsilon_k-\mathrm{re}\,a_m)/R(x) \sim \lambda|A(x)| \sim \lambda|a_0^{-1/n}(x)|$ as $x \to \infty$, with some constant λ, and the difference $|b_{mm}(x)|-1$ has a constant sign for sufficiently large x. Moreover,

$$\begin{aligned}
\lim_{x\to\infty} b_{mj}(x)/\big(|b_{mm}(x)|-1\big) &= \lim_{x\to\infty} \frac{\sum\limits_{v-1}^{n}\left(\varepsilon_{v-1}^{m-j}/\psi_k(x-v+1)\right)R(x)}{2n\big(A(x)-1\big)|A(x)|(\mathrm{re}\,\varepsilon_k-\mathrm{re}\,a_n)} \\
&= \lim_{x\to\infty} \frac{\sum\limits_{v=1}^{n}\varepsilon_{v-1}^{m-j}\cdot c}{2n\big(A(x)-1\big)(\mathrm{re}\,\varepsilon_k-\mathrm{re}\,a_m)} = 0
\end{aligned}$$

for $j = 1, \ldots, n$ and $j \neq m$, since $\varphi_{mj}(x) = o\big(1/\psi_k(x)\big)$ as $x \to \infty$ and $\lim\limits_{x\to\infty} A(x)$ $\psi_k(x-v+1) = 1$.

By (7) we obtain the series $\sum\limits_{s=0}^{\infty}\big||b_{mm}(x+s)|-1\big|$ is uniformly convergent or uniformly divergent to ∞ for $x_0 \leqslant x \leqslant x_0+1$. If for the index considered above m we have $\sum\limits_{s=0}^{\infty}\big||b_{mm}(x+s)|-1\big| = \infty$ uniformly for $x_0 \leqslant x \leqslant x_0+1$, then hypothesis (3b) is satisfied. (If the series $\sum\limits_{s=0}^{\infty}\big|\,|b_{mm}(x+s)|-1\big|$ is uniformly convergent for $x_0 \leqslant x \leqslant x_0+1$, then there is satisfied hypothesis (3a).)

In the case $\operatorname{re}\alpha_m = \operatorname{re}\varepsilon_k$ we get $|b_{mm}(x)|-1 \sim \{2|A(x)|\operatorname{re}\varphi_m(x)-2\operatorname{re}(\varphi_m(x)/\alpha_m)\}/R(x)$ and we obtain $\sum_{s=0}^{\infty}\big||b_{mm}(x+s)|-1\big| < \infty$ uniformly for $x_0 \leqslant x \leqslant x_0+1$, since the series $\sum_{s=0}^{\infty}|\varphi_m(x+s)|$ is uniformly convergent in this interval. In this case hypothesis (3a) is satisfied.

Applying Theorem 1 we obtain that the system of equations (4) with coefficients $b_{mj}(x)$ defined by (35) has for sufficiently large x a solution $\overline{u}_1(x), \ldots, \overline{u}_n(x)$ such that $\lim_{x\to\infty}\overline{u}_1(x) = 1$ and $\lim_{x\to\infty}\overline{u}_m(x) = 0$ for $m = 2, \ldots, n$. The solution $\{\overline{u}_m(x)\}$ may be extended to the point $x = x_0$.

Setting $\overline{w}_m(x) = \sum_{v=1}^{n}\varepsilon_{m-1}^{n-v+1}\overline{u}_v(x)$, $m = 1, \ldots, n$, we obtain that the functions $\overline{w}_m(x)$ satisfy for $x \geqslant x_0$ the system of equations (34) and $\lim_{x\to\infty}\overline{w}_m(x) = 1$ for $m = 1, \ldots, n$. Then the function

$$y_k(x) = \overline{w}_1(x)\varphi_1(x)P(x)\gamma_3(x)/\gamma_2(x) \qquad \text{(cf. Lemmas 3 and 4)}$$

satisfies the difference equation (1) and by Lemmas 3 and 4 satisfies the asymptotic relations (11). By (1) the solution $y_k(x)$ may be extended to the point $x = 0$.

Now, we shall prove indirectly that the functions $y_k(x)$, $k = 1, \ldots, n$, are linearly independent. Suppose that there exist constants c_k, $k = 1, \ldots, n$, such that $\sum_{k=1}^{n}|c_k| > 0$ and

$$\sum_{k=1}^{n} c_k y_k(x) = 0 \qquad \text{for } x > 0. \tag{36}$$

Setting

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1(x) & \ldots & y_n(x) \\ y_1(x-1) & \ldots & y_n(x-1) \\ \cdots & \cdots & \cdots \\ y_1(x-n+1) & \ldots & y_n(x-n+1) \end{vmatrix},$$

we obtain

$$W(y_1, \ldots, y_n) = C\begin{vmatrix} y_1(x) & \ldots & y_n(x) \\ \Delta y_1(x-1) & \ldots & \Delta y_n(x-1) \\ \cdots & \cdots & \cdots \\ \Delta^{n-1}y_1(x-n+1) & \ldots & \Delta^{n-1}y_n(x-n+1) \end{vmatrix},$$

where $C = (-1)\binom{n}{2}$, and

$$W(y_1, \ldots, y_n)/P_1(x)\ldots P_n(x) = C\begin{vmatrix} a_{11}(x) & \ldots & a_{1n}(x) \\ \cdots & \cdots & \cdots \\ a_{n1}(x) & \ldots & a_{nn}(x) \end{vmatrix},$$

where $a_{ik}(x) = \Delta^{i-1}y_k(x-i+1)a_0^{(n+1-2i)/2n}(x)/P_k(x)$. By (11) we obtain $\lim_{x\to\infty} a_{ik}(x) = \varepsilon_k^{i-1}$ for $i, k = 1, \ldots, n$. Since the determinant $|\varepsilon_k^{i-1}|$ is different from 0 and $P_k(x) \neq 0$ for $x \geqslant n$ and $k = 1, \ldots, n$, we infer that $W(y_1, \ldots, y_n) \neq 0$ for sufficiently large x, in contradiction with (36).

Finally, by (11) and (6) we get for $x\to\infty$, $m = 0, \ldots, n-1$ and $k = 1, \ldots, n$:

$$|\Delta^m y_k(x-m)/\Delta^m y_k(x-m-1)| \sim |y_k(x)/y_k(x-1)| \sim 1/|1-\varepsilon_k a_0^{1/n}(x)| \to \lambda_k.$$

If $\lim_{x\to\infty} a_0(x) = a$ and $0 < |a| < \infty$, then $\lambda_k = 1/|1-\alpha_k - i\beta_k| = \big((1-\alpha_k)^2 + \beta_k^2\big)^{-1/2}$, where $\varepsilon_k a^{1/n} = \alpha_k + i\beta_k$.

By (11) the functions $\Delta^m y_k(x)$ satisfy the inequality $0 < \eta \leqslant |\Delta^m y_k(x)| \leqslant M$ in the interval $\langle x_1, x_1+1\rangle$ for sufficiently large x_1 and with some η, M. Writing

$$\Delta^m y_k(x) = \Delta^m y_k(x-[x-x_1]) \prod_{v=0}^{[x-x_1]-1} \Delta^m y_k(x-v)/\Delta^m y_k(x-v-1)$$

we obtain that if $\alpha_k > \frac{1}{2}(\alpha_k^2+\beta_k^2)$, then $\lambda_k > 1$ and $|\Delta^m y_k(x)| \to \infty$ as $x\to\infty$; if $\alpha_k < \frac{1}{2}(\alpha_k^2+\beta_k^2)$, then $\lambda_k < 1$ and $\Delta^m y_k(x) \to 0$. In the case when $\lim_{x\to\infty} |a_0(x)| = \infty$ (i.e., $\lambda_k = 0$) we get similarly $\lim_{x\to\infty} \Delta^m y_k(x) = 0$.

Proof of Theorem 3. By hypothesis the functions $P_k(x) = 1/\prod_{s=0}^{[x]-n} \big(1-\varepsilon_k a_0^{1/n}(x-s)\big)$ are continuous for $x > n$, $x \neq n+1, n+2, \ldots$, and right-hand side continuous at the points $x = n, n+1, \ldots$ For $x = v = n+1, n+2, \ldots,$ we have

$$\begin{aligned}\lim_{\varepsilon\to+0} P_k(v-\varepsilon) &= \lim_{\varepsilon\to+0} 1\Big/\prod_{s=0}^{v-n-1} \big(1-\varepsilon_k a_0^{1/n}(v-\varepsilon-s)\big) \\ &= 1\Big/\prod_{s=0}^{v-n-1} \big(1-\varepsilon_k a_0^{1/n}(v-s)\big) \\ &= 1\Big/\prod_{s=0}^{v-n} \big(1-\varepsilon_k a_0^{1/n}(v-s)\big) = P_k(v),\end{aligned}$$

by hypothesis. It follows that the functions $P_k(x)$ are continuous for $x \geqslant n$ and $k = 1, \ldots, n$. We set for $x \geqslant n$

$$\begin{aligned}\cos\alpha(x) &= \big(1-\operatorname{re}\varepsilon_k a_0^{1/n}(x)\big)/|1-\varepsilon_k a_0^{1/n}(x)|, \\ \sin\alpha(x) &= \operatorname{im}\varepsilon_k a_0^{1/n}(x)/|1-\varepsilon_k a_0^{1/n}(x)|,\end{aligned}$$

where $0 \leqslant \alpha(x) < 2\pi$. Then we have for $x \geqslant n$

$$\begin{aligned} P_k(x) &= |P_k(x)| \prod_{v=0}^{[x]-n} \big(\cos\alpha(x-v) + i\sin\alpha(x-v)\big) \\ &= |P_k(x)| \prod_{s=n}^{[x]} \big(\cos\alpha(\xi+s) + i\sin\alpha(\xi+s)\big) \\ &= |P_k(x)| \exp iB_k(x), \end{aligned}$$

where $\xi = x-[x]$ and the function $B_k(x) = \sum_{s=n}^{[x]} \alpha(\xi+s) = \arg P_k(x)$ is continuous for $x \geqslant n$.

If $\lim_{x\to\infty} |a_0(x)| = \infty$, then $\lim_{x\to\infty} \sin\alpha(x) = \sin(2k+\theta)\pi/n \neq 0$, by hypothesis; if $\lim_{x\to\infty} a_0(x) = a$ and $|a| < \infty$, then

$$\lim_{x\to\infty} \sin\alpha(x) = \operatorname{im} \varepsilon_k a^{1/n}/|1-\varepsilon_k a^{1/n}| = \sin\{(2k+\theta)\pi/n\}|a^{1/n}|/|1-\varepsilon_k a^{1/n}| \neq 0.$$

Since the function $\alpha(x) = \arg P_k(x) - \arg P_k(x-1)$ is continuous for $x \geqslant n$, we obtain in both cases that there exists $\alpha \neq 0$ such that $\lim_{x\to\infty} \alpha(x) = \alpha$. It follows that $\lim_{x\to\infty} |B_k(x)| = \infty$.

We set $y_k(x) = y_k^*(x) + iy_k^{**}(x)$. By (11) we have for $x\to\infty$

$$\big(\Delta^m y_k^*(x-m) + i\Delta^m y_k^{**}(x-m)\big)\big(\cos(B_k+\lambda_{km}) - i\sin(B_k+\lambda_{km})\big)T(x)\to 1,$$

$$\big(\Delta^m y_k^*(x-m)\cos(B_k+\lambda_{km}) + \Delta^m y_k^{**}(x-m)\sin(B_k+\lambda_{km})\big)T(x) - 1\to 0, \tag{37}$$

$$\big(-\Delta^m y_k^*(x-m)\sin(B_k+\lambda_{km}) + \Delta^m y_k^{**}(x-m)\cos(B_k+\lambda_{km})\big)T(x)\to 0, \tag{38}$$

where $T(x) = |a_0^{(n-1-2m)/2n}(x)/P_k(x)|$. Multiplying relation (37) by $\cos(B_k+\lambda_{km})$ and (38) by $-\sin(B_k+\lambda_{km})$ and adding we obtain that $\Delta^m y_k^*(x-m)T(x) - \cos\big(B_k(x)+\lambda_{km}\big)\to 0$. Multiplying (37) by $\sin(B_k+\lambda_{km})$ and (38) by $\cos(B_k+\lambda_{km})$ and adding we obtain that $\Delta^m y_k^{**}(x-m)T(x) - \sin\big(B_k(x)+\lambda_{km}\big)\to 0$ as $x\to\infty$.

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