ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XX (1977)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XX (1977)

JAN MILEWSKI (Kraków)

On a certain limit problems for poliparabolic equation

In paper [1] some limit problems for the bicaloric equation in the quart plane and in paper [2] the similar problem for the rectangular are solved.

In this paper we shall construct the solutions of the poliparabolic problems in the unbounded stripe of $2m$-dimensional time space.

To solve these problems we shall use suitable Green functions.

1. In the paper we shall give the solution $u(X)$ of the equation

(1) $$P^m u(X) = F(X), \qquad X = (x, t) = (x_1, x_2, \ldots, x_n, t)$$

in the unbounded stripe

$$\Omega = \{X\colon -\infty < x_i < \infty, |x_n| < c, t > 0, i = 1, 2, \ldots, n-1\},$$

where

$$P = a^2 \sum_{i=1}^{n} D_{x_i}^2 - D_t - b, \qquad P^m = P(P^{m-1}),$$

a and c being positive constants, b non-negative constant and m is positive integer.

Let H denote the set of functions $u(X)$ continuous with the derivatives $D_t^\beta D_x^{|\alpha|} u(X)$, $|\alpha| + 2\beta \leqslant 2m$, satisfying equation (1) in Ω, where α denotes a suitable multiindex.

We shall construct the function $u(X) \in H$, which satisfies the initial conditions

(2) $$D_t^k u(x, 0) = f_k(x)$$

and boundary conditions

(3) $$-D_{x_n} P^k u(X) - \gamma P^k u(X) = h_{1,k}(X') \quad \text{for } x_n = c,$$

(4) $$D_{x_n} P^k u(X) + \gamma P^k u(X) = h_{2,k}(X') \quad \text{for } x_n = -c,$$

where $X' = (x_1, x_2, \ldots, x_{n-1}, t)$, $k = 0, 1, \ldots, m-1$ and γ is an arbitrary real number.

Moreover, we shall construct the function $u(X) \epsilon H$ satisfying the initial conditions (2) and boundary conditions

$$(5) \qquad P^k u(X) = h_{1,k}(X') \quad \text{for } x_n = c,$$

$$(6) \qquad D_{x_n} P^k u(X) = h_{2,k}(X') \quad \text{for } x_n = -c \ (k = 0, 1, \ldots, m-1).$$

2. Let us consider the sets

$$S_0 = \{X: \ -\infty < x_i < \infty, \ |x_n| < c, t = 0\},$$

$$S^{(q)} = \{X: \ -\infty < x_i < \infty, \ x_n = (-1)^{q+1} c, t > 0\},$$

where $i = 1, 2, \ldots, n-1$, $q = 1, 2$.

For the construction of the solutions of problems (1), (2), (3), (4) and (1), (2), (5), (6) we shall use Green functions for the equation

$$(7) \qquad Pu(X) = 0.$$

Let X denote an arbitrary point of Ω. Let $X_1^{(1)}$ be the symmetric image of the point X with respect to the plane $S^{(1)}$, $X_{2p}^{(1)}$ the symmetric image of the point $X_{2p-1}^{(1)}$ with respect to $S^{(2)}$ and $X_{2p+1}^{(1)}$ the symmetric image of the point $X_{2p}^{(1)}$ with respect to $S^{(1)}$, $p = 1, 2, \ldots$ Further let $X_1^{(2)}$ be the symmetric image of the point X with respect to $S^{(2)}$, $X_{2p}^{(2)}$ the symmetric image of the point $X_{2p-1}^{(2)}$ with respect to $S^{(1)}$, $X_{2p+1}^{(2)}$ the symmetric image of the point $X_{2p}^{(2)}$ with respect to $S^{(2)}$, $p = 1, 2, \ldots$ Put

$$X_p^{(q)} = (x_1, x_2, \ldots, x_{n-1}, x_{n,p}^{(q)}, t), \qquad q = 1, 2.$$

It is easy to verify by induction that

$$x_{n,p}^{(q)} = (-1)^p [x_n + (-1)^q 2pc].$$

Put

$$\bar{X}_p^{(q)} = (x_1, x_2, \ldots, x_{n-1}, \bar{x}_{n,p}^{(q)}, t),$$

where

$$\bar{x}_{n,p}^{(q)} = (-1)^p [x_n + (-1)^q (2pc + \xi)], \qquad \xi \geqslant 0, \ p = 1, 2, \ldots, \ q = 1, 2.$$

Let

$$U(X; Y) = \begin{cases} (t-s)^{-n/2} \exp[4a^2 (s-t)^{-1} |x-y|^2] & \text{for } s < t, \\ 0 & \text{for } s \geqslant t, \ X \neq Y \end{cases}$$

and

$$(8) \quad g(X; Y) = U(X; Y) + \sum_{p=1}^{\infty} [U(X_p^{(1)}; Y) + U(X_p^{(2)}; Y) - V(X_{2p-1}^{(1)}; Y) + \\ + V(X_{2p-1}^{(2)}; Y)],$$

where

$$V(X_{2p-1}^{(q)}; Y) = 2\gamma \int_0^{\infty} e^{\gamma\xi} U(\bar{X}_{2p-1}^{(q)}; Y) d\xi \quad (q = 1, 2),$$

$$Y = (y, s) = (y_1, y_2, \ldots, y_n, s) \in \Omega \cup S^{(1)} \cup S^{(2)}, \qquad X \in \Omega.$$

Let

$$G(X; Y) = \begin{cases} e^{-b(t-s)} g(X; Y) & \text{for } s < t, \\ 0 & \text{for } s \geqslant t,\ X \neq Y. \end{cases} \tag{9}$$

Let us consider the series

$$Q_1^{(q)}(X; Y) = \sum_{p=2}^{\infty} |D_{x,y,t,s}^{|\alpha|} U(X_p^{(q)}; Y)| \tag{10}$$

and

$$Q_2^{(q)}(X; Y) = \sum_{p=2}^{\infty} \int_0^{\infty} e^{\gamma\xi} |D_{x,y,t,s}^{|\alpha|} U(\bar{X}_p^{(q)}; Y)| d\xi \quad (q = 1, 2). \tag{11}$$

Let

$$\Omega^{(l+1)} = \{X\colon |x_i| \leqslant c_i,\ |x_n| \leqslant 2^l c,\ 0 \leqslant t \leqslant T,\ i = 1, 2, \ldots, n-1\},$$

where $l = 0, 1$ and c_i, T are positive constants.

LEMMA 1. *The series defined by formulas* (10), (11) *are uniformly convergent for* $X \in \Omega^{(2)}$, $Y \in \Omega^{(1)}$.

Proof. Applying the estimation [5], p. 489, we get a common majorant of the series (10), (11) of the form $C(t-s)^{\mu}$, where C and μ are suitable positive constants.

We shall prove the following

THEOREM 1. *The function* $G(X; Y)$ *given by formula* (9) *is the Green function with the pole* X *for equation* (7) *in the domain* Ω *and for boundary condition*

(*) $(-1)^q [D_{y_n} G(X; Y) + \gamma G(X; Y)] = 0$ *as* $Y \in S^{(q)}, X \neq Y$ $(q = 1, 2)$.

Proof. By definition of the Green function and properties of the fundamental solution it is sufficient to prove that the function $G(X; Y)$ satisfies the boundary condition (*).

Since the integral $V(X_{2p-1}^{(q)}; Y)$ is uniformly convergent, we have

$$D_{y_n} V(X_{2p-1}^{(q)}; Y) = 2\gamma \int_0^{\infty} e^{\gamma\xi} D_{y_n} U(\bar{X}_{2p-1}^{(q)}; Y) d\xi \quad (q = 1, 2).$$

By the equality

$$D_{y_n} U(\bar{X}_{2p-1}^{(q)}; Y) = (-1)^{q+1} D_{\xi} U(\bar{X}_{2p-1}^{(q)}; Y)$$

integrating by parts we have

$$D_{y_n} V(X_{2p-1}^{(q)}; Y) = (-1)^{q+1}\gamma V(X_{2p-1}^{(q)}; Y) + (-1)^{q+1} 2\gamma U(X_{2p-1}^{(q)}; Y) \tag{12}$$
$$(p = 1, 2, \ldots).$$

By formula (12) and symmetry of the suitable points with respect to the plane $y_n = c$ we get

$$-D_{y_n}[U(X; Y) + U(X_1^{(1)}; Y) - V(X_1^{(1)}; Y)] - \gamma[U(X; Y) + U(X_1^{(1)}; Y) - V(X_1^{(1)}; Y)] = 0,$$

$$-D_{y_n}[U(X_p^{(1)}; Y) + U(X_{p+1}^{(1)}; Y) - V(X_{p+1}^{(1)}; Y)] - \gamma[U(X_p^{(1)}; Y) + U(X_{p+1}^{(1)}; Y) - V(X_{p+1}^{(1)}; Y)] = 0 \quad \text{for } y_n = c,\ p = 2, 4, 6, \ldots$$

and

$$-D_{y_n}[U(X_p^{(2)}; Y) + U(X_{p+1}^{(2)}; Y) + V(X_p^{(2)}; Y)] - \gamma[U(X_p^{(2)}; Y) + U(X_{p+1}^{(2)}; Y) + V(X_p^{(2)}; Y)] = 0 \quad \text{for } y_n = c,\ p = 1, 3, 5, \ldots,$$

hence

$$-D_{y_n} G(X; Y) - \gamma G(X; Y) = 0 \quad \text{for } Y \in S^{(1)},\ X \neq Y.$$

Similarly we can prove that

$$D_{y_n} G(X; Y) + \gamma G(X; Y) = 0 \quad \text{for } Y \in S^{(2)},\ X \neq Y.$$

3. By Ω_t, $S_t^{(1)}$, $S_t^{(2)}$ we shall denote the subdomains of the domains Ω, $S^{(1)}$, $S^{(2)}$ lying under the characteristic $s = t$. Let $dy = dy_1 dy_2 \ldots dy_n$, $dY' = dy_1 dy_2 \ldots dy_{n-1} ds$ and $dY = dy_1 dy_2 \ldots dy_n ds$.

Let

$$\varphi_j(y) = \frac{1}{j!}\psi_j(y) - \frac{1}{(j-1)!1!}\psi_{j-1}(y) + \ldots + (-1)^j \frac{1}{0!j!}\psi_0(y), \tag{13}$$

where

$$\psi_j(y) = f_j(y), \quad \psi_{j-1}(y) = C_1^0(-b) f_{j-1}(y) + C_1^1 a^2 \Delta f_{j-1}(y), \ldots, \psi_0(y)$$
$$= C_j^0(-b)^j f_0(y) + C_j^1(-b)^{j-1} a^2 \Delta f_0(y) + \ldots + C_j^j (a^2\Delta)^j f_0(y),$$

$$\Delta = \sum_{i=1}^{n} D_{y_i}^2, \quad \Delta^j = \Delta(\Delta^{j-1}), \quad C_n^k = \binom{n}{k} \quad (j = 0, 1, \ldots, m-1).$$

Let us consider the functions

$$v_j(X) = A \int_{S_0} \varphi_j(y) G(X; Y)|_{s=0} dy,$$

$$w_{q,j}(X) = (-1)^{j+1} a^2 A \int_{S_t^{(q)}} h_{q,j}(Y') \frac{(t-s)^j}{j!} G(X; Y)|_{y_n = (-1)^{q+1}} dY',$$

$A = (2a\sqrt{\pi})^{-n}$, $q = 1, 2$, $j = 0, 1, \ldots, m-1$ and let

$$z(X) = (-1)^m A \int_{\Omega_t} F(Y) \frac{(t-s)^{m-1}}{(m-1)!} G(X; Y) dY, \tag{14}$$

$$v(X) = \sum_{j=0}^{m-1} t^j v_j(X), \tag{15}$$

$$w(X) = \sum_{j=0}^{m-1} [w_{1,j}(X) + w_{2,j}(X)]. \tag{16}$$

We shall prove that the function

$$u(X) = v(X) + w(X) + z(X) \tag{17}$$

is the solution of problem (1), (2), (3), (4) in the domain Ω.

Similarly as in [6] we can verify by induction

LEMMA 2. *If* $\omega(X; Y)$ *satisfies the equation* $P_X \omega(X; Y) = 0$ *for all* Y, *then*

$$P_X^k\left(\frac{(t-s)^j}{j!}\,\omega(X; Y)\right) = \begin{cases} \dfrac{(-1)^k (t-s)^{j-k}}{(j-k)!}\,\omega(X, Y) & \text{for } k \leqslant j, \\ 0 & \text{for } k > j, \end{cases}$$

k and j being positive integers.

4. Let

$$\tilde{\Omega} = \{X:\ -\infty < x_i < \infty,\ |x_n| \leqslant c,\ t \geqslant 0,\ i = 1, 2, \ldots, n-1\},$$

$$\Omega^{(3)} = \{x:\ -\infty < x_i < \infty,\ |x_n| \leqslant c,\ i = 1, 2 \ldots, n-1\},$$

$$\Omega^{(4)} = \{X':\ -\infty < x_i < \infty,\ t \geqslant 0,\ i = 1, 2, \ldots, n-1\}.$$

Let us consider the integrals

$$I_{1,l}^{(q)}(X) = \int_{S_0} f(y) Q_l^{(q)}(X; Y)|_{s=0}\, dy,$$

$$I_{2,l}^{(q)}(X) = \int_{S_t^{(1)}} h(Y') Q_l^{(q)}(X; Y)|_{y_n=c}\, dY',$$

$$I_{3,l}^{(q)}(X) = \int_{S_t^{(2)}} h(Y') Q_l^{(q)}(X; Y)|_{y_n=-c}\, dY',$$

$$I_{4,l}^{(q)}(X) = \int_{\Omega_t} F(Y) Q_l^{(q)}(X; Y)\, dY,$$

where $q = 1, 2$, $l = 1, 2$ and the series $Q_l^{(q)}(X; Y)$ are defined by formulas (10) and (11).

Now we shall prove

LEMMA 3. *If the functions $f(y), h(Y')$ and $F(Y)$ are bounded and measurable in the sets $\Omega^{(3)}$, $\Omega^{(4)}$, $\tilde{\Omega}$ respectively, then the integrals $I_{\nu,l}^{(q)}(X)$ $(q = 1, 2,\ l = 1, 2,\ \nu = 1, 2, 3, 4)$ are uniformly convergent in every set $\Omega^{(2)}$.*

Proof. Similarly as in [7], p. 132, we can verify that the function

$$Ct^{\mu} \tag{18}$$

is a common majorant of the functions $|I_{\nu,l}^{(q)}(X)|$, C and μ being positive constants. Hence the integrals $I_{\nu,l}^{(q)}(X)$ are uniformly convergent in every set $\Omega^{(2)}$.

Let us write

$$J_1^{(k)}(X) = B \int\limits_{\Omega^{(3)}} f(y) D_t^k U(X; y, 0)\, dy, \quad \text{where } B = Ae^{-bt},\ k = 0, 1, 2, \ldots$$

We shall prove

LEMMA 4. *If the functions $D_y^{|\alpha|} f(y)$, $|\alpha| \leqslant 2k$ are bounded and measurable for $y \in \Omega^{(3)}$ and continuous in a neighbourhood of the point $x_0 = (\overset{0}{x}_1, \overset{0}{x}_2, \ldots, \overset{0}{x}_n)$, then*

$$\lim J_1^{(k)}(X) = (a^2 \Delta)^k f(x_0) \quad \textit{as } X \to X_0 \in S_0,\ X \in \Omega,\ X_0 = (x_0, 0)$$
$$(k = 1, 2, \ldots).$$

Proof. Let K_ϱ denote the n-dimentional ball with radius ϱ, centered at the point x_0. We choose a number ϱ such that $K_\varrho \subset \Omega^{(3)}$ and the functions $D_y^{|\alpha|} f(y)$ are continuous for $y \in \bar{K}_\varrho$. Let

$$J_1^{(k)}(X) = J_{1,1}^{(k)}(X) + J_{1,2}^{(k)}(X),$$

where

$$J_{1,1}^{(k)}(X) = B \int\limits_{K_\varrho} f(y) D_t^k U(X; y, 0)\, dy,$$

$$J_{1,2}^{(k)}(X) = B \int\limits_{\Omega^{(3)} - K_\varrho} f(y) D_t^k U(X; y, 0)\, dy.$$

Let us consider the integral

$$J_{1,1}^{(1)}(X) = -B \int\limits_{K_\varrho} f(y) D_s U(X; y, 0)\, dy = B \sum_{i=1}^{n} \int\limits_{K_\varrho} f(y) a^2 D_{y_i}^2 U(X; y, 0)\, dy.$$

Integrating by parts, twice we obtain

$$J_{1,1}^{(1)}(X) = B \sum_{i=1}^{n} \Big[\int\limits_{K_\varrho} a^2 \big(D_{y_i}^2 f(y)\big) U(X; y, 0)\, dy + I_{1,i}(X) + I_{2,i}(X) \Big],$$

where

$$I_{1,i}(X) = Ba^2 \int\limits_{\partial K_\varrho} f(y) D_{y_i} U(X; y, 0) \cos(n, y_i) d\sigma_y,$$

$$I_{2,i}(X) = Ba^2 \int\limits_{\partial K_\varrho} [D_{y_i} f(y)] U(X; y, 0) \cos(n, y_i) d\sigma_y.$$

Since $X \to X_0 \in S_0$ thus we can suppose that $|xx_0| \leqslant \delta/2$, δ being a positive number. Hence

$$\lim I_{\lambda,i}(X) = 0 \quad \text{as } X \to X_0 \in S_0,\ i = 1, 2, \ldots, n,\ \lambda = 1, 2.$$

From the last formulas it follows that

$$J_{1,1}^{(1)}(X) = B \int\limits_{E_n} \bar{f}(y) U(X; y, 0) dy,$$

where

$$\bar{f}(y) = \begin{cases} a^2 \Delta f(y) & \text{for } y \in K_\varrho, \\ 0 & \text{for } y \in E_n - K_\varrho. \end{cases}$$

Applying the theorem of Weierstrass ([5], p. 466) we get

$$\lim J_1^{(1)}(X) = a^2 \Delta f(x_0) \quad \text{as } X \to X_0 \in S_0,\ X \in \Omega. \tag{19}$$

The proof of the implication

$$\lim_{X \to X_0} J_1^{(k)}(X) = (a^2 \Delta)^k f(x_0) \Rightarrow \lim_{X \to X_0} J_1^{(k+1)}(X) = (a^2 \Delta)^{k+1} f(x_0)$$

$(k = 1, 2, \ldots)$ is similar to the proof of condition (19).

LEMMA 5. *If the functions $D_y^{|\alpha|} f(y)$, $|\alpha| \leqslant 2k$, are measurable and bounded in $\Omega^{(3)}$ and are continuous in a neighbourhood of the point* $x_0 = (\overset{0}{x}_1, \overset{0}{x}_2, \ldots, \overset{0}{x}_n)$, *then*

$$\lim D_t^k J_1^{(0)}(X) = \sum_{i=0}^{k} C_k^i (-b)^{k-i} (a^2 \Delta)^i f(x_0) \quad \textit{as } X \to X_0 \in S_0,\ X \in \Omega,$$

$X_0 = (x_0, 0)$, $k = 1, 2, \ldots$

Proof. Similarly as in (7), p. 132, we can verify that the integral $J_1^{(0)}(X)$ belongs to the class C^∞ in Ω. Hence

$$D_t^k J_1^{(0)}(X) = \sum_{i=0}^{k} C_k^i (-b)^{k-i} J_1^{(i)}(X).$$

From the last formula and Lemma 4 follows the assertion of Lemma 5.

Let

$$J_2(X) = A \int\limits_{-\infty}^{t} \int\limits_{E_{n-1}} h(Y') \frac{c - x_n}{t - s} e^{-b(t-s)} U(X; Y)|_{y_n = c} dY'.$$

Now we shall prove

LEMMA 6. *If the function* $h(Y')$ *is measurable, bounded for* $Y' \in \Omega^{(4)}$ *and continuous at the point* $X_0' = (\overset{0}{x}_1, \overset{0}{x}_2, \ldots, \overset{0}{x}_{n-1}, t_0)$, *then*

$$\lim J_2(X) = h(X_0') \quad \textit{as } X \to X_0 \in S^{(1)},\ X \in \Omega,\ X_0 = (\overset{0}{x}_1, \overset{0}{x}_2, \ldots, \overset{0}{x}_{n-1}, c, t_0).$$

Proof. Applying suitable formulas, by [4], p. 500, and [5], p. 456, we get

$$A \int_{-\infty}^{t} \int_{E_{n-1}} \frac{c-x_n}{t-s} e^{-b(t-s)} U(X;\, Y)|_{y_n=c}\, dY' = \exp\left(-\frac{\sqrt{b}(c-x_n)}{a}\right).$$

Hence

$$J_2(X) = h(X_0')\exp\left(-\frac{\sqrt{b}(c-x_n)}{a}\right) + J_3(X),$$

where

$$J_3(X) = A \int_{-\infty}^{t} \int_{E_{n-1}} [h(Y') - h(X_0')] \frac{c-x_n}{t-s} e^{-b(t-s)} U(X;\, Y)|_{y_n=c} dY'.$$

Similarly as in [7], p. 141, we obtain

$$\lim J_3(X) = 0 \quad \text{as } X \to X_0 \in S^{(1)}$$

and consequently the assertion of Lemma 6.

5. Let

$$\Omega^{(5)} = \{x\colon -\infty < x_i < \infty,\ |x_n| \leqslant c,\ i = 1, 2, \ldots, n-1\}.$$

We shall prove

LEMMA 7. *If the functions* $F(Y)$ *and* $D_{y_i}F(Y)$ $(i = 1, 2, \ldots, n-1)$ *are bounded and continuous in the set* $\tilde{\Omega}$, *then the function* $z(X)$ *defined by formula* (14) *is of class* H.

Proof. By [5], p. 536, and Lemma 3, the integrals $z(X)$ and

$$\int_{\Omega_t} F(Y) D_t^{\beta} D_x^{|\alpha|} \left[\frac{(t-s)^{m-1}}{(m-1)!} G(X;\, Y)\right] dY, \qquad |\alpha| + 2\beta \leqslant 2m$$

are almost uniformly convergent in Ω. From the conditions

$$\lim_{s \to t^-} \int_{S_0} F(Y)(t-s)^p G(X;\, Y)\, dy = 0 \quad \text{for } X \neq Y,\ p = 1, 2, \ldots \tag{20}$$

it follows that the function $z(X)$ and its derivatives $D_t^{\beta} D_x^{|\alpha|} z(X)$, $|\alpha| + 2\beta \leqslant 2m$ are continuous in Ω. Moreover, from Lemma 2 and condition (20) it follows that

$$P^m z(X) = A\int_{S_0} F(Y)G(X;\,Y)|_{s=t}dy = \lim_{s\to t^-} A\int_{S_0} F(Y)G(X;\,Y)dy.$$

Let

$$\bar{F}(y,s) = \begin{cases} F(y,s) & \text{for } y\in\Omega^{(5)},\ s\geqslant 0,\\ 0 & \text{for } y\in E_n-\Omega^{(5)},\ s\geqslant 0.\end{cases}$$

Similarly as in [5], p. 466, we obtain

$$\lim_{s\to t^-} A\int_{S_0} F(Y)e^{-b(t-s)}U(X;\,Y)dy = \lim_{s\to t^-} A\int_{E_n} \bar{F}(Y)e^{-b(t-s)}U(X;\,Y)dy = F(X)$$

for $X\in\Omega$. Moreover,

$$\lim_{s\to t^-} A\int_{S_0} F(Y)e^{-b(t-s)}\sum_{p=1}^{\infty}[U(X_p^{(1)};\,Y)+U(X_p^{(2)};\,Y)-V(X_{2p-1}^{(1)};\,Y)+ \\ +V(X_{2p-1}^{(2)};\,Y)] = 0 \quad \text{for } X\in\Omega.$$

Hence the function $z(X)$ satisfies equation (1) in Ω.

Let

$$|a^j| = a_1^j+a_2^j+\ldots+a_n^j, \qquad D_y^{|a^j|} = D_{y_n}^{a_n^j}\ldots D_{y_2}^{a_2^j}D_{y_1}^{a_1^j}.$$

LEMMA 8. *If the functions $D_y^{|a^j|}f_j(y)$, $|a^j|\leqslant 2(m-j-1)$, $h_{1,j}(Y')$, $h_{2,j}(Y')$ $(j=0,1,\ldots,m-1)$ are measurable and bounded for $y\in\Omega^{(3)}$, $Y'\in\Omega^{(4)}$ and the functions $F(Y)$, $D_{y_i}F(Y)$ $(i=1,2,\ldots,n)$ are bounded and continuous in $\tilde{\Omega}$, then the function $u(X)$ defined by formula* (17) *belongs to the class H.*

The proof of Lemma 8 is analogous to the proof of Lemma 7.

LEMMA 9. *If* a) *the functions $D_y^{|a^j|}f_j(y)$, $|a^j|\leqslant 2(m-j-1)$, $h_{1,j}(Y')$, $h_{2,j}(Y')$ $(j=0,1,\ldots,m-1)$ and $F(Y)$ are measurable and bounded in the sets $\Omega^{(3)}$, $\Omega^{(4)}$, $\tilde{\Omega}$, respectively,* b) *the functions $D_y^{|a^j|}f_j(y)$, $|a^j|\leqslant 2(m-j-1)$ $(j=0,1,\ldots,m-2)$ are continuous in the neighbourhood of the point $x_0\in\Omega^{(3)}$,* c) *the function $f_{m-1}(y)$ is continuous at the point $x_0\in\Omega^{(3)}$, then the function $u(X)$ defined by formula* (17) *satisfies the initial conditions* (2).

Proof. The function $v(X)$ defined by formula (15) is the sum of the functions $I_l(X)$, $l=1,2$, where

$$I_1(X) = \sum_{j=0}^{m-1} Bt^j\int_{S_0}\varphi_j(y)\,U(X;\,y,0)\,dy,$$

$$I_2(X) = \sum_{j=0}^{m-1} Bt^j\int_{S_0}\varphi_j(y)\sum_{p=1}^{\infty}[U(X_p^{(1)};\,Y)+U(X_p^{(2)};\,Y)-V(X_{2p-1}^{(1)};\,Y)+ \\ +V(X_{2p-1}^{(2)};\,Y)].$$

Similarly to [3] we can verify that the function

$$I_3(X) = \sum_{j=0}^{m-1} t^j u_j(X),$$

where

$$u_j(X) = \frac{B}{j!}\int_{S_0} f_j(y)\,U(X;y,0)\,dy - \frac{B}{(j-1)!1!}D_t\int_{S_0} f_{j-1}(y)\,U(X;y,0)\,dy + \ldots$$
$$\ldots + \frac{(-1)^j B}{0!j!}D_t^j\int_{S_0} f_0(y)\,U(X;y,0)\,dy,$$

satisfies the initial conditions (2). By formula (20), Lemmas 4, 5 and Weierstrass theorem ([5], p. 466), we can verify that the function $I_1(X)$ satisfies the initial conditions (2). Analogously to [7], p. 132, we can prove that the function defined by formula (18) is the common majorant of the functions $|D_t^k I_2(X)|$, $|D_t^k w(X)|$ and $|D_t^k z(X)|$ $(k = 0, 1, \ldots, m-1)$. Hence the functions $D_t^k I_2(X)$, $D_t^k w(X)$ and $D_t^k z(X)$ tend to zero as $X \to X_0 \in S_0$.

We shall prove

LEMMA 10. *If the functions* $h_{k,j}(Y')$ $(k = 1, 2;\ j = 0, 1, \ldots, m-1)$ *are measurable and bounded in* $\Omega^{(4)}$ *and the functions* $h_{1,j}(Y')$ *are continuous at the point* $X_0' = (\overset{0}{x}_1, \overset{0}{x}_2, \ldots, \overset{0}{x}_{n-1}, t_0)$, *then the function* $w(X)$ *given by formula* (16) *satisfies the boundary conditions* (3).

Proof. Let

$$w(X) = w_1(X) + w_2(X),$$

where

$$w_\nu(X) = \sum_{j=0}^{m-1} w_{\nu,j}(X) \qquad (\nu = 1, 2).$$

Similarly as in the proof of Theorem 1 we get

$$D_{x_n} V(X_{2p-1}^{(q)}; Y) = (-1)^{q+1}\gamma V(X_{2p-1}^{(q)}; Y) + (-1)^{q+1} 2\gamma\, U(X_{2p-1}^{(q)}; Y) \tag{21}$$
$$(q = 1, 2;\ p = 1, 2, \ldots).$$

In consequence by Lemma 2 we obtain

$$-D_{x_n} P^k w_{\nu,j}(X) - \gamma P^k w_{\nu,j}(X) = \begin{cases} L_{j,k}^{(\nu)}(X) & \text{for } j \geqslant k, \\ 0 & \text{for } j < k, \end{cases}$$

where

$$L_{j,k}^{(\nu)}(X) = (-1)^{j+k} a^2 A \int_{S_t^{(\nu)}} h_{\nu,j}(Y')\,\frac{(t-s)^{j-k}}{(j-k)!}\Big\{D_{x_n} U(X; Y) +$$
$$+ \sum_{p=1}^{\infty} D_{x_n}[U(X_p^{(1)}; Y) + U(X_p^{(2)}; Y)]\Big\}\Big|_{y_n = (-1)^{\nu+1}c}\, dY' \qquad (\nu = 1, 2).$$

In the same way as in the proof of Lemma 6 we have

$$\lim[-D_{x_n}P^k w_1(X)-\gamma P^k w_1(X)] = h_{1,k}(X_0') \quad \text{as } X\to X_0\in S^{(1)},\ X\in\Omega,$$
$$(k = 0,1,\ldots,m-1).$$

The integral $L^{(2)}_{j,k}(X)$ is uniformly convergent in every set

$$\Omega^{(6)} = \{X:\ |x_i|\leqslant c_i,\ d_1\leqslant x_n\leqslant 2c,\ 0\leqslant t\leqslant T,\ i=1,2,\ldots,n-1\},$$

where c_i, d_1, T are real numbers for which $c_i>0$, $-c<d_1<2c$, $T>0$. Hence the function $L^{(2)}_{j,k}(X)$ is continuous at the point $X_0\in S^{(1)}$. Since

$$D_{x_n}[U(X;Y)+U(X_1^{(1)};Y)]|_{y_n=-c} = 0,$$

and

$$D_{x_n}[U(X^{(1)}_{p+1};Y)+U(X^{(2)}_p;Y)]|_{y_n=-c}=0 \quad \text{for } X=X_0\in S^{(1)},\quad p=1,2,\ldots$$

therefore $\lim L^{(2)}_{j,k}(X) = 0$ as $X\to X_0\in S^{(1)}$ and finally we obtain

$$\lim[-D_{x_n}P^k w_2(X)-\gamma P^k w_2(X)] = 0 \quad \text{as } X\to X_0\in S^{(1)},$$
$$k = 0,1,\ldots,m-1.$$

LEMMA 11. *If the functions $h_{k,j}(Y')$ $(k=1,2)$ are bounded and measurable in $\Omega^{(4)}$ and the functions $h_{2,j}(Y')$ are continuous at the point $X_0' = (\overset{0}{x}_1,\overset{0}{x}_2,\ldots,\overset{0}{x}_{n-1},t_0)$, then the function $w(X)$ defined by formula* (16) *satisfies the boundary conditions* (4).

The proof is similar to that of Lemma 10.

LEMMA 12. *If the functions $D_y^{|a^j|}f_j(y)$, $|a^j|\leqslant 2(m-j-1)$ $(j=0,1,\ldots,m-1)$ are bounded and measurable in $\Omega^{(3)}$, then the function $v(X)$ given by formula* (15) *satisfies the conditions*

$$\lim[D_{x_n}P^k v(X)+\gamma P^k v(X)] = 0 \quad \textit{as } X\to X_0\in S^{(1)}\cup S^{(2)},$$
$$k = 0,1,\ldots,m-1.$$

Proof. In virtue of Lemma 2 and formula (21) we get

$$D_{x_n}P^k v_j(X)+\gamma P^k v_j(X) = \begin{cases} L^{(3)}_{j,k}(X) & \text{for } j\geqslant k,\\ 0 & \text{for } j<k,\end{cases}$$

where

$$L^{(3)}_{j,k}(X) = \frac{(-1)^{j+k}t^{j-k}A}{(j-k)!}\int\limits_{S_0}\varphi_j(y)[D_{x_n}G(X;Y)+\gamma G(X;Y)]|_{s=0}\,dy.$$

The integral $L^{(3)}_{j,k}(X)$ is uniformly convergent in every set

$$\Omega^{(7)} = \{X:\ |x_i|\leqslant c_i,\ |x_n|\leqslant 2c,\ 0<T_0\leqslant t\leqslant T,\ i=1,2,\ldots,n-1\},$$

where c_i, T_0, T are positive numbers. Hence the function $L^{(3)}_{j,k}(X)$ is continuous at the point $X_0 \in S^{(1)}$, by virtue of formula (21), and by formulas

$$D_{x_n}[U(X;Y)+U(X_1^{(1)};Y)]|_{s=0}=0, \quad D_{x_n}[U(X_{p+1}^{(1)};Y)+U(X_p^{(2)};Y]|_{s=0}=0$$

for $X = X_0 \in S^{(1)}$ $(p = 1, 2, \ldots)$, we get

$$\lim L^{(3)}_{j,k}(X) = 0 \quad \text{as } X \to X_0 \in S^{(1)}.$$

Hence

$$\lim [D_{x_n} P^k v(X) + \gamma P^k v(X)] = 0 \quad \text{as } X \to X_0 \in S^{(1)}.$$

Analogously we can verify that

$$\lim [D_{x_n} P^k v(X) + \gamma P^k v(X)] = 0 \quad \text{as } X \to X_0 \in S^{(2)}.$$

LEMMA 13. *Let the function $F(Y)$ be bounded and measurable in the set $\tilde{\Omega}$, then the function $z(X)$ defined by formula* (14) *satisfies the conditions*

$$\lim [D_{x_n} P^k z(X) + \gamma P^k z(X)] = 0 \quad \text{as } X \to X_0 \in S^{(1)} \cup S^{(2)} \quad (k = 0, 1, \ldots, m-1).$$

The proof of Lemma 13 is analogous to that of Lemma 12.

By Lemmas 8–13 we obtain

THEOREM 2. *Let the functions $D_y^{|\alpha^j|} f_j(y)$, $|\alpha^j| \leqslant 2(m-j-1)$, $h_{1,j}(Y')$, $h_{2,j}(Y')$, $F(Y)$ and $D_{y_i} F(Y)$ $(j = 0, 1, \ldots, m-1;\ i = 1, 2, \ldots, n)$ be bounded and continuous for $y \in \Omega^{(3)}$, $Y' \in \Omega^{(4)}$ and $Y \in \tilde{\Omega}$; then the function $u(X)$ given by formula* (17) *is of class H and satisfies the limit conditions* (2), (3) *and* (4).

6. Let

$$G_1(X;Y) = \begin{cases} e^{-b(t-s)} g_1(X;Y) & \text{for } s < t, \\ 0 & \text{for } s \geqslant t,\ X \neq Y, \end{cases} \tag{22}$$

where

$$g_1(X;Y) = U(X;Y) + U(X_1^{(2)};Y) + \sum_{p=1}^{\infty} (-1)^p [U(X_{2p-1}^{(1)};Y) + U(X_{2p}^{(1)};Y) + U(X_{2p}^{(2)};Y) + U(X_{2p+1}^{(2)};Y)].$$

THEOREM 3. *The function $G_1(X;Y)$ defined by formula* (22) *is the Green function with the pole X for equation* (7) *in the domain Ω and for boundary condition of Dirichlet–Neuman type $G_1(X;Y) = 0$ for $Y \in S^{(1)}$, $X \neq Y$ and $D_{y_n} G_1(X;Y) = 0$ for $Y \in S^{(2)}$, $X \neq Y$.*

The proof is analogous to that of Theorem 1.

Let

$$(23)\quad u(X) = \sum_{j=0}^{m-1} \Big\{ At^j \int_{S_0} \varphi_j(y) G_1(X;\, Y)|_{s=0} dy +$$

$$+(-1)^{j+1} a^2 A \int_{S_t^{(1)}} h_{1,j}(Y') \frac{(t-s)^j}{j!} D_{y_n} G_1(X;\, Y)|_{y_n=c} dY' +$$

$$+(-1)^{j+1} a^2 A \int_{S_t^{(2)}} h_{2,j}(Y') \frac{(t-s)^j}{j!} G_1(X;\, Y)|_{y_n=-c} dY' \Big\} +$$

$$+(-1)^m A \int_{\Omega_t} F(Y) \frac{(t-s)^{m-1}}{(m-1)!} G_1(X;\, Y) dY,$$

where the functions $\varphi_j(y)$ are defined by formula (13).

THEOREM 4. *If the functions* $D_y^{|a^j|} f_j(y)$, $|a^j| \leqslant 2(m-j-1)$, $h_{1,j}(Y')$, $h_{2,j}(Y')$, $F(Y)$ *and* $D_{y_i} F(Y)$ $(j = 0, 1, \ldots, m-1;\ i = 1, 2, \ldots, n)$ *are bounded and continuous in the sets* $\Omega^{(3)}$, $\Omega^{(4)}$, $\tilde{\Omega}$, *respectively, then the function* $u(X)$ *defined by formula* (23) *is of class* H *and satisfies the limit conditions* (2), (5), (6).

The proof is similar to that of Theorem 2.

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