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On the mixed problem for the Poisson equation in the quart-plane

1. In the present paper we shall solve the boundary problem for the equation

\[ \Delta u(X) = 0, \quad X = (x, y), \]

\[ \Delta u(X) = F(X) \]

with boundary conditions

\[ D_i u(0, y) = f_i(y) \quad (i = 0, 1), \]

\[ D_j u(x, 0) + hu(x, 0) = f_i(x) \]

in the quart-plane \( E^+ = \{ X : x > 0, y > 0 \} \), where \( h \) denotes the negative constant and \( f_i (i = 0, 1) \), \( F \) are the given functions.

We shall call problems (1), (2), (3); (1), (2), (3); (1a), (2), (3) and (1a), (2), (3) conventionally \( M_i (i = 1, 2, 3, 4) \) problem.

2. For the solution of the problems \( M_i \) we shall use the convenient Green-functions.

Let \( Y = (s, t) \) denote an arbitrary point belonging to \( E_2 \).

Let

\[ r_1^2 = (x-s)^2(y-t)^2, \quad r_2^2 = (x-s)^2 + (y+t)^2, \quad r_3^2 = (x+s)^2 + (y-t)^2, \]

\[ r_4^2 = (x+s)^2 + (y+t)^2, \quad r_5^2 = (x-s)^2 + (y+t+v)^2, \quad r_6^2 = (x+s)^2 + (y+t+v)^2, \]

and let \( K_i(X, Y) = \ln r_i^2 \ (i = 1, \ldots, 4) \), \( K_i(X, Y) = \int_0^\infty e^{\nu x}(\ln r_i^2) d\nu (i = 5, 6) \),

\[ K_7(X, t) = x(x^2 + (y-t)^2)^{-1}, \]

\[ K_8(K, t) = x(x^2 + (y+t)^2)^{-1}, \]

\[ K_9(X, t) = \int_0^\infty e^{\nu x}(x^2 + (y+t+v)^2)^{-1} d\nu, \]

\[ K_{10}(X, s) = y((x-s)^2 + y^2)^{-1}, \]

\[ \int_0^\infty e^{\nu x}(x^2 + (y+t+v)^2)^{-1} d\nu, \]

\[ K_{10}(X, s) = y((x-s)^2 + y^2)^{-1}, \]
\[ K_{11}(X, s) = y((x + s)^2 + y^2)^{-1}, \quad K_{12}(X, s) = \ln((x - s)^2 + y^2), \]
\[ K_{13}(X, s) = \ln((x + s)^2 + y^2), \]
\[ K_{14}(X, s) = \int_0^\infty e^{hv} \ln((x - s)^2 + (y + v)^2) dv, \]
\[ K_{15}(X, s) = \int_0^\infty e^{hv} \ln((x + s)^2 + (y + v)^2) dv, \]
\[ K_{16}(X, t) = \ln(x^2 + (y - t)^2), \]
\[ K_{17}(X, t) = \ln(x^2 + (y + t)^2), \]
\[ K_{18}(X, t) = \int_0^\infty e^{hv} \ln(x^2 + (y + v + t)^2) dv, \]
\[ K_{19}(X, t) = (y - t)((y - t)^2 + x^2)^{-1}, \]
\[ K_{20}(X, t) = (y + t)((y - t)^2 + x^2)^{-1}, \]
\[ K_{21}(X, s) = (x - s)((x - s)^2 + y^2)^{-1}, \]
\[ K_{22}(X, s) = (x + s)((x + s)^2 + y^2)^{-1}, \]
\[ K_{23}(X, s) = \int_0^\infty e^{hv}(x - s)((x - s)^2 + (y + v)^2)^{-1} dv, \]
\[ K_{24}(X, s) = \int_0^\infty e^{hv}(x + s)((x - s)^2 + (y + v)^2)^{-1} dv, \]
\[ K^i(X, Y) = \ln r_i^2 \quad (i = 5, 6) \]

and
\[ H_4(X) = 2C_1 \int_0^\infty f_2(s) K_4(X, s) ds \quad (i = 10, \ldots, 15). \]

Let
\[ G_1(X, Y) = C_1(K_1(X, Y) + K_2(X, Y)) + C_1 I_1(X, Y), \]
\[ G_2(X, Y) = C_1(K_3(X, Y) + K_4(X, Y)) + C_2 I_2(X, Y), \]
where \( I_i(X, Y) = K_{i+4}(X, Y) \) \((i = 1, 2)\) and \( C_1 = (2\pi)^{-1}, C_2 = 2hC_1. \)

Let
\[ W = \{ (X, Y) : 0 < A < x < B, 0 < A < y < B, 0 \leq t < B, 0 \leq s \leq B \}, \]
where \( A, B \) are positive constants.

Let
\[ I_{ipmn}^{(R)}(X, Y) = C_2 \int_0^\infty e^{hv} |D_{x=1}^{\text{pgmn}} \ln r_{i+4}^2| dv \quad (i = 1, 2), \]
m, n, p, q being non-negative integers for which \( 0 \leq p + q + m + n \leq 4. \)
Let $\varepsilon$ be an arbitrary positive number. Now we shall prove some lemmas which we shall use in the sequel.

**Lemma 1.** There exists the number $R_0(\varepsilon)$ such that for every $R > R_0(\varepsilon)$ and every point $(X, Y) \in W$ we have $I_{pqmn}^R < \varepsilon$ ($j = 1, 2$).

**Proof.** If $p + q + m + n > 0$, then the majorant of the integrals $I_{pqmn}^R$ are the finite sum of the integrals of the form

$$J = C \int_\mathbb{R} e^{\nu|x|} \left| y + t + v \right|^a(r_i)^{-2c} dv$$

(i = 5, 6),

where $2c \geq a + b$ and $C$ is a positive constants. Since $(x + (-1)^i y)^2(r_i)^{-2} \leq 1$ ($i = 5, 6$), $(y + t + v)^2(r_i)^{-2} \leq 1$ ($i = 5, 6$), thus the integral

$$J^1(R) = C_1 A^a \int_\mathbb{R} e^{\nu \ln v} dv,$$

$C_1$ being the convenient constant and $a = a + b - 2c$, is the majorant of the integral $J$.

The integral $J^1(R)$ is arbitrary small for sufficiently great $R$.

If $p + q + m + n = 0$, then the integral $C \int_\mathbb{R} e^{\nu \ln v} dv$ is the majorant of the integrals $\int_\mathbb{R} e^{\nu \ln r_i^2} dv$ ($i = 5, 6$) for $R$ sufficiently great.

**Corollary 1.** The integrals

$$I_{pqmn}(X, Y) = \int_0^\infty e^{\nu \ln r_i^2} dv$$

are uniformly convergent in every set $W$, there exist the derivatives

$$D_{xyst} I_j(X, Y) = I_{pqmn}(X, Y)$$

and

$$D_{xyst} I_j(X, Y) = I_{pqmn}(X, Y)$$

for $j = 1, 2$.

**Theorem 1.** The functions

$$G_i(X, Y) = G_1(X, Y) + (-1)^i G_2(X, Y)$$

are the Green function for the problem $(M_i)$ ($i = 1, 2$), in the domain $E^+_2$ with the pole $X$.

**Proof.** The functions $G_i$ ($i = 1, 2$) are harmonic functions of the point $Y$.

Indeed, the functions $\ln r_i^2$ ($i = 1, 2, 3, 4$) are harmonic and by Corollary 1 we obtain

$$\Delta X G_1(X, Y) = C_1 A_x K_1(X, Y) + C_2 A_x K_2(X, Y)$$

and similarily $\Delta Y G_2(X, Y) = 0$. Hence the functions $G_i$ ($i = 1, 2$) are harmonic as the sum of two harmonic functions.
Moreover, we shall prove that the functions $G_i (i = 1, 2)$ satisfy the homogeneous boundary conditions

\begin{align}
(4) \quad (D_t + h)G_i (X, s, 0) &= 0 \quad (i = 1, 2), \\
(5) \quad G^1 (X, 0, t) &= 0, \\
(6) \quad D_s G^2 (X, 0, t) &= 0.
\end{align}

Condition (5) is obvious. We shall prove condition (4) for the function $G_1$ and for $i = 1$. The proof of condition (4) for the function $G_2$ and for $i = 2$ is the same. By Corollary 1 and Theorem 1 for $t = 0$ we get

\[ D_t G_1 (X, s, t)|_{t=0} = C_1 D_t (K_1 (X, Y) + K_2 (X, Y)) + C_2 D_t (I_1 (X, Y))|_{t=0} \]

\[ = C_2 \int_0^\infty e^{hv} D_t (K^1_5 (X, Y))|_{t=0} dv = C_2 \int_0^\infty e^{hv} D_v (K^1_5 (X, Y))|_{v=0} dv \]

\[ = (C_2 K_2 (X, Y) - C_2 h K_5 (X, Y))|_{t=0} \]

\[ = - C_2 K_2 (X, s) - h C_2 K_5 (X, s); \]

because for $t = 0$ we have $D_t (K_1 (X, Y) + K_2 (X, Y)) = 0$ and $K_2 (X, Y) = K_1 (X, s), K_5 (X, Y) = K_1 (X, s)$. Moreover, for $t = 0$ we have

\[ h G_1 (X, s, 0) = 2 h C_1 K_{12} (X, s) + h C_2 K_{14} (X, s). \]

Hence

\[ h G_1 (X, s, 0) + D_t G_1 (X, s, 0) = 0. \]

We omit the simple proof of condition (6).

3. Under certain assumptions concerning the functions $f_i (i = 0, 1, 2)$ we shall prove that the functions

\begin{align}
(I) \quad &u_1 (X) = \int_0^\infty f_2 (s) G^1 (X, s, 0) ds - \int_0^\infty f_0 (t) D_s G^1 (X, 0, t) dt \\
(II) \quad &u_2 (X) = \int_0^\infty f_2 (s) G^2 (X, s, 0) ds + \int_0^\infty f_1 (t) G^2 (X, 0, t) dt
\end{align}

are the solutions of the problems $(M_i) (i = 1, 2)$.

We shall prove now that the function $u_1$ given by formula (I) is the solution of the problem $(M_1)$.

By (I) and $G^1$ we have

\[ u(x) = \sum_{i=1}^7 J_i (X), \]
where
\[ J_i(X) = (-1)^{i+1} C_1 \int_0^\infty f_2(s) K_j(X, s) ds \quad (i = 1, 2; j = i+11), \]
\[ J_i(X) = (-1)^{i+1} C_2 \int_0^\infty f_2(s) K_j(X, s) ds \quad (i = 3, 4; j = i+11), \]
\[ J_i(X) = C_i \int_0^\infty f_0(t) K_{i+2}(X, t) dt \quad (i = 5, 6, 7), \]
where
\[ C_i = \begin{cases} -4C_1 & \text{for } i = 5, 6, \\ -4C_2 & \text{for } i = 7. \end{cases} \]

Let
\[ W_1 = \{(X): |x| < A; 0 < A < y < B\}; \]
\[ W_2 = \{(X): 0 < A < x < B; |y| < A\} \]
and let
\[ J_{ipq}(X) = \int_0^\infty f_0(t) D_{xy}^{pq} K_j(X, t) dt; \quad J_{ipq}^R(X) = \int_R^\infty |f_0(t) D_{xy}^{pq} K_j(X, t)| dt \]
for \( i = 5, 6, 7; j = i+2, \)
\[ J_{ipq}(X) = \int_0^\infty f_2(s) D_{xy}^{pq} K_j(X, s) ds; \quad J_{ipq}^R(X) = \int_R^\infty |f_2(s) D_{xy}^{pq} K_j(X, s)| ds \]
for \( i = 1, 2, 3, 4; j = i+11, \)
p, q being non-negative integers and \( 0 \leq p + q \leq 2. \)

**Lemma 2.** If 1° the function \( f_0 \) is continuous and absolutely integrable for \( t \geq 0, \) 2° the integral
\[ \int_a^\infty |f_2(s)\ln s| ds \]
is convergent for any \( a > 0, \) 3° the function \( f_2 \) is continuous for \( s \geq 0, \) then the integrals \( J_{ipq}(X) \) \( (i = 1, \ldots, 4) \) are uniformly convergent in the set \( W_1 \) and the integrals \( J_{ipq}(X) \) \( (i = 5, 6, 7) \) in the set \( W_2. \)

**Proof.** The common majorant of the integrals \( J_{ipq}^R(X) \) \( (i = 5, 6, 7) \) is the integral
\[ C \int_R^\infty |f_0(t)| dt \]
and for the integrals $J_{ipq}(X)$ ($i = 1, \ldots, 4$) the integral

$$C \int_{\mathbb{R}} |f_2(s) \ln |s|| ds,$$

where $C$ is a positive constant.

**Corollary 2.** If the assumptions of Lemma 2 are satisfied, then there exist the derivatives

$$D^{pq}_{xy} J_i(X) \quad \text{for } i = 1, 2, 3, 4 \text{ and } X \in W_1,$$

and

$$D^{pq}_{xy} J_i(X) \quad \text{for } i = 5, 6, 7 \text{ and } X \in W_2,$$

and

$$D^{pq}_{xy} J_i(X) = \int_0^\infty f_2(s) D^{pq}_{xy} K_j(X, s) ds \quad (i = 1, \ldots, 4; \quad j = 11 + i),$$

$$D^{pq}_{xy} J_i(X) = \int_0^\infty f_0(t) D^{pq}_{xy} K_j(X, t) dt \quad (i = 5, 6, 7; \quad j = i + 2).$$

From Corollary 2 and Lemma 2 follows

**Lemma 3.** If the functions $f_i$ ($i = 1, 2$) satisfy the assumptions of Lemma 2, then the function $u_1$ is the harmonic function in the quart-plane $E_2^\infty$.

**Proof.** Since $G^1$ is the harmonic function with respect to the point $X$ we get

$$\Delta u_1(X) = \Delta_X \left( \int_0^\infty f_2(s) G^1(X, s, 0) ds \right) - \Delta_X \left( \int_0^\infty f_0(t) G^1(X, 0, t) dt \right)$$

$$= \int_0^\infty f_2(s) \Delta_X G^1(X, s, 0) ds - \int_0^\infty f_0(t) \Delta_X G^1(X, 0, t) dt = 0.$$

4. In order to prove that the function $u_1$ satisfies the boundary conditions (2) ($i = 0, 1$) and (3) shall give some lemmas.

**Lemma 4.** If the function $f_2$ satisfies assumptions $2^0, 3^0$ of Lemma 2 and $y_0 > 0$, then

$$g = \lim_{X \to (0^+, y_0)} \sum_{i=1}^4 J_i(X) = 0 \quad \text{as } X \to (0^+, y_0).$$

**Proof.** By Lemma 2 and continuity of the integrals $J_i(X)$ ($i = 1, \ldots, 4$), we obtain

$$g = \sum_{i=1}^4 J_i(0, y_0) = 0.$$
Let \( m_1 = \sup |f_i| \) (i = 0, 1, 2) and let
\[
\tilde{f}_0(t) = \begin{cases} 
  f_0(t) & \text{for } t \geq 0, \\
  0 & \text{for } t < 0.
\end{cases}
\]

**Lemma 5.** If the function \( f_0 \) is continuous, bounded and absolutely integrable, for \( t \geq 0, y_0 > 0 \), then
(a) \( \lim J_5(X) = f_0(y_0) \) as \( X \to (0^+, y_0) \),
(b) \( \lim J_i(X) = 0 \) (i = 6, 7) as \( X \to (0^+, y_0) \).

**Proof.** Ad (a). By Krzyżanński's book (1) we get
\[
J_5(X) = \int_{-\infty}^{\infty} \tilde{f}_0(t) K_7(X, t) \, dt \to f_0(y_0) \text{ as } X \to (0^+, y_0).
\]
Ad (b). For \( J_6(X) \) we have
\[
|J_6(X)| \leq m_1 C_2 \int_0^{\infty} (\frac{1}{2} y_0 + t)^{-2} \, dt \to 0 \text{ as } X \to (0^+, y_0)
\]
and for \( J_7(X) \) we obtain the estimation
\[
|J_7(X)| \leq 2m_1 C_2 \int_{E_2^+} e^{h_2(\frac{1}{2} y_0 + t)-2} \, dv \, dt \to 0 \text{ as } X \to (0^+, y_0).
\]

**Lemma 6.** If 1° the functions \( f_i \) (i = 0, 2) are bounded and continuous for \( t \geq 0, s \geq 0 \), the function \( f_0 \) is absolutely integrable for \( t \geq 0 \), 2° the integral
\[
\int_a^{\infty} |f_2(x)| \ln a \, dx
\]
is convergent for any \( a > 0 \) and \( a_0 > 0 \), then
(c) \( (D_y + h) \sum_{i=1}^4 J_i(X) \to f_2(x_0) \text{ as } X \to (x_0, 0^+) \)
and
(d) \( (D_y + h) \sum_{i=5}^7 J_i(X) \to 0 \text{ as } X \to (x_0, 0^+) \).

**Proof.** By (Ia) and Corollary 2 we have
\[
(D_y + h) \sum_{i=1}^4 J_i(X) = 2H_{10}(X) - hH_{12}(X) - h^2 H_{14}(X) + H_{12}(X) + h^2 H_{14}(X) +
+ 2H_{11}(X) - hH_{13}(X) - h^2 H_{15}(X) + hH_{13}(X) + h^2 H_{14}(X)
= 2 (H_{10}(X) + H_{11}(X)) \to f_2(x_0) \text{ as } X \to (x_0, 0^+).
\]

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Indeed \((t)\), \(\lim 2H_{10}(X) = f_2(x_0)\) as \(X \to (x_0, 0^+)\) and

\[
|H_{11}(X)| \leq A \int_0^\infty y \left((\frac{1}{2} x_0^2 + s^2)^{-1}\right) ds \to 0 \quad \text{as } X \to (x_0, 0^+);
\]

\(A\) being a positive constant.

Ad (d). By continuity of the integrals \(J_i (i = 5, 6, 7)\) we get

\[
(D_y + h) \sum_{i=5}^7 J_i(X) = 8C_1x \int_0^\infty f_0(t)(y-t)(x_0^2 + (y-t)^2)^{-1} +
\]

\[+(y + t)(x_0^2 + (y + t)^2)^{-1} dt + 4C_1h \int_0^\infty f_0(t) (K_8(X, t) - K_7(X, t)) dt
\]

\[-8C_1x_0 \int_0^\infty f_0(t) \left(t(x_0^2 + t^2)^{-1} - t(x_0^2 + t^2)^{-1}\right) dt + 4C_1h \int_0^\infty f_0(t) \left(x_0(x_0^2 + t^2)^{-1} -
\]

\[-x_0(x_0^2 + t^2)^{-1}\right) dt = 0.

By Lemmas 2, ..., 6 follows

**Theorem 2.** Let the functions \(f_i (i = 0, 2)\) be bounded and continuous for \(t \geq 0, s \geq 0, x_0 > 0\), the function \(f_0\) be absolutely integrable for \(t \geq 0\) and the integral

\[
\int_a^\infty |f_3(s)\ln s| ds
\]

be convergent for any \(a > 0\), then the function \(u_1(X)\) defined by formula (I) or (Ia) is the solution of the problem \((M_1)\).

5. We shall give now the solution of the problem \((M_2)\). By formula (II) we get

\[
(IIa) \quad u_2(X) = \int_0^\infty f_2(s) \left(2C_1K_{12}(X, s) + 2C_2K_{13}(X, s) + C_2K_{14}(X, s) +
\]

\[+C_2K_{15}(X, s)\right) ds + \int_0^\infty f_1(t) \left(2C_1K_{16}(X, t) + 2C_1K_{17}(X, t) + 2C_2K_{18}(X, t)\right) dt.
\]

We shall prove that the function \(u_2\) defined by formula (IIa) is the solution of the problem \((M_2)\).

**Lemma 3a.** If \(1^o\) the functions \(f_i (i = 1, 2)\) are continuous for \(s \geq 0, t \geq 0\), \(2^o\) the integrals

\[
\int_a^\infty |f_i(z)\ln z| dz \quad (i = 1, 2),
\]

be convergent for any \(a > 0, z > 0\), then the function \(u_2(X, t)\) defined by formula (IIa) is the solution of the problem \((M_2)\).
are convergent for any \( a > 0 \), then the function \( u_2 \) given by formula (IIa) is the harmonic function in \( E^+_2 \).

The proof is similar to the proof of Lemma 3.

We shall prove now that the function \( u_2 \) satisfies the boundary conditions (2), (3)

**Proof.** By (IIa) we have

\[
D_x u_2(X) = \sum_{i=1}^{7} Q_i(X),
\]

where

\[
Q_i(X) = 2C_i \int_{0}^{\infty} f_i(s) K_{i+20}(X, s) ds \quad (i = 1, 2, 3, 4),
\]

\[
Q_i(X) = 2C_1 \int_{0}^{\infty} f_1(t) K_{i+2}(X, t) dt \quad (i = 5, 6, 7),
\]

and

\[
C_i = C_1 \quad \text{for } i = 1, 2, \quad C_i = C_2 \quad \text{for } i = 3, 4.
\]

By Krzyżański’s book (*) we get

\[
\lim_{X \to (0^+, y_0)} Q_6(X) = f_1(y_0)
\]

By continuity of the integrals \( Q_i(X) \) \((i = 1, 2, 3, 4)\) we have

\[
\lim_{X \to (0^+, y_0)} \sum_{i=1}^{4} Q_i(X) = \sum_{i=1}^{4} Q_i(0, y_0) = 0
\]

Moreover,

\[
|Q_6(X)| \leq 2m_1 C_1 x \int_{0}^{\infty} (t + \frac{1}{2} y_0)^{-2} dt \to 0 \quad \text{as } x \to 0^+
\]

and

\[
|Q_7(X)| \leq 2m_1 C_2 x \int_{E^+_2} e^{hv} (t + \frac{1}{2} y_0)^{-2} dv dt \to 0 \quad \text{as } x \to 0^+.
\]

The proof of the condition

\[
\lim_{X \to (0^+, y_0)} (D_y + h) u_2(X) = f_2(x_0)
\]

is similar to the proof of Lemma 6.

Finally we get

**Lemma 7.** If the functions \( f_i \) \((i = 1, 2)\) are bounded for \( s \geq 0, t \geq 0 \) and satisfy conditions 1°, 2° of Lemma 3a and \( x_0 > 0, y_0 > 0 \), then the function \( u_2 \) satisfies the boundary conditions

\[
\lim_{X \to (x_0, 0^+)} (D_y + h) u_2(X) = f_2(x_0),
\]

\[
\lim_{X \to (0^+, y_0)} D_x u_2(X) = f_1(y_0).
\]
By Lemma 3a and 7 we obtain

**Theorem 3.** If the functions \( f_i (i = 1, 2) \) satisfy the assumptions of Lemma 3a and 7, then the function \( u_2 \) given by formula (IIa) is the solution of the problem \((M_2)\).

6. We shall give now the solution of the problems \((M_i) (i = 3, 4)\). Under certain assumptions concerning the function \( F \) we shall prove that the functions

\[ v_i(X) = u_i(X) + T_i(X) \quad (i = 1, 2), \]

where the functions \( u_i \) are defined by formulae (I) and (II) respectively and

\[ T_i(X) = \int \int F(Y) G_i(X, Y) ds dt \quad (i = 1, 2) \]

are the solutions of the problem \((M_i) (i = 3, 4)\),

7. We shall prove some lemmas concerning the integrals \( T_i (i = 1, 2) \).

Let

\[ S_i(X) = 2C_1 \int \int F(Y) \ln r_i^2 ds dt \quad (i = 1, \ldots, 4), \]

and

\[ S_i(X) = C_2 \int \int F(Y) e^{h_i} \ln r_i^2 ds dt dv \quad (i = 5, 6). \]

Let the function \( f \) be bounded and continuous in \( E_2 \) and \( f(Y) = F(Y) \) for \( Y \in E_2^+ \) and let

\[
|f(x + \rho \cos \varphi, y + \rho \sin \varphi)| \leq F_1(\rho), \quad |f(x + \rho \cos \varphi, -y - v + \rho \sin \varphi)| \leq F_2(\rho, v),
\]

for \( x^2 + y^2 \geq \delta^2 > 0, v \geq 0, \rho \geq 0, 0 \leq \varphi \leq 2\pi, \delta \) being a positive number.

Let

\[ S_i^{pq}(X) = \int \int F(Y) D_i^{pq} K_i(X, Y) dY, \]

where \( p, q = 0, 1, 2; 0 \leq p + q \leq 2; i = 2, \ldots, 6. \)

Let \( K_R \) denote the circle with the centre \( X \) and the radius \( R. \)

We shall prove now
Lemma 8. If the functions $F_i$ ($i = 1, 2$) are continuous and bounded for $q > 0$, $v > 0$ and the integrals

$$s_1 = \int_a^\infty F_1(q) e^{lnq} dq, \quad s_2 = \int_a^\infty F_1(q) dq, \quad s_3 = \int_a^\infty F_1(q) q^{-1} dq,$$

$$s_4 = \int_a^\infty \int_a^\infty F_2(q, v) e^{\nu q} lnq dq dv, \quad s_5 = \int_a^\infty \int_a^\infty e^{\nu q} F_2(q, v) dq dv,$$

$$s_6 = \int_a^\infty \int_a^\infty e^{\nu q} F_2(q, v) q^{-1} dq dv$$

are convergent for any $a > 0$, then the integrals $S_{pq}^i$ ($i = 2, \ldots, 6$) are uniformly convergent for $x > 0$, $y > 0$.

Proof. It is sufficient to verify the thesis for the integrals $S_{pq}^i(X)$ ($i = 2, \ldots, 6$).

Applying to the integrals $S_{pq}^i(X)$ ($i = 2, 3, 4$) the transformation

$s = \pm x + \rho \cos \varphi$, \quad $t = \pm y + \rho \sin \varphi$

and to the integrals $S_{pq}^i$ ($i = 5, 6$) the transformation

$s = \pm x + \rho \cos \varphi$, \quad $t = \pm y - v + \rho \sin \varphi$,

where $\rho > \rho_0 > 0$, $0 \leq \varphi \leq 2\pi$, we can easily verify that the common majorant of the integrals $S_{pq}^i(X)$ ($p + q = 0$; $i = 2, 3, 4$) is the integral $s_1$ and for the integrals $S_{pq}^i(X)$ ($p + q = 0$; $i = 5, 6$) is the integral $s_4$, for the integrals $S_{pq}^i(X)$ ($i = 2, 3, 4; p + q = 1$) is the integral $s_2$, for the integrals $S_{pq}^i(X)$ ($i = 5, 6; p + q = 1$) is the integral $s_5$, for the integrals $S_{pq}^i(X)$ ($i = 2, 3, 4; p + q = 2$) is the integral $s_3$, for the integrals $S_{pq}^i(X)$ ($i = 5, 6; p + q = 2$) is the integral $s_6$.

By Lemma 8 follows

Corollary 3. There exist the derivatives $D_{xq}^{pq} S_i(X)$ and

$$D_{xq}^{pq} \left( \int_{E_2^+} F(Y) K_i(X, Y) dY \right) = \int_{E_2^+} F(Y) D_{xq}^{pq} K_i(X, Y) dY$$

$$(i = 2, \ldots, 6; 0 \leq p + q \leq 2).$$

Let

$S_1(X) = S_1^1(X) + S_1^2(X),$

where

$$S_1^1(X) = C_1 \int_{K_R} F(Y) K_1(X, Y) dY,$$

$$S_1^2(X) = C_1 \int_{E_2^+ \setminus K_R} F(Y) K_1(X, Y) dY.$$
We shall prove

**Lemma 9.** If the assumptions of Lemma 8 are satisfied and \( F \in C^1(E_2^+) \), then the functions \( T_i \) \((i = 1, 2)\) satisfy equation (1a) for \( X \in E_2^+ \).

**Proof.** For \( T_1 \) by Corollary 3 we have

\[
\Delta S_i(X) = 2C_1 \int \int_{E_2^+} F(Y) \Delta_x K_i(X, Y) dy = 0 \quad (i = 2, \ldots, 6),
\]

because \( K_i(X, Y) \) \((i = 2, \ldots, 6)\) are the harmonic functions and \(|XY| > \delta > 0\), \( \delta \) being a positive number.

From Krzyżan’s book(1), p. 328, we have

\[
\Delta S_1(X) = \Delta S_1^1(X) + \Delta S_1^2(X) = F(X).
\]

Hence the function \( T_1 \) satisfies equation (1a) in the set \( E_2^+ \).

Similarly we can prove that the function \( T_2 \) satisfies equation (1a) in the set \( E_2^+ \).

8. Let \( Z = E_2^+ \cup Z_1 \cup Z_2 \), where \( Z_1 = \{X: x = 0, y > 0\}, \ Z_2 = \{X: x > 0, y = 0\} \). We shall prove

**Lemma 10.** If the functions \( F_i \) \((i = 1, 2)\) are continuous and bounded for \( q > 0, r \geq 0 \), the integrals \( s_i \) \((i = 1, 2, 4, 5)\) are convergent for any \( a > 0 \), then the integrals \( S_i(X) \) \((i = 1, \ldots, 6)\) and its first derivatives are locally uniformly convergent for every \( X \in Z \).

**Proof.** By Krzyżan’s(1), p. 326, and from the assumptions concerning the functions \( F_i \) \((i = 1, 2)\) we get the thesis of Lemma 10.

**Lemma 11.** Let the assumptions of Lemma 10 be satisfied; then the functions \( T_j \) \((j = 1, 2)\) satisfy the homogeneous boundary conditions

(a) \( \lim D^{j}_{x} T_{i+1}(X) = 0 \quad \text{as} \quad X \to (0^+, y_0), \ y_0 > 0 \quad (i = 0, 1) \)

and

(b) \( \lim (D_y + h) T_i(X) = 0 \quad \text{as} \quad X \to (x_0, 0^+), \ x_0 > 0 \). \]

**Proof.** We shall give the proof for the function \( T_1 \); for the function \( T_2 \) the proof is similar. By uniform convergence of the integrals \( S_i \) \((i = 1, \ldots, 6)\) and its first derivatives in the set \( Z \) follows the continuity of these functions in the set \( Z_1 \cup Z_2 \) and consequently we get

\[
\lim T_1(X) = \sum_{i=1}^{6} S_i(0, y_0) = \int \int_{E_2^+} F(Y) G^1(0, y_0, s, t) dY = 0
\]

as \( X \to (0^+, y_0), \ y_0 > 0 \),
and
\[
\lim_{X \to (x_0, 0^+)} (D_y + h) T_1(X) = \int \int_{\mathbb{E}^2_+} F(Y) (D_y + h) G^1(x_0, 0, s, t) \, dY = 0
\]
as \( X \to (x_0, 0^+), x_0 > 0 \)
because \( G^1(0, y, s, t) = (D_y + h) G^1(x_0, 0, s, t) = 0 \).

By Theorems 2, 3 and Lemmas 8–11 we obtain

**Theorem 4.** If the assumptions of Theorems 2, 3 and Lemma 8 are satisfied, then the functions \( v_i \) \((i = 1, 2)\) are solutions of the problems \( M_i \) \((i = 3, 4)\) respectively.