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On problem of density of $C_0^\infty(\Omega)$ in generalized Orlicz–Sobolev space $W_M^k(\Omega)$ for every open set $\Omega \subsetneq \boldsymbol{R}^n$

Let Ω be an open set in $\boldsymbol{R}^n$, μ = Lebesgue measure in $\boldsymbol{R}^n$, $\boldsymbol{R}_+ = \langle 0, \infty)$, W_+ = the set of all non-negative rational numbers and let M be a real-valued function defined in $\Omega \times \boldsymbol{R}_+$ and satisfying the following conditions:

1° $M(u, v)$ is an N-function(1) of the variable v for almost every $u \in \Omega$,

2° $M(u, v)$ is a continuous function of the variable u for each $v \in \boldsymbol{R}_+$,

3° there is a constant $c > 0$ such that $M(u, 1) \geqslant c$ for almost every $u \in \Omega$.

We denote by F_1 the set of all complex-valued functions f defined and Lebesgue measurable in Ω. F_0 will denote the set of functions $f \in F_1$ such that $f(x) = 0$ almost everywhere and F will be defined as $F = F_1/F_0$. According to the assumptions, the function $M(x, |f(x)|)$ is Lebesgue measurable for each $f \in F_1$ and $M(x, |f_1(x)|) = M(x, |f_2(x)|)$ almost everywhere, if $f_1 - f_2 \in F_0$.

Given a non-negative integer k, we denote

$$W_M^k(\Omega) = \{f \in F: \bigvee_{|\alpha| \leqslant k} \exists D^\alpha f \in L_M^*(\Omega)\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i are non-negative integers, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D^\alpha = \partial^{|\alpha|}/\partial x^{\alpha_1} \ldots \partial x_n^{\alpha_n}$ is the generalized derivative operator (i.e., derivative in the sense of the theory of distributions) of order $|\alpha|$ and $L_M^*(\Omega)$ is the generalized Orlicz space (see [8]). Let

$$N(u, v) = \sup_{\sigma \geqslant 0} \{\sigma v - M(u, \sigma)\} = \sup_{\sigma \in W_+} \{\sigma v - M(u, \sigma)\}, \quad u \in \Omega,$$

i.e., $N(u, v)$ is the N-function of the variable v complementary to N-func-

(1) See [6], p. 12–75.

tion $M(u,v)$ in the sense of Young with respect to the variable v. Further, let us denote

$$\varrho(f) = \int_\Omega M\big(x, |f(x)|\big)dx, \quad f\in L_M^*(\Omega),$$

$$\varrho_a(f) = \int_\Omega M\big(x, |D^a f(x)|\big)dx, \quad f\in W_M^k(\Omega), \quad |a|\leqslant k,$$

$$\bar\varrho(f) = \sum_{|a|\leqslant k} \varrho_a(f), \quad f\in W_M^k(\Omega),$$

$$\|f\|_{L_M(\Omega)} = \inf\left\{\varepsilon > 0:\ \varrho\left(\frac{f}{\varepsilon}\right)\leqslant 1\right\}, \quad f\in L_M^*(\Omega),$$

$$\|f\|_{L_M(\Omega)}^1 = \sup_{\|g\|_{L_N(\Omega)}\leqslant 1}\left\{\left|\int_\Omega f(x)g(x)dx\right|\right\}, \quad f\in L_M^*(\Omega)\ (^2),$$

$$\|f\|_{W_M^k(\Omega)} = \inf\left\{\varepsilon > 0:\ \bar\varrho\left(\frac{f}{\varepsilon}\right)\leqslant 1\right\}, \quad f\in W_M^k(\Omega),$$

$$\Omega_\varepsilon = \{x\in\Omega:\ \inf_{y\in\Gamma(\Omega)}\|x-y\|\geqslant\varepsilon\}, \quad \text{where } \|x\| = \Big(\sum_{i=1}^n x_i^2\Big)^{1/2},$$

and $\Gamma(\Omega)$ is the boundary of Ω.

The Hölder's inequality (see [6], p. 98)

$$\left|\int_\Omega f(x)g(x)dx\right|\leqslant \|f\|_{L_M(\Omega)}^1\,\|g\|_{L_N(\Omega)}, \quad f\in L_M^*(\Omega),\ g\in L_N^*(\Omega), \tag{1}$$

and the inequality (see [6], p. 97)

$$\|f\|_{L_M(\Omega)}\leqslant\|f\|_{L_M(\Omega)}^1\leqslant 2\,\|f\|_{L_M(\Omega)}, \quad f\in L_M^*(\Omega) \tag{2}$$

hold.

First, we shall prove some lemmas.

LEMMA 1. *Let $\Omega\subset \boldsymbol{R}^n$, $\Omega_1\subset\boldsymbol{R}^m$ be open sets and let $f(x,y)$ be a complex-valued function defined and Lebesgue measurable in $\Omega\times\Omega_1$ and $f(\cdot,y)\in L_M^*(\Omega)$ for almost every $y\in\Omega_1$. Then the following inequality* (³) *holds:*

$$\left\|\int_{\Omega_1} f(\cdot,y)dy\right\|_{L_M(\Omega)}^1\leqslant\int_{\Omega_1}\|f(\cdot,y)\|_{L_M(\Omega)}^1dy. \tag{3}$$

Proof. Let $J(x) = \int_{\Omega_1} f(x,y)dy$, $g\in L_N^*(\Omega)$ and $\|g\|_{L_N(\Omega)}\leqslant 1$, where $N(u,v)$ is the complementary function to the N-function $M(u,v)$ in the sense of Young with respect to the variable v. Then

$$\begin{aligned}\left|\int_\Omega J(x)g(x)dx\right| &\leqslant \int_\Omega |g(x)|\Big(\int_{\Omega_1}|f(x,y)|\,dy\Big)dx\\ &\leqslant \int_\Omega \|f(\cdot,y)\|_{L_M(\Omega)}^1\|g\|_{L_N(\Omega)}dy\leqslant\int_\Omega\|f(\cdot,y)\|_{L_M(\Omega)}^1dy.\end{aligned}$$

(²) See [6], p. 98.

(³) This inequality is a generalization of Minkowski inequality (see [2], p. 148).

Hence and by definition of $\|J\|^1_{L_M(\Omega)}$, we obtain (3).

LEMMA 2. *Let $f \in W^k_M(\Omega)$. If for every compact set S in $\boldsymbol{R}^n$,*

$$\|f\|_{L_1[S\cap(\Omega\setminus\Omega_\varepsilon)]} = O(\varepsilon^k), \quad \text{as } \varepsilon\to 0, \tag{4}$$

then the function

$$\Phi(x) = \begin{cases} f(x), & x\in\Omega, \\ 0, & x\notin\Omega \end{cases} \tag{5}$$

belongs to $W^k_M(\boldsymbol{R}^n)$.

Proof. It is known (see [1], p. 55–56) that there exist generalized derivatives $D^\alpha\Phi$ for $|\alpha|\leqslant k$ and

$$D^\alpha\Phi(x) = \begin{cases} D^\alpha f(x), & x\in\Omega, \\ 0, & x\notin\Omega. \end{cases}$$

Hence, Lemma 2 follows immediately.

LEMMA 3. *Let $f\in W^k_M(\Omega)$. If there exists a sequence of functions $\varphi_s\in C^\infty_0(\Omega)$ such that*

$$\lim_{s\to\infty}\|f-\varphi_s\|_{W^k_M(\Omega)} = 0,$$

then $\Phi\in W^k_M(\boldsymbol{R}^n)$.

Proof. It can be assumed that $\varphi_s\in C^\infty_0(\boldsymbol{R}^n)$ for $s = 1, 2, 3, \ldots$ Thus, $\{\varphi_s\}^\infty_{s=1}\subset W^k_M(\boldsymbol{R}^n)$, and

$$\|\varphi_s-\varphi_t\|_{W^k_M(\boldsymbol{R}^n)} \leqslant \|\varphi_s-f\|_{W^k_M(\Omega)} + \|f-\varphi_t\|_{W^k_M(\Omega)}\to 0, \quad \text{as } s, t\to\infty.$$

Thus, the sequence $\{\varphi_s\}^\infty_{s=1}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{W^k_M(\boldsymbol{R}^n)}$. But the space $W^k_M(\boldsymbol{R}^n)$ is complete (see [3]), so there exists a function $F\in W^k_M(\boldsymbol{R}^n)$ such that

$$\lim_{s\to\infty}\|F-\varphi_s\|_{W^k_M(\boldsymbol{R}^n)} = 0.$$

Hence, in particular, it follows that

$$\lim_{s\to\infty}\|F-\varphi_s\|_{L_M(\boldsymbol{R}^n)} = 0. \tag{6}$$

Since $\varphi_s(x) = 0$ for $x\in\boldsymbol{R}^n\setminus\Omega$, so from the last condition it follows that $\|F\|_{L_M(\boldsymbol{R}^n\setminus\Omega)} = 0$. Thus, $F(x) = 0$ for almost every $x\in\boldsymbol{R}^n\setminus\Omega$. From (6) it follows that

$$\|F-f\|_{L_M(\Omega)} \leqslant \|F-\varphi_s\|_{L_M(\boldsymbol{R}^n)} + \|f-\varphi_s\|_{L_M(\Omega)}\to 0, \quad \text{as } s\to\infty.$$

Hence $F(x) = f(x)$ for almost every $x\in\Omega$, and so $F(x) = \Phi(x)$ for almost every $x\in\boldsymbol{R}^n$ and $\Phi\in W^k_M(\boldsymbol{R}^n)$. Thus, the proof of Lemma 3 is complete.

Now, we denote:

$$f_\varepsilon(x) = \varepsilon^{-n} \int_{\Omega_{2\varepsilon}} \psi\left(\frac{x-y}{\varepsilon}\right) f(y)\,dy, \quad f \in L^1_{\mathrm{loc}}(\Omega)$$

$$f'_\varepsilon(x) = \varepsilon^{-n} \int_{\Omega} \psi\left(\frac{x-y}{\varepsilon}\right) f(y)\,dy, \quad f \in L^1_{\mathrm{loc}}(\Omega)\ (^4),$$

where $\psi \in C^\infty(\boldsymbol{R}^n)$, $\operatorname{supp}\psi = \bar{K}(0,1) = \{x \in \boldsymbol{R}^n\colon \|x\| \leqslant 1\}$, $\psi(y) = \psi(x)$ for $\|x\| = \|y\|$, $\psi(x) \geqslant 0$, $\int_{\boldsymbol{R}^n} \psi(x)\,dx = 1$. Obviously, $f_\varepsilon, f'_\varepsilon \in C^\infty(\boldsymbol{R}^n)$. If Ω is bounded, then $f_\varepsilon \in C_0^\infty(\Omega)$, $f'_\varepsilon \in C_0^\infty(\boldsymbol{R}^n)$. Further, let us introduce the following condition:

4° there is a constant $\varkappa \geqslant 2$ such that inequality

$$(\Delta_2) \qquad M(u, 2v) \leqslant \varkappa M(u, v)$$

is satisfied for almost every $u \in \Omega$ and each $v \in \boldsymbol{R}_+$.

Let $p(u)$ be a continuous function in $\boldsymbol{R}^n$ satisfying the following conditions: there exist positive constants c_1, c_2 such that $p(u) \geqslant c_1$ for almost every $u \in \boldsymbol{R}^n$, $p(u^{(1)} + u_2^{(2)}) \leqslant c_2 p(u^{(1)}) p(u^{(2)})$ for $\|u^{(1)}\| \leqslant \|u^{(2)}\|$ (⁵). Let us put

$$M(u, v) = p(u) M_1(v),$$

where $M_1(v)$ is a real-valued N-function satisfying condition (Δ_2) for every $v \geqslant 0$ (see [7], p. 35)(⁶). As well known(⁷), such a function $M(u, v)$ satisfies the following condition

5° $\int_A M(x, |f'_\varepsilon(x)|)\,dx \leqslant \boldsymbol{C} \int_A M(x, |f(x)|)\,dx$, $0 < \varepsilon \leqslant 1$, $A \subset \Omega$, is a measurable set, where $\boldsymbol{C}$ is an absolute constant. For such $M(u, v)$ we shall prove the following

THEOREM 1. *If $f \in W_M^k(\Omega)$ and the condition*

$$(7) \qquad \|f\|_{L_M(\Omega \setminus \Omega_\varepsilon)} = O(\varepsilon^k) \quad \textit{holds, as } \varepsilon \to 0,$$

then there exists a sequence of functions $\varphi_s \in C_0^\infty(\Omega)$ such that

$$(8) \qquad \|f - \varphi_s\|_{W_M^k(\Omega)} \to 0, \quad \textit{as } s \to \infty.$$

(⁴) If $f \in L_M^*(\Omega)$, then $f \in L^1_{\mathrm{loc}}(\Omega)$ (see [3], Lemma 1).

(⁵) See [4], Examples (22), (23), (24).

(⁶) In the case $\mu(\Omega) < \infty$ it suffices to assume that condition (Δ_2) is satisfied for large v.

(⁷) See [4], Example 1.

Proof. We shall consider two cases.

1° Ω is an open and bounded set in $\boldsymbol{R}^n$. Then $f_\varepsilon \in C_0^\infty(\Omega)$ for $\varepsilon > 0$. We show that the functions f_ε satisfy condition (8), as $\varepsilon \to 0$. From (7) and the inequality $\|f\|_{L_1(\Omega)} \leqslant C\|f\|_{L_M(\Omega)}$ (see [3], Lemma 1) it follows that $\Phi \in W_M^k(\boldsymbol{R}^n)$ (see Lemma 2). Furthermore, $f_\varepsilon(x) = f_\varepsilon'(x)$ for $x \in \Omega_{3\varepsilon}$. Indeed, if $x \in \Omega_{3\varepsilon}$ and $y \in \Omega \setminus \Omega_{2\varepsilon}$, then $\|x-y\| \geqslant \varepsilon$. Hence $\psi\left(\frac{x-y}{\varepsilon}\right) = 0$ for $y \in \Omega \setminus \Omega_{2\varepsilon}$, $x \in \Omega_{3\varepsilon}$ and

$$f_\varepsilon'(x) = \varepsilon^{-n} \int\limits_{\Omega_{2\varepsilon}} \psi\left(\frac{x-y}{\varepsilon}\right) f(y)\,dy = f_\varepsilon(x).$$

Thus

$$\|\Phi - \Phi_\varepsilon\|_{W_M^k(\boldsymbol{R}^n)} = \|\Phi - \Phi_\varepsilon'\|_{W_M^k(\boldsymbol{R}^n)} \to 0, \quad \text{as } \varepsilon \to 0 \ (^8),$$

and

$$\|f - f_\varepsilon\|_{W_M^k(\Omega_{3\varepsilon})} = \|f - f_\varepsilon'\|_{W_M^k(\Omega_{3\varepsilon})} \leqslant \|\Phi - \Phi_\varepsilon'\|_{W_W^k(\boldsymbol{R}^n)} \to 0, \quad \text{as } \varepsilon \to 0.$$

Next, we can write

$$\|f - f_\varepsilon\|_{W_M^k(\Omega)} \leqslant \|f - f_\varepsilon\|_{W_M^k(\Omega_{3\varepsilon})} + \|f\|_{W_M^k(\Omega\setminus\Omega_{3\varepsilon})} + \|f_\varepsilon\|_{W_M^k(\Omega\setminus\Omega_{3\varepsilon})}.$$

We have $\|f\|_{W_M^k(\Omega\setminus\Omega_{3\varepsilon})} \to 0$ as $\varepsilon \to 0$, by virtue of Lebesgue bounded convergence theorem, and by condition 4°. Now, we prove that $\|f_\varepsilon\|_{W_M^k(\Omega\setminus\Omega_{3\varepsilon})} \to 0$ as $\varepsilon \to 0$. Since the function $M(u, v)$ satisfies condition 4° and the following inequality holds

$$\max_{|\alpha|\leqslant k} \|D^\alpha f\|_{L_M(\Omega)} \leqslant \|f\|_{W_M^k(\Omega)} \leqslant c_{n,k} \max_{|\alpha|\leqslant k} \|D^\alpha f\|_{L_M(\Omega)} \ (^9), \tag{9}$$

where $c_{n,k} = \sum_{|\alpha|\leqslant k} 1$, it suffices to show that $\int_{\Omega\setminus\Omega_{3\varepsilon}} M\big(x, |D^\alpha f_\varepsilon(x)|\big)\,dx \to 0$, as $\varepsilon \to 0$, for $|\alpha| \leqslant k$. Since $D^\alpha f_\varepsilon(x) = 0$ for $x \in \Omega \setminus \Omega_\varepsilon$, $|\alpha| \leqslant k$, so we have

$$\int\limits_{\Omega\setminus\Omega_{3\varepsilon}} M\big(x, |D^\alpha f_\varepsilon(x)|\big)\,dx = \int\limits_{\Omega_\varepsilon\setminus\Omega_{3\varepsilon}} M\big(x, |D^\alpha f_\varepsilon(x)|\big)\,dx \quad \text{for } |\alpha| \leqslant k,\ \varepsilon > 0.$$

Clearly, we have

$$D^\alpha f_\varepsilon(x) = \frac{1}{\varepsilon^{n+|\alpha|}} \int\limits_{\substack{x-z\in\Omega_{2\varepsilon}\\ |z|\leqslant\varepsilon}} D^\alpha \Psi\left(\frac{z}{\varepsilon}\right) f(x-z)\,dz.$$

(8) See [4], Theorem 5 for $\Omega = \boldsymbol{R}^n$.

(9) See [3], Proposition 2.

Now, let us put

$$c_a = \int_{|z|\leqslant 1} |D^a\Psi(z)|\,dz \quad \text{for } |a| \leqslant k\,(^{10}).$$

Then $\varepsilon^{-n} \int_{|z|\leqslant\varepsilon} |D^a\Psi(z/\varepsilon)|dz = c_a$. Applying the integral Jensen's inequality, we obtain

$$M\big(x, |D^a f_\varepsilon(x)|\big) \leqslant M\left(x, \frac{1}{\varepsilon^{|a|}} \int_{|z|\leqslant\varepsilon} \frac{|D^a\Psi(z/\varepsilon)|}{\varepsilon^n c_a} \cdot c_a |f(x-z)|\,dz\right)$$
$$\leqslant \int_{|z|\leqslant\varepsilon} \frac{|D^a\Psi(z/\varepsilon)|}{\varepsilon^n c_a} M\left(x, \frac{c_x|f(x-z)|}{\varepsilon^{|a|}}\right)dz.$$

Hence, by virtue of Tonelli theorem, we have

(10)

$$\int_{\Omega_\varepsilon\setminus\Omega_{3\varepsilon}} M\big(x, |D^a f_\varepsilon(x)|\big)\,dx \leqslant \int_{\Omega_\varepsilon\setminus\Omega_{3\varepsilon}} \left\{\int_{|z|\leqslant\varepsilon} \frac{|D^a f(x-z)|}{\varepsilon^n c_a} M\left(x, \frac{c_a|f(x-z)|}{\varepsilon^{|a|}}\right)dz\right\}dx$$
$$\leqslant \varkappa^{n_a} \int_{\Omega_\varepsilon\setminus\Omega_{3\varepsilon}} \left\{\int_{|z|\leqslant\varepsilon} \frac{D^a\Psi(z/\varepsilon)}{\varepsilon^n c_a} M\left(x, \frac{|f(x-z)|}{\varepsilon^{|a|}}\right)dz\right\}dx$$
$$= \varkappa^{n_a} \int_{|z|\leqslant\varepsilon} \frac{D^a\Psi(z/\varepsilon)}{\varepsilon^n c_a} \left\{\int_{\Omega_\varepsilon\setminus\Omega_{3\varepsilon}} M\left(x, \frac{|f(x-z)|}{\varepsilon^{|a|}}\right)dx\right\}dz,$$

where n_a is a natural number such that $c_a \leqslant 2^{n_a}$, the constant $\varkappa$ is defined by condition 4°. Let us assume

$$J_1(z) = \int_{\Omega_\varepsilon\setminus\Omega_{3\varepsilon}} M\left(x, \frac{|f(x-z)|}{\varepsilon^{|a|}}\right)dx \quad \text{for } |z| \leqslant \varepsilon \leqslant 1,$$

and let us estimate $J_1(z)$. Let $u = x-z$; then $|du| = |dz|$. If $\|z\| \leqslant \varepsilon$, then $(\Omega_\varepsilon\setminus\Omega_{3\varepsilon})-z \subset \Omega\setminus\Omega_{4\varepsilon}$. Hence for $\|z\| \leqslant \varepsilon \leqslant 1$, we have

(11) $J_1(z)$

$$= \int_{\Omega_\varepsilon\setminus\Omega_{3\varepsilon}} M\left(x, \frac{|f(x-z)|}{\varepsilon^{|a|}}\right)dx = \int_{(\Omega_\varepsilon\setminus\Omega_{3\varepsilon})-z} M\left(u+z, \frac{|f(u)|}{\varepsilon^{|a|}}\right)du$$
$$\leqslant c_2 \int_{\Omega\setminus\Omega_{4\varepsilon}} p(u)p(z)M_1\left(\frac{|f(u)|}{\varepsilon^{|a|}}\right)du \leqslant c_2 p(1) \int_{\Omega\setminus\Omega_{4\varepsilon}} M\left(x, \frac{|f(x)|}{\varepsilon^{|a|}}\right)dx,$$

where $p(1) = \sup_{\|z\|=1} p(z)$.

(^{10}) It can be assumed that c_a for $|a| \leqslant k$.

By (10) and (11), for $0 < \varepsilon \leqslant \frac{1}{4}$, we have

$$\int_{\Omega_\varepsilon \setminus \Omega_{3\varepsilon}} M\big(x, |D^\alpha f_\varepsilon(x)|\big)\,dx \leqslant c_2 \varkappa^{n_\alpha} p(1) \int_{\Omega \setminus \Omega_{4\varepsilon}} M\left(x, \frac{|f(x)|}{\varepsilon^{|\alpha|}}\right) dx$$

$$\leqslant c_2 \varkappa^{n_\alpha + 2|\alpha|} p(1) \int_{\Omega \setminus \Omega_{4\varepsilon}} M\left(x, \frac{|f(x)|}{(4\varepsilon)^{|\alpha|}}\right) dx \leqslant c_2 \varkappa^{n_\alpha + 2|\alpha|} p(1) \int_{\Omega \setminus \Omega_{4\varepsilon}} M\left(x, \frac{|f(x)|}{(4\varepsilon)^{k}}\right) dx \to 0,$$

as $\varepsilon \to 0$, $|\alpha| \leqslant k$. Thus, the proof in case 1° is complete.

2° Ω is an arbitrary open set in $\boldsymbol{R}^n$. Let $g_\varepsilon(x) = f_\varepsilon(x)\chi_\varepsilon(x)$, where $\chi_\varepsilon(x) = \chi(\varepsilon x)$, $\chi \in C^\infty(\boldsymbol{R}^n)$ and $\chi(x) = 1$ for $\|x\| \leqslant 1$, $\chi(x) = 0$ for $\|x\| \geqslant 2$, $0 \leqslant \chi(x) \leqslant 1$. Obviously, $g_\varepsilon \in C_0^\infty(\Omega)$ and $\operatorname{supp} g_\varepsilon \subset \bar{\Omega}_\varepsilon \cap \{x \in \boldsymbol{R}^n\colon \|x\| \leqslant 2/\varepsilon\}$. We shall prove that the sequence g_ε satisfies condition (8), as $\varepsilon \to 0$. We have

$$(12) \quad D^\alpha g_\varepsilon(x) = D^\alpha f_\varepsilon(x)\chi_\varepsilon(x) + \sum_{\substack{\beta+\gamma=\alpha \\ \gamma \neq 0}} C_{\beta\gamma} D^\beta f_\varepsilon(x) D^\gamma \chi_\varepsilon(x), \qquad |\alpha| \leqslant k.$$

Similarly as in case 1°, we have

$$\|f - g_\varepsilon\|_{W_M^k(\Omega)} \leqslant \|f - g_\varepsilon\|_{W_M^k(\Omega_{3\varepsilon})} + \|f\|_{W_M^k(\Omega \setminus \Omega_{3\varepsilon})} + \|g_\varepsilon\|_{W_M^k(\Omega \setminus \Omega_{3\varepsilon})}.$$

It is known that $\|g_\varepsilon\|_{W_M^k(\Omega \setminus \Omega_{3\varepsilon})} \to 0$, as $\varepsilon \to 0$ (see case 1°). In order to prove that $\|g_\varepsilon\|_{W_M^k(\Omega \setminus \Omega_{3\varepsilon})} \to 0$, as $\varepsilon \to 0$ it suffices to show that each component of the sum (12) tends to zero with respect to $\varrho_{L_M(\Omega \setminus \Omega_{3\varepsilon})}$-convergence. We shall apply also case 1°. Let $|\alpha| \leqslant k$. Since for every $x \in \Omega$,

$$|D^\alpha f_\varepsilon(x)\chi_\varepsilon(x)| \leqslant |D^\alpha f_\varepsilon(x)|,$$

so by case 1°, $\|D^\alpha f_\varepsilon \chi_\varepsilon\|_{W_M^k(\Omega \setminus \Omega_{3\varepsilon})} \to 0$, as $\varepsilon \to 0$. Now, let $\beta + \gamma = \alpha$, $\gamma \neq 0$, then $D^\gamma[\chi_\varepsilon(x)] = \varepsilon^{|\gamma|}\chi(\varepsilon x)$.

Denoting

$$_{1/\varepsilon}\boldsymbol{P}_{2/\varepsilon} = \left\{x \in \boldsymbol{R}^n\colon \frac{1}{\varepsilon} \leqslant \|x\| \leqslant \frac{2}{\varepsilon}\right\},$$

we have $D^\gamma[\chi_\varepsilon(x)] = 0$ for $x \in \boldsymbol{R}^n \setminus {}_{1/\varepsilon}\boldsymbol{P}_{2/\varepsilon}$. Moreover, if $\gamma \neq 0$ the following inequality holds:

$$(13) \qquad |D^\beta f_\varepsilon(x) D^\gamma[\chi_\varepsilon(x)]| \leqslant c_\gamma \varepsilon^{|\gamma|} |D^\beta f_\varepsilon(x)| \chi_{{}_{1/\varepsilon}\boldsymbol{P}_{2/\varepsilon}}(x),$$

where $c_\gamma = \sup_{1 \leqslant \|x\| \leqslant 2} |D^\gamma \chi(x)|$.

Hence, by virtue of the proof of case 1° and condition 4°, for $\gamma \neq 0$, $\beta+\gamma = \alpha$, we have

$$\int_{\Omega\setminus\Omega_{3\varepsilon}} M\big(x, |D^\beta f_\varepsilon(x) D^\gamma[\chi_\varepsilon(x)]|\big)dx \to 0, \quad \text{as } \varepsilon\to 0.$$

Thus

$$\|D^\beta f_\varepsilon D^\gamma \chi_\varepsilon\|_{L_M(\Omega\setminus\Omega_{3\varepsilon})} \to 0, \quad \text{as } \varepsilon\to 0 \text{ for } |\beta+\gamma| = |\alpha| \leqslant k,\ \gamma\neq 0.$$

Now, we prove that

$$\|f-g_\varepsilon\|_{W^k_M(\Omega_{3\varepsilon})} \to 0, \quad \text{as } \varepsilon\to 0. \tag{14}$$

Since $f'_\varepsilon(x) = f_\varepsilon(x)$ for $x\in\Omega_{3\varepsilon}$, so

$$\|f-g_\varepsilon\|_{W^k_M(\Omega_{3\varepsilon})} = \|\Phi-\Phi_\varepsilon\chi_\varepsilon\|_{W^k_M(\Omega_{3\varepsilon})} \leqslant \|\Phi-\Phi'_\varepsilon\chi_\varepsilon\|_{W^k_M(\mathbf{R}^n)},$$

where Φ is defined by (5). It is known, that $\Phi\in W^k_M(\mathbf{R}^n)$ and

$$\|\Phi-\Phi'_\varepsilon\|_{W^k_M(\mathbf{R}^n)} \to 0, \quad \text{as } \varepsilon\to 0. \tag{15}$$

By virtue of (9) it suffices to show that

$$\|D^\alpha\Phi - D^\alpha(\Phi'_\varepsilon\chi_\varepsilon)\|_{L_M(\mathbf{R}^n)} \to 0, \quad \text{as } \varepsilon\to 0,\ |\alpha|\leqslant k.$$

Further, we have

$$\int_{\mathbf{R}^n} M\big(x, |\Phi(x)-\Phi'_\varepsilon(x)\chi_\varepsilon(x)|\big)dx = \int_{\|x\|\leqslant 1/\varepsilon} M\big(x, |\Phi(x)-\Phi'_\varepsilon(x)|\big)dx + \tag{16}$$
$$+\int_{1/\varepsilon<\|x\|\leqslant 2/\varepsilon} M\big(x, |\Phi(x)-\Phi'_\varepsilon(x)\chi_\varepsilon(x)|\big)dx + \int_{\|x\|>2/\varepsilon} M\big(x, |\Phi(x)|\big)dx.$$

By (15),

$$\int_{\|x\|\leqslant 1/\varepsilon} M\big(x, |\Phi(x)-\Phi'_\varepsilon(x)|\big)dx \leqslant \int_{\mathbf{R}^n} M\big(x, |\Phi(x)-\Phi'_\varepsilon(x)|\big)dx \to 0, \quad \text{as } \varepsilon\to 0.$$

We have also

$$\int_{\|x\|>2/\varepsilon} M\big(x, |\Phi(x)|\big)dx \to 0, \quad \text{as } \varepsilon\to 0. \tag{17}$$

Indeed, by virtue of condition 4°, the function $M\big(x, |\Phi(x)|\big)$ belongs to $L_1(\mathbf{R}^n)$. Furthermore, $\Phi(x)\chi_{\mathbf{R}^n\setminus\overline{K}(0,2/\varepsilon)}(x)$, where $\overline{K}(0,2/\varepsilon) = \{x\in\mathbf{R}^n\colon \|x\|\leqslant 2/\varepsilon\}$, is convergent to zero for almost every $x\in\mathbf{R}^n$, as $\varepsilon\to 0$. By continuity of the function $M(u,v)$ with respect to v, $M\big(x, |\Phi(x)\chi_{\mathbf{R}^n\setminus\overline{K}(0,2/\varepsilon)}(x)|\big)\to 0$, as $\varepsilon\to 0$ for almost every $x\in\mathbf{R}^n$. Moreover, $M\big(x, |\Phi(x)\chi_{\mathbf{R}^n\setminus\overline{K}(0,2/\varepsilon)}(x)|\big) \leqslant M\big(x, |\Phi(x)|\big)$ for every $x\in\mathbf{R}^n$. Thus, by virtue of the Lebesgue bounded-convergence theorem we obtain (17). Now, we consider the integral

$$\int_{1/\varepsilon<\|x\|\leqslant 2/\varepsilon} M\big(x, |\Phi(x)-\Phi'_\varepsilon(x)\chi_\varepsilon(x)|\big)dx, \quad \text{as } \varepsilon\to 0. \tag{17'}$$

We have, $|\Phi(x)-\Phi_\varepsilon'(x)\chi_\varepsilon(x)| \leqslant |\Phi(x)|+|\Phi_\varepsilon'(x)|$, for $x\in \mathbf{R}^n$. Hence, by convexity of $M(u,v)$ with respect to the variable v and by condition 5°, for $0<\varepsilon\leqslant 1$, we have

$$\int\limits_{1/\varepsilon<\|x\|\leqslant 2/\varepsilon} M\big(x,\tfrac{1}{2}|\Phi(x)-\Phi_\varepsilon'(x)\chi_\varepsilon(x)|\big)dx \leqslant \int\limits_{\|x\|>1/\varepsilon} M\big(x,|\Phi(x)|+|\Phi_\varepsilon'(x)|\big)dx$$
$$\leqslant \tfrac{1}{2}\int\limits_{\|x\|>1/\varepsilon} M\big(x,|\Phi(x)|\big)dx+\tfrac{1}{2}\int\limits_{\|x\|>1/\varepsilon} M\big(x,|\Phi_\varepsilon'(x)|\big)dx$$
$$\leqslant \tfrac{1}{2}(1+C)\int\limits_{\|x\|>1/\varepsilon} M\big(x,|\Phi(x)|\big)dx\to 0, \quad \text{as } \varepsilon\to 0.$$

Thus, integral (17′) tends to zero, as $\varepsilon\to 0$. By (12) for $|\alpha|\leqslant k$, we have

$$\|D^\alpha(\Phi-g_\varepsilon)\|_{L_M(\mathbf{R}^n)} = \Big\|D^\alpha\Phi-D^\alpha\Phi_\varepsilon'\chi_\varepsilon-\sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq 0}} c_{\beta\gamma}D^\beta\Phi_\varepsilon' D^\gamma\chi_\varepsilon\Big\|_{L_M(\mathbf{R}^n)}$$
$$\leqslant \|D^\alpha\Phi-D^\alpha\Phi_\varepsilon'\chi_\varepsilon\|_{L_M(\mathbf{R}^n)}+\sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq 0}} c_{\beta\gamma}\|(D^\beta\Phi_\varepsilon'-D^\beta\Phi)D^\gamma\chi_\varepsilon\|_{L_M(\mathbf{R}^n)}+$$
$$+\sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq 0}} c_{\beta\gamma}\|D^\beta\Phi D^\gamma\chi_\varepsilon\|_{L_M(\mathbf{R}^n)}.$$

Similarly as (17′), we have also $\|D^\alpha\Phi-D^\alpha\Phi_\varepsilon'\chi_\varepsilon\|_{L_M(\mathbf{R}^n)}\to 0$ as $\varepsilon\to 0$. Let $\beta+\gamma=\alpha$, $\gamma\neq 0$. Then, by virtue of (13) and (15), we obtain

$$\|(D^\beta\Phi-D^\beta\Phi_\varepsilon')D^\gamma\chi_\varepsilon\|_{L_M(\mathbf{R}^n)}\leqslant \varepsilon^{|\gamma|}c_\gamma\|D^\beta\Phi-(D^\beta\Phi)_\varepsilon'\|_{L_M(\mathbf{R}^n)}\to 0,$$

as $\varepsilon\to 0$. Moreover, by (13), we obtain

$$\|D^\beta\Phi D^\gamma\chi_\varepsilon\|_{L_M(\mathbf{R}^n)}\leqslant c_\gamma\varepsilon^{|\gamma|}\|D^\beta\Phi\|_{L_M(1/\varepsilon, P_{2/\varepsilon})}\to 0, \quad \text{as } \varepsilon\to 0,$$
$$|\beta+\gamma|=|\alpha|\leqslant k,\ \gamma\neq 0.$$

Hence

$$\|D^\alpha(\Phi-g_\varepsilon)\|_{L_M(\mathbf{R}^n)}\to 0, \quad \text{as } \varepsilon\to 0 \text{ and for every } |\alpha|\leqslant k.$$

Thus, by virtue of (9), it follows that $\|f-g_\varepsilon\|_{W^k_M(\Omega_{3\varepsilon})}\to 0$ as $\varepsilon\to 0$, and the proof of case 2° is complete.

Now, let us assume that $M(u,v)=M(v)$, where $M(v)$ is an N-function satisfying condition (Δ_2) for every $v\geqslant 0$. Thus, there exists[11] a real-valued and convex function K defined on $\mathbf{R}_+$ such that: $K(u)=0$ if and only if $u=0$, $K(1)=1$, K is an increasing, continuous function for $u\geqslant 0$ and

$$(\Delta_2)\qquad M(u\cdot v)\leqslant K(u)M(v) \quad \text{for every } u,v\geqslant 0.$$

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[11] $K(u)=\sup\limits_{v>0}\dfrac{M(u\cdot v)}{M(v)}$ for every $u\geqslant 0$, see [7], p. 57–58.

Then conditions 1°–5° hold. Let K^{-1} denote the inverse function to K. For such function $M(v)$ we shall prove that if for $f \in W_M^k(\Omega)$ there exists a sequence of functions $\varphi_s \in C_0^\infty(\Omega)$ such that condition (8) holds, then condition (7) is satisfied. First, we shall prove some lemmas.

LEMMA 3. *If the function M is such that $\int_0^1 \frac{dt}{K^{-1}(t^n)} < \infty$, $f \in C^1(\Omega)$, $S \subset \Omega \subset R^n$, $\mu(S) > 0$, Ω is bounded and starlike with respect to S, then the following inequality holds*

$$(18)\qquad \|f\|_{L_M(\Omega)}^1 \leqslant \frac{\mu(\Omega) N^{-1}\left(\frac{1}{\mu(\Omega)}\right)}{\mu(S) N^{-1}\left(\frac{1}{\mu(S)}\right)} \|f\|_{L_M(S)}^1 + C(n, M) d \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{L_M(\Omega)}^1,$$

where $d = \sup_{x', x'' \in \Omega} \|x' - x''\|$, N complementary to the N-function M in the sense of Young.

Proof. Since the set Ω is starlike with respect to S, so for $x \in S$, $y \in \Omega$, $t \in \langle 0, 1\rangle$, we have $x + t(y-x) \in \Omega$. First, we shall prove that

$$(19)\qquad \left\|f\big(x+t(y-x)\big)\right\|_{L_M(\Omega), y} \leqslant \{K^{-1}(t^n)\}^{-1} \|f\|_{L_M(\Omega)}.$$

Let us put $x + t(y-x) = z$. Then

$$\int_\Omega M\left[\left|f\big(x+t(y-x)\big)\right|\right] dy = t^{-n} \int_\Omega M\left[K^{-1}(t^n) \frac{|f(z)|}{K^{-1}(t^n)}\right] dz \leqslant \int_\Omega M\left[\frac{|f(z)|}{K^{-1}(t^n)}\right] dz.$$

Thus, we have

$$\int_\Omega M\left[\frac{\left|f\big(x+t(y-x)\big)\right|}{\|f\|_{L_M(\Omega)}/K^{-1}(t^n)}\right] dy \leqslant \int_\Omega M\left(\frac{|f(z)|}{\|f\|_{L_M(\Omega)}}\right) dz.$$

Consequently, inequality (19) holds. From (19) it follows that

$$\int_0^1 \left\| \frac{\partial f}{\partial x_i}\big(x+t(y-x)\big) \right\|_{L_M(\Omega)} dt \leqslant \left(\int_0^1 \frac{dt}{K^{-1}(t^n)}\right) \left\| \frac{\partial f}{\partial x_i} \right\|_{L_M(\Omega)}.$$

It is known that the following equality holds for $f \in C^1(\Omega)$ (see [1], p. 52):

$$f(y) = f(x) + \sum_{i=1}^{n} \int_0^1 \frac{\partial f}{\partial x_i}\big(x+t(y-x)\big)(y_i - x_i)\, dt.$$

Hence

$$|f(y)| \leqslant |f(x)| + d \sum_{i=1}^{n} \int_0^1 \left| \frac{\partial f}{\partial x_i}\big(x+t(y-x)\big)\right| dt \qquad \text{for } x \in S,\, y \in \Omega,$$

and thus applying inequalities (2) and (3), we obtain

$$\|f\|^1_{L_M(\Omega),y} \leqslant \|f(x)\|^1_{L_M(\Omega),y} + d\sum_{i=1}^{n}\int_0^1 \left\| \frac{\partial f}{\partial x_i}(x+t(y-x)) \right\|^1_{L_M(\Omega),y} dt$$

$$\leqslant \|f(x)\|^1_{L_M(\Omega),y} + 2d\left(\int_0^1 \frac{dt}{K^{-1}(t^n)}\right)\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|^1_{L_M(\Omega)}.$$

From [6], equality (9.11), we have

$$\|f(x)\|^1_{L_M(\Omega),y} = |f(x)|\,\mu(\Omega)N^{-1}\left[\frac{1}{\mu(\Omega)}\right],$$

where N^{-1} is the inverse function to $N(v)$, $N(v)$ is the complementary function to the N-function $M(v)$ in the sense of Young. Thus,

$$\|f\|^1_{L_M(\Omega)} \leqslant |f(x)|\,\mu(\Omega)N^{-1}\left[\frac{1}{\mu(\Omega)}\right] + c(n, M)d\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|^1_{L_M(\Omega)}.$$

Taking the norm $\|\cdot\|^1_{L_M(S)}$, we have

$$\|f\|^1_{L_M(\Omega)}\mu(S)N^{-1}\left[\frac{1}{\mu(S)}\right] \leqslant \mu(\Omega)N^{-1}\left[\frac{1}{\mu(\Omega)}\right]\|f\|^1_{L_M(\Omega)} +$$

$$+ C(n, M)d\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|^1_{L_M(\Omega)}\mu(S)N^{-1}\left[\frac{1}{\mu(S)}\right],$$

i.e., we obtain inequality (18).

By inequality (2), we get

$$(20)\quad \|f\|_{L_M(\Omega)} \leqslant 2\,\frac{\mu(\Omega)N^{-1}\left[\dfrac{1}{\mu(\Omega)}\right]}{\mu(S)N^{-1}\left[\dfrac{1}{\mu(s)}\right]}\|f\|_{L_M(S)} + 2C(n, M)d\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_{L_M(\Omega)}.$$

LEMMA 4. *If $f\in C^k(\Omega)$ and Ω is as in Lemma 3, then the followin inequality holds:*

$$(21)\quad \|f\|_{L_M(\Omega)} \leqslant C_3\Big\{(1+d)^{k-1}C(\Omega, S, M)\|f\|_{W_M^{k-1}(S)} + d^k\sum_{|\alpha|=k}\|D^\alpha f\|_{L_M(\Omega)}\Big\},$$

where the constant C_3 depends on n, k, M only, and

$$C(\Omega, S, M) = \frac{2\mu(\Omega)N^{-1}\left[\dfrac{1}{\mu(\Omega)}\right]}{\mu(S)N^{-1}\left[\dfrac{1}{\mu(S)}\right]}.$$

Proof. From (20) it follows that for $k=1$ inequality (21) holds. Let (21) be true for $k-1$. But $D^\alpha f \in C^1(\Omega)$ for $|\alpha| = k-1$, so by virtue of (18), we have

$$\|f\|_{L_M(\Omega)} \leqslant c_3 \Big\{(1+d)^{k-2} C(\Omega, S, M)\|f\|_{W_M^{k-2}(S)} +$$

$$+ d^{k-1} \sum_{|\alpha|=k-1} \Big[C(\Omega, S, M)\|D^\alpha f\|_{L_M(S)} + 2C(n, M) d \sum_{i=1}^{n} \Big\| \frac{\partial f}{\partial x_i}(D^\alpha f) \Big\|_{L_M(\Omega)} \Big]\Big\}$$

$$\leqslant 2c_3 \max(1, C(n, M)) \Big\{(1+d)^{k-1} C(\Omega, S, M)\|f\|_{W^{k-1}_{M(S)}} + d^k \sum_{|\alpha|=k} \|D^\alpha f\|_{L_M(\Omega)}\Big\}.$$

Corollary 1. *If $f \in W_M^k(\Omega)$, Ω is bounded and starlike with respect to S and there exists a sequence of functions $\varphi_s \in C_0^\infty(\Omega)$ satisfying* (8), *then f satisfies* (21).

Corollary 2. *If $f \in W_M^k(\Omega)$, Ω is bounded and starlike with respect to S, then f satisfies* (21).

Proof. This follows from the density of $C^\infty(\Omega)$ in $W_M^k(\Omega)$ (see [5]) and from Lemma 4.

Corollary 3. *If $f \in W_M^k(\Omega)$, where Ω is as in Lemma 3 and $f(x) = 0$ for $x \in S$, then*

$$\|f\|_{L_M(\Omega)} \leqslant c_3 d^k \sum_{|\alpha|=k} \|D^\alpha f\|_{L_M(\Omega)}. \tag{22}$$

Lemma 5. *Let M be such that $\int_0^1 \frac{dt}{K^{-1}(t^n)} < \infty$ and let $\Omega \subset \boldsymbol{R}^n$, $\Omega \neq \boldsymbol{R}^n$ be an open set. Then for every $f \in C_0^\infty(\Omega)$ the following inequality holds:*

$$\|f\|_{L_M(\Omega \setminus \Omega_\varepsilon)} \leqslant c_4 \varepsilon^k \sum_{|\alpha|=k} \|D^\alpha f\|_{L_M(\Omega\setminus\Omega_{2\varepsilon})}, \tag{23}$$

where the constant c_4 depends on n, k, M only.

Proof. Let $B = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $B^n = \overbrace{B \times B \times \ldots \times B}^{n\text{-times}}$, and

$$f_1(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

Further, let $\frac{\varepsilon}{2} l = \left(\frac{\varepsilon}{2} l_1, \ldots, \frac{\varepsilon}{2} l_n\right)$ for $l = (l_1, \ldots, l_n)$ and $K\left(\frac{\varepsilon}{2} l, \frac{\varepsilon}{2}\right) = \left\{x \in \boldsymbol{R}^n : \left\|x - \frac{\varepsilon}{2} l\right\| < \frac{\varepsilon}{2}\right\}$. Then $\bigcup_{l \in B^n} K\left(\frac{\varepsilon}{2} l, \frac{\varepsilon}{2}\right) = \boldsymbol{R}^n$. Let us consider those balls $K\left(\frac{\varepsilon}{2} l, \frac{\varepsilon}{2}\right)$ that $\Omega \setminus \Omega_\varepsilon \cap K\left(\frac{\varepsilon}{2} l, \frac{\varepsilon}{2}\right) \neq \emptyset$ (if Ω is bounded, then there exists a finite number of such balls, if Ω is not bounded, then there exists a countable set of such balls). Let us denote these balls by L_i and their centers by $x^{(i)}$. Further, let $y^{(i)}$ be a point from $\Gamma(\Omega)$ such that $d(x^{(i)}, \Gamma(\Omega)) = \|x^{(i)} - y^{(i)}\|$. Let us denote by Q_i the balls with centers $y^{(i)}$ and radius 2ε. Since $L_i \subset Q_i$, so

$$\|f\|_{L_M(\Omega\setminus\Omega_\varepsilon)} \leqslant \sum_i \|f_1\|_{L_M(L_i)} \leqslant \sum_i \|f_1\|_{L_M(Q_i)}.$$

We have $f(x) = 0$ in $\Omega - \Omega_h$ for sufficiently small h. Hence there exist smaller balls in the balls Q_i in which $f(x) = 0$. By inequality (22), we obtain

$$\|f_1\|_{L_M(Q_i)} \leqslant c_3(2\varepsilon)^k \sum_{|\alpha|=k} \|D^\alpha f_1\|_{L_M(Q_i)}.$$

Thus

$$\|f\|_{L_M(\Omega\setminus\Omega_\varepsilon)} \leqslant c_4' \varepsilon^k \sum_{|\alpha|=k} \sum_i' \|D^\alpha f\|_{L_M(\Omega\setminus\Omega_i)}.$$

Since every point $x \in Q_i$ belongs to at most 16^n balls of the family $\{Q_j\}_{j=1}^\infty$ (see [1], p. 54–55), so

$$\sum_i \chi_{\Omega\cap Q_i}(x) \leqslant 16^n \chi_{\bigcup_i(\Omega\cap Q_i)}(x), \qquad x \in \boldsymbol{R}^n.$$

Hence

$$\|f\|_{L_M(\Omega\setminus\Omega_\varepsilon)} \leqslant c_4' \varepsilon^k 16^n \sum_{|\alpha|=k} \|D^\alpha f\|_{L_M(\Omega\setminus\Omega_{2\varepsilon})},$$

i.e., inequality (23) holds.

Corollary 4. *For every function $f \in W_M^k(\Omega)$ such that there exists a sequence of functions $\varphi_s \in C_0^\infty(\Omega)$ satisfying* (8), *inequality* (23) *holds.*

Proof. If (8) holds, then

$$\|f - \varphi_s\|_{L_M(\Omega)} \to 0 \quad \text{and} \quad \sum_{|\alpha|=k} \|D^\alpha f - D^\alpha \varphi_s\|_{L_M(\Omega\setminus\Omega_\varepsilon)} \to 0, \quad \text{as } \varepsilon \to \infty.$$

Hence, we obtain

$$(24) \quad \|\varphi_s\|_{L_M(\Omega)} \to \|f\|_{L_M(\Omega)} \quad \text{and} \quad \sum_{|\alpha|=k} \|D^\alpha \varphi_s\|_{L_M(\Omega\setminus\Omega_\varepsilon)} \to \sum_{|\alpha|=k} \|D^\alpha f\|_{L_M(\Omega\setminus\Omega_\varepsilon)}.$$

Since (22) is satisfied by the functions φ_s, as $\varepsilon \to 0$, so, by (24), it is satisfied by f, too.

Theorem 2. *If $f \in W_M^k(\Omega)$, where Ω is an arbitrary open set in $\boldsymbol{R}^n$, $M(v)$ satisfies condition (Δ_2), $\int_0^1 \frac{dt}{K^{-1}(t^n)} dt < \infty$ and if there exists a sequence of functions $\varphi_s \in C_0^\infty(\Omega)$ such that* (8) *holds, then condition* (7) *is satisfied.*

Proof. If $\Omega = \boldsymbol{R}^n$, then (7) is satisfied immediately. Let $\Omega \neq \boldsymbol{R}^n$ and choose $\delta > 0$. From (8) it follows that there exists s_0 such that

$$\|f - \varphi_{s_0}\|_{W_M^k(\Omega)} \leqslant \delta.$$

Let ε_0 be such that

$$\|\varphi_{s_0}\|_{L_M(\Omega \setminus \Omega_\varepsilon)} = 0 \quad \text{for } 0 < \varepsilon \leqslant \varepsilon_0.$$

Then, for $0 < \varepsilon \leqslant \varepsilon_0$, by Corollary 4, we have

$$\|f\|_{L_M(\Omega \setminus \Omega_\varepsilon)} \leqslant \|f - \varphi_{s_0}\|_{L_M(\Omega \setminus \Omega_\varepsilon)} + \|\varphi_{s_0}\|_{L_M(\Omega \setminus \Omega_\varepsilon)}$$
$$\leqslant c_4 \varepsilon^k \sum_{|\alpha| = k} \|D^\alpha f - D^\alpha \varphi_{s_0}\|_{L_M(\Omega \setminus \Omega_{2\varepsilon})} \leqslant c_4 \delta \varepsilon^k.$$

Thus, the proof of Theorem 2 is complete.

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