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The Green function for the Lauricelli problem and for polyharmonic equation in the half-space

1. Let $P = (x_1, x_2, \ldots, x_m)$, $Q = (y_1, y_2, \ldots, y_m)$ denote arbitrary points of the space E_m. Let

$$E_m^+ = \{P\colon\ -\infty < x_i < \infty,\ i = 1, 2, \ldots, m-1,\ x_m > 0\}$$

and

$$r^2 = \sum_{i=1}^{m} (x_i - y_i)^2, \qquad r_1^2 = \sum_{i=1}^{m-1} (x_i - y_i)^2 + (x_m + y_m)^2.$$

We shall construct the Green function $G(P, Q)$ for the polyharmonic equation

$$\Delta^p u(Q) = 0 \tag{1}$$

in the domain E_m^+ satisfying the homogeneous boundary conditions of the Lauricelli type

$$D_{y_m}^i G(P, Q)|_{y_m=0} = 0, \qquad i = 0, 1, \ldots, p-1. \tag{2}$$

We shall consider two cases:

(a) $m > 2p$, (b) $m < 2p$ and m is odd.

2. We shall prove some lemmas.

LEMMA 1. *If a sequence* $\{A_n^k\}$ $(n = 1, 2, \ldots;\ k = 1, 2, \ldots, [n/2]+1)$ *satisfies the conditions*:

1° $A_n^1 = 1$ *for* $n = 1, 2, \ldots;\ A_2^2 = 1$,

2° $A_n^k = (n-2k+3)A_{n-1}^{k-1} + A_{n-1}^k$ *for* $n \geqslant 4;\ k = 2, 3, \ldots, [n/2]$,

3° *for* $n \geqslant 3$

$$A_n^{[n/2]+1} = \begin{cases} A_{n-1}^{[n/2]} & \text{for } n \text{ even}, \\ 2A_{n-1}^{[n/2]} + A_{n-1}^{[n/2]+1} & \text{for } n \text{ odd}, \end{cases}$$

then

$$(3)\qquad A_n^k = \frac{1}{\big(2(k-1)\big)!!}\prod_{j=0}^{2k-3}(n-j) \quad \textit{for } n = 2, 3, \ldots$$

and $k = 2, 3, \ldots, [n/2]+1$, *and* $A_n^1 = 1$ *for* $n = 1, 2, \ldots$

Proof. We use induction. For $n = 1$ the assertion is obvious. For $n = 2$ we get $k = 1, 2$ and $A_2^k = 1$. For $n = 3$ we get $k = 1, 2$ and $A_3^1 = 1$, $A_3^2 = 2A_2^1 + A_2^2 = 3$, and from (3) we have $A_3^1 = 1$, $A_3^2 = 3$. Let us suppose that the assertion of Lemma 1 is true for $n-1 \geqslant 3$. Then for n we get $k = 1, 2, \ldots, [n/2]+1$. If $k = 1$ the assertion is obvious. For $k = 2$, by assumption,

$$A_n^2 = (n-1)A_{n-1}^1 + A_{n-1}^2 = (n-1) + \tfrac{1}{2}(n-1)(n-2) = \tfrac{1}{2}n(n-1) = \tfrac{1}{2}\prod_{j=0}^{1}(n-j).$$

If $3 \leqslant k \leqslant [n/2]$, then by assumption

$$A_n^k = (n-2k+3)A_{n-1}^{k-1} + A_{n-1}^k = (n-2k+3)\frac{1}{\big(2(k-2)\big)!!}\prod_{j=0}^{2k-5}(n-1-j) + \frac{1}{\big(2(k-1)\big)!!}\prod_{j=0}^{2k-3}(n-1-j) = \frac{1}{\big(2(k-1)\big)!!}[(n-2k+3)(2k-2) + (n-2k+3)(n-2k+2)]\prod_{j=0}^{2k-5}(n-1-j) = \frac{1}{\big(2(k-1)\big)!!}\prod_{j=0}^{2k-3}(n-j).$$

If $k = [n/2]+1$ and n is even, then

$$A_n^{[n/2]+1} = A_{n-1}^{[n/2]} = \frac{1}{(n-2)!!}\prod_{j=0}^{n-3}(n-1-j) = \frac{(n-1)!}{(n-2)!!} = \frac{n!}{n!!} = \frac{1}{n!!}\prod_{j=0}^{n-1}(n-j).$$

If n is odd, then $\left[\frac{n}{2}\right] = \frac{n-1}{2}$ and

$$A_n^{[n/2]+1} = 2A_{n-1}^{(n-1)/2} + A_{n-1}^{(n+1)/2} = \frac{2}{(n-3)!!}\prod_{j=0}^{n-4}(n-1-j) + \frac{1}{(n-1)!!}\prod_{j=0}^{n-2}(n-1-j) = \frac{(n-1)!}{(n-3)!!} + \frac{(n-1)!}{(n-1)!!} = \frac{n!}{(n-1)!!} = \frac{1}{(n-1)!!}\prod_{j=0}^{n-2}(n-j)$$

and finally we get the assertion of Lemma 1.

LEMMA 2. *If q is an arbitrary real number, then*

$$(4)\quad D^n_{y_m} r^q = (-1)^n \sum_{k=1}^{\left[\frac{n}{2}\right]+1} A^k_n q(q-2)\ldots(q-2n+2k) r^{q-2n+2k-2}(x_m-y_m)^{n-2k+2}.$$

Proof. For $n = 1$ the assertion is obvious. Let us assume that formula (4) holds for $n-1 \geqslant 1$. Thus

$$D^n_{y_m} r^q = D_{y_m}\Big\{(-1)^{n-1} \sum_{k=1}^{\left[\frac{n-1}{2}\right]+1} [A^k_{n-1} q(q-2)\ldots(q-2n+2k+2) r^{q-2n+2k} \times$$

$$\times (x_m-y_m)^{n-2k+1}]\Big\} = (-1)^{n-1} \sum_{k=1}^{\left[\frac{n-1}{2}\right]+1} \{A^k_{n-1} q(q-2)\ldots(q-2n+2k+2)\times$$

$$\times D'_{y_m}[r^{q-2n+2k}(x_m-y_m)^{n-2k+1}]\}$$

$$= (-1)^{n-1} \sum_{k=1}^{\left[\frac{n-1}{2}\right]+1} \{A^k_{n-1} q(q-2)\ldots(q-2n+2k+2)\times$$

$$\times [-(q-2n+2k) r^{q-2n+2k-2}(x_m-y_m)^{n-2k+2} -$$

$$-(n-2k+1) r^{q-2n+2k}(x_m-y_m)^{n-2k}]\}$$

$$= (-1)^n \sum_{k=1}^{\left[\frac{n-1}{2}\right]+1} A^k_{n-1} q(q-2)\ldots(q-2n+2k) r^{q-2n+2k-2}(x_m-y_m)^{n-2k+2} +$$

$$+(-1)^n \sum_{k=1}^{\left[\frac{n-1}{2}\right]+1} A^k_{n-1} q(q-2)\ldots(q-2n+2k+2)\times$$

$$\times (n-2k+1) r^{q-2n+2k}(x_m-y_m)^{n-2k}.$$

If n is even, then $\left[\frac{n-1}{2}\right] = \frac{n-2}{2}$, $\left[\frac{n}{2}\right] = \frac{n}{2}$ and in this case we obtain

$$D^n_{y_m} r^q = (-1)^n \Big\{A^1_{n-1} q(q-2)\ldots(q-2n+2) r^{q-2n}(x_m-y_m)^n +$$

$$+\sum_{k=2}^{\frac{n}{2}} A^k_{n-1} q(q-2)\ldots(q-2n+2k) r^{q-2n+2k-2}(x_m-y_m)^{n-2k+2} +$$

$$+\sum_{k=1}^{\frac{n-2}{2}} A^k_{n-1} q(q-2)\ldots(q-2n+2k+2)(n-2k+1) r^{q-2n+2k}(x_m-y_m)^{n-2k} +$$

$$+A_{n-1}^{n/2}q(q-2)\ldots(q-n+2)r^{q-n}\Big\}$$

$$=(-1)^n\Big\{A_{n-1}^1 q(q-2)\ldots(q-2n+2)r^{q-2n}(x_m-y_m)^n+$$

$$+\sum_{k=2}^{\frac{n}{2}} q(q-2)\ldots(q-2n+2k)r^{q-2n+2k-2}(x_m-y_m)^{n-2k+2}[A_{n-1}^k+$$

$$+(n-2k+3)A_{n-1}^{k-1}]+A_{n-1}^{n/2}q(q-2)\ldots(q-n+2)r^{q-n}\Big\},$$

and by assumptions $1^\circ, 2^\circ, 3^\circ$ we get

$$D_{y_m}^n r^q=(-1)^n\Big\{A_n^1 q(q-2)\ldots(q-2n+2)r^{q-2n}(x_m-y_m)^n+$$

$$+\sum_{k=2}^{\frac{n}{2}} A_n^k q(q-2)\ldots(q-2n+2k)r^{q-2n+2k-2}(x_m-y_m)^{n-2k+2}+$$

$$+A_n^{\frac{n}{2}+1}q(q-2)\ldots(q-n+2)r^{q-n}\Big\}$$

$$=(-1)^n\sum_{k=1}^{\left[\frac{n}{2}\right]+1} A_n^k q(q-2)\ldots(q-2n+2k)r^{q-2n+2k-2}(x_m-y_m)^{n-2k+2}.$$

If n is odd, then $\left[\frac{n-1}{2}\right]=\frac{n-1}{2}$, $\left[\frac{n}{2}\right]=\frac{n-1}{2}$ and we get

$$D_{y_m}^n r^q=(-1)^n\Big\{A_{n-1}^1 q(q-2)\ldots(q-2n+2)r^{q-2n}(x_m-y_m)^n+$$

$$+\sum_{k=2}^{\frac{n+1}{2}} A_{n-1}^k q(q-2)\ldots(q-2n+2k)r^{q-2n+2k-2}(x_m-y_m)^{n-2k+2}+$$

$$+\sum_{k=1}^{\frac{n-1}{2}} A_{n-1}^k q(q-2)\ldots(q-2n+2k+2)(n-2k+1)r^{q-2n+2k}(x_m-y_m)^{n-2k}\Big\}$$

$$=(-1)^n\Big\{A_{n-1}^1 q(q-2)\ldots(q-2n+2)r^{q-2n}(x_m-y_m)^n+$$

$$+\sum_{k=2}^{\frac{n+1}{2}} q(q-2)\ldots(q-2n+2k)r^{q-2n+2k-2}(x_m-y_m)^{n-2k+2}[A_{n-1}^k+$$

$$+(n-2k+3)A_{n-1}^{k-1}]\Big\}.$$

By assumptions $1^\circ, 2^\circ, 3^\circ$ we get

$$D_{y_m}^n r^q=\sum_{k=1}^{\left[\frac{n}{2}\right]+1} A_n^k q(q-2)\ldots(q-2n+2k)r^{q-2n+2k-2}(x_m-y_m)^{n-2k+2}$$

for n odd. Finally we get the assertion of Lemma 2.

Similarly, we can verify the formula

$$(5)\qquad D^n_{y_m} r_1^q = \sum_{k=1}^{[\frac{n}{2}]+1} A^k_n q(q-2)\ldots(q-2n+2k) r_1^{q-2n+2k-2}(x_m+y_m)^{n-2k+2}.$$

LEMMA 3. *If i is even and $A^1_0 = 1$, then*

$$\sum_{k=2s-2}^{i-1} A^s_k \binom{i}{k} 2^{i-k}(-1)^k = 0$$

for $s = 1, 2, \ldots, \frac{1}{2}i$.

Proof. If $i \geqslant 2$ and $s = 1$, then

$$\sum_{k=0}^{i-1} A^1_k \binom{i}{k} 2^{i-k}(-1)^k = \sum_{k=0}^{i} \binom{i}{k} 2^{i-k}(-1)^k - 1 = 0.$$

If $s \geqslant 2$, then by Lemma 1 we get

$$\sum_{k=2s-2}^{i-1} A^s_k \binom{i}{k} 2^{i-k}(-1)^k$$
$$= \frac{1}{\big(2(s-1)\big)!!} \sum_{k=2s-2}^{i-1} 2^{i-k}(-1)^k \binom{i}{k} \prod_{j=0}^{2s-3} (k-j)$$
$$= \frac{1}{\big(2(s-1)\big)!!} \sum_{k=2s-2}^{i-1} \frac{i!\,2^{i-k}(-1)^k}{(i-k)!\,(k-2s+2)!}$$
$$= \frac{i!}{(i-2s+2)!\,\big(2(s-1)\big)!!} \Big[\sum_{k=0}^{i-2s+2} \binom{i-2s+2}{k} 2^{i-2s+2-k}(-1)^k - 1\Big]$$
$$= 0.$$

Similarly, we can prove

LEMMA 4. *If i is odd and $A^1_0 = 1$, then*

$$\sum_{k=2s-2}^{i} A^s_k \binom{i}{k} 2^{i-k}(-1)^k = A^s_i$$

for $s = 1, 2, \ldots, \frac{1}{2}(i+1)$.

3. Applying Lemmas 1, 2, 3, 4 we shall construct the Green function for equation (1) and for the domain E^+_m satisfying the homogeneous boundary conditions (2).

Let

$$C_k = \frac{2^k q(q-2)\ldots(q-2k+2)}{k!},$$

where $q = 2p-m$. We shall consider cases (a) and (b) only.

Let

$$G(P,Q) = r^q - r_1^q + \sum_{k=1}^{p-1} C_k (x_m y_m)^k r^{q-2k}. \tag{6}$$

THEOREM. *The function $G(P,Q)$ defined by formula (6) is the Green function for equation (1) in the domain E_m^+ satisfying the boundary conditions (2) with a pole at the point P.*

Proof. By Krzyżański's book(1) if the function $u(Q)$ is a harmonic function, then the function $y_m u(Q)$ is a biharmonic one and consequently the function $G(P,Q)$ satisfies equation (1) for $Q \neq P$. If $Q = P$, then $G(P,Q) = O(r^{-m+2}) + H(P,Q)$, where $H(P,Q)$ is a function satisfying equation (1) with respect to the point Q.

Now, we shall verify the boundary conditions (2). If $i = 0$, then for $y_m = 0$ we obtain $r = r_1$ and $G(P,Q)|_{y_m=0} = 0$.

Let $1 \leqslant i \leqslant p-1$ and

$$R^2 = \sum_{k=1}^{m-1} (x_k - y_k)^2 + x_m^2.$$

By Lemma 2, formula (5) and by Leibniz formula, we get

$$\begin{aligned}
D_{y_m}^i G(P,Q)\Big|_{y_m=0} &= \Big\{D_{y_m}^i r^q - D_{y_m}^i r_1^q + \sum_{k=1}^{p-1} C_k x_m^k D_{y_m}^i (y_m^k r^{q-2k})\Big\}\Big|_{y_m=0} \\
&= \Big\{D_{y_m}^i r^q - D_{y_m}^i r_1^q + \sum_{k=1}^{p-1} C_k x_m^k \sum_{j=0}^{i} \binom{i}{j} D_{y_m}^i (y_m^k) D_{y_m}^{i-j} r^{q-2k}\Big\}\Big|_{y_m=0} \\
&= [(-1)^i - 1] \sum_{k=1}^{\left[\frac{i}{2}\right]+1} A_i^k q(q-2)\dots(q-2i+2k) R^{q-2i+2k-2} x_m^{i-2k+2} + \\
&\qquad + \sum_{k=1}^{i} C_k x_m^k \binom{i}{k} k!\, D_{y_m}^{i-k} r^{q-2k}\Big|_{y_m=0} \\
&= [(-1)^i - 1] \sum_{k=1}^{\left[\frac{i}{2}\right]+1} A_i^k q(q-2)\dots(q-2i+2k) R^{q-2i+2k-2} x_m^{i-2k+2} +
\end{aligned}$$

$$+ \sum_{k=1}^{i} \sum_{s=1}^{\left[\frac{i-k}{2}\right]+1} C_k A_{i-k}^s \binom{i}{k} k! (-1)^{i-k} (q-2k)\dots(q-2i+2s) R^{q-2i+2s-2} x_m^{i-2s+2}.$$

(1) M. Krzyżański, *Równania różniczkowe cząstkowe rzędu drugiego*, cz. II, Warszawa 1962, p. 194.

If i is even, then by Lemma 3, we get

$$\sum_{k=1}^{i}\sum_{s=1}^{\left[\frac{i-k}{2}\right]+1} C_k A_{i-k}^s \binom{i}{k} k!(-1)^{i-k}(q-2k)\dots(q-2i+2s)R^{q-2i+2s-2}x_m^{i-2s+2}$$
$$=\sum_{s=1}^{i/2} x_m^{i-2s+2}R^{q-2i+2s-2}q(q-2)\dots(q-2i+2s)\sum_{k=2s-2}^{i-1} A_k^s\binom{i}{k}2^{i-k}(-1)^k = 0$$

and $D_{y_m}^i G(P,Q)|_{y_m=0} = 0$.

If i is odd, then by Lemma 4, we get

$$D_{y_m}^i G(P,Q)|_{y_m=0}$$
$$= -2\sum_{k=1}^{[i/2]+1} A_i^k q(q-2)\dots(q-2i+2k)R^{q-2i+2k-2}x_m^{i-2k+2}+$$
$$+\sum_{k=1}^{i}\sum_{s=1}^{\left[\frac{i-k}{2}\right]+1} A_{i-k}^s\binom{i}{k}2^k(-1)^{i-k}q(q-2)\dots(q-2i+2s)R^{q-2i+2s-2}x_m^{i-2s+2}$$
$$= -\sum_{k=1}^{[i/2]+1} A_i^k q(q-2)\dots(q-2i+2k)R^{q-2i+2k-2}x_m^{i-2k+2}+$$
$$+\sum_{k=0}^{i}\sum_{s=1}^{\left[\frac{i-k}{2}\right]+1} A_{i-k}^s\binom{i}{k}2^k(-1)^{i-k}q(q-2)\dots(q-2i+2s)R^{q-2i+2s-2}x_m^{i-2s+2}$$
$$= -\sum_{k=1}^{[i/2]+1} A_i^k q(q-2)\dots(q-2i+2k)R^{q-2i+2k-2}x_m^{i-2k+2}+$$
$$+\sum_{s=1}^{[i/2]+1} q(q-2)\dots(q-2i+2s)R^{q-2i+2s-2}x_m^{i-2s+2}\sum_{k=2s-2}^{i} A_k^s\binom{i}{k}2^{i-k}(-1)^k$$

and $D_{y_m}^i G(P,Q)|_{y_m=0} = 0$.