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On derivatives of vector measures into $l_p(X)$, $0 < p < 1$

Let (Ω, Σ, μ) be a finite positive measure space and let $m: \Sigma \to l_p(X)$, $0 < p < 1$, be a (σ-additive) vector measure, where $l_p(X)$ is the non-locally convex space of all p-absolutely summable sequences in a Banach space $(X, \| \|_X)$, endowed with the p-homogeneous F-norm $\|(x_n)\|_p := \sum_{n=1}^{\infty} (\|x_n\|_X)^p$. As the inclusion $l_p(X) \subset l_1(X)$ is continuous, m can also be viewed as a vector measure $\tilde{m}: \Sigma \to l_1(X)$.

If X has the Radon–Nikodym-Property (RNP), then $l_1(X)$ also has RNP ([1]). So if $\tilde{m}$ is μ-continuous and has finite (resp. σ-finite) variation, it has a μ-measurable, Bochner- (resp. Pettis-) integrable derivative $f: \Omega \to l_1(X)$ (cf. [6]). It is natural to ask under which conditions the derivative f has values in $l_p(X)$, too, and furthermore, whether then $f: \Omega \to l_p(X)$ is μ-measurable and $\|f(\cdot)\|_p$ is μ-integrable.

These questions were posed to us by Professor Joe Diestel. This note provides some answers into this direction.

In the sequel, $(X, \| \|_X)$ will denote a Banach space with RNP and E an F-space with F-norm $\| \|_E$ (cf. [8]). (Ω, Σ, μ) is a finite positive measure space (cf. [2]). By a *vector measure* we understand a σ-additive set function $m: \Sigma \to E$.

For $B \in \Sigma$ with $\mu(B) > 0$ the set $\mathscr{A}_B(m) := \left\{ \frac{m(D)}{\mu(D)} : D \in \Sigma,\ D \subset B,\ \mu(D) > 0 \right\}$ is called the *average range* of m on B.

A function $f: \Omega \to E$ is said to be *weakly measurable* if $x^* f$ is measurable for all $x^* \in E^*$, and is *μ-measurable* if there exists a sequence (f_n) of simple functions with $\lim_{n\to\infty} \|f_n(t) - f(t)\|_E = 0$ μ-a.e.

The variation of m is defined by $|m|(\Omega) := \sup_{\pi} \sum_{B \in \pi} \|m(B)\|_E$, where the supremum is taken over all finite partitions π of Ω.

The notion of the variation is no more useful in F-spaces with non-homogeneous F-norms. Even in the case of the spaces $l_p(\boldsymbol{R})$, $0 < p < 1$, which have a p-homogeneous F-norm, the variation is never σ-finite unless the vector measure is purely atomic (cf. [3]).

Also, there are vector measures $m\colon \Sigma \to l_p(X)$, such that $m\colon \Sigma \to l_1(X)$ has a Bochner-integrable derivative (in $l_1(X)$), which does not take values in $l_p(X)$.

EXAMPLE 1. Fix $0 \neq x \in X$, set $f_n(t) := \frac{1}{n^{1/p}} r_n(t)x$, where r_n is the n-th Rademacher function, and define $f\colon \Omega \to l_1(X)$ by $f(t) := (f_n(t))_{n\in\mathbf{N}}$. Then f is Bochner-integrable in $l_1(X)$, $\tilde{m}(E) := (B) - \int_E f(t)\,d\mu(t) \in l_p(X)$ and $f(t) \notin l_p(X)$ for all $t \in \Omega$.

As we remarked already, m cannot have σ-finite variation but replacing this notion by another one [which is equivalent to the σ-finiteness of variation in Banach spaces with RNP], we get an RN-type result analogous to the Banach space case.

PROPOSITION 1. *Let $m\colon \Sigma \to l_p(X)$, $0 < p < 1$, be a μ-continuous vector measure with locally bounded average range (i.e., for every $A \in \Sigma$, $\mu(A) > 0$, there is a $B \subset A$, $\mu(B) > 0$, such that $\mathscr{A}_B(m)$ is bounded in $l_p(X)$). Then there exists a μ-measurable function $f\colon \Omega \to l_p(X)$ with $m(E) = (P) - \int_E f(t)\,d\mu(t)$.*

Proof. We first remark that the notion of a Pettis-integral may be extended to the space $l_p(X)$, since it has a separating dual. Moreover, we note that $l_p(X)^* = l_1(X)^*$ [4].

By exhaustion one can find pairwise disjoint sets $A_i \in \Sigma$ and a μ-null set N such that $\mathscr{A}_{A_i}(m)$ is bounded in $l_p(X)$ for all $i \in N$ and $\Omega = N \cup (\bigcup_{i=1}^{\infty} A_i)$. As the topology of $l_1(X)$ is coarser than that of $l_p(X)$, boundedness of $\mathscr{A}_{A_i}(m)$ in $l_p(X)$ implies that $|\tilde{m}|(A_i)$ is finite. Therefore, as $l_1(X)$ has RNP, $\tilde{m}$ has a Bochner-derivative f on every A_i with $f(t) \in \overline{\mathscr{A}_{A_i}(m)}^1$ (the closure of $\mathscr{A}_{A_i}(m)$ in $l_1(X)$) for all $t \in A_i$ (cf. [7]).

Setting $f(t) = 0$ on N, we get the representation $m(E) = (P) - \int_E f(t)\,d\mu(t)$, where f is μ-measurable in $l_1(X)$. Furthermore, the boundedness of $\mathscr{A}_{A_1}(m)$ implies that $\overline{\mathscr{A}_{A_1}(m)}^1 \subset l_p(X)$, i.e., $f(t) \in l_p(X)$ for all $t \in \Omega$.

It remains to show that f is μ-measurable in $l_p(X)$. As f has μ-essentially separable range in $l_1(X)$, it is easy to see that this is also true in $l_p(X)$.

Now we show that $\|f(\cdot)\|_p$ is measurable. This is done if we know that for every $\varepsilon > 0$ $U_\varepsilon := \{t\colon \|f(t)\|_p \leqslant \varepsilon\} \in \Sigma^*$, where Σ^* is the Lebesgue-extension of Σ. But since f is μ-measurable in $l_1(X)$, we have for every closed set $A \subset l_1(X)$ that $f^{-1}(A) \in \Sigma^*$ ([2], III. 6.9). The closed ball $B_\varepsilon^p(0) := \{x \in l_p(X)\colon \|x\|_p \leqslant \varepsilon\}$ is still closed in $l^1(X)$ and $f^{-1}(B_\varepsilon^p(0)) = U_\varepsilon$, which implies $U_\varepsilon \in \Sigma^*$. The rest of the proof now is the same as in the Banach space case (see e.g. [2], III. 6. 11).

The last part of the proof implies the following

COROLLARY. *A function $f\colon \Omega \to l_p(X)$ is μ-measurable if and only if*

(a) *f is μ-essentially separably valued, and*

(b) *f is weakly measurable.*

There is no direct analogue of Proposition 1 for Bochner-integrable functions, as such a notion makes no sense in a non-locally convex space, because there exists a sequence of simple functions uniformly tending to zero such that the corresponding sequence of integrals does not tend to zero ([8], p. 84). But we can still ask whether $\int_\Omega \|f(t)\|_E d\mu(t) < \infty$.

For a μ-measurable, Pettis-integrable function $f\colon \Omega \to l_1(X)$ we have $\int_\Omega \|f(t)\|_1 d\mu(t) < \infty$ if and only if $\tilde{m}\colon \Sigma \to l_1(X)$ with $\tilde{m}(E) := (P) - \int_E f(t)\,d\mu(t)$ has bounded variation (cf. [6]), i.e., if and only if the vector measure $\overline{m}\colon \Sigma \to l_\infty(\boldsymbol{R})$, $\overline{m}(E) := \left(\int_E \|f_n(t)\|_X d\mu(t)\right)_{n\in\mathbf{N}}$ has values in $l_1(R)$. This gives us a condition for the integrability of $\|f(\cdot)\|_p$.

PROPOSITION 2. *Let $f\colon \Omega \to l_p(X)$, $0 < p < 1$, be μ-measurable and Pettis-integrable, i.e., $m(E) = (P) - \int_E f(t)\,d\mu(t) \in l_p(X)$. Then we have $\int_\Omega \|f(t)\|_p d\mu(t) < \infty$, if the vector measure $\overline{m}$ has values in $l_p(\boldsymbol{R})$.*

Proof. For positive functions we have

$$(*)\qquad \left(\int_\Omega h(t)\,d\mu(t)\right)^p \geqslant \frac{1}{\mu(\Omega)^{1-p}} \int_\Omega h(t)^p\,d\mu(t) \qquad ([5], 13.6).$$

This implies

$$\int_\Omega \|f(t)\|_p d\mu(t) = \int_\Omega \sum_{n=1}^\infty \left(\|f_n(t)\|_X\right)^p d\mu(t) = \sum_{n=1}^\infty \int_\Omega \left(\|f_n(t)\|_X\right)^p d\mu(t)$$

$$\leqslant \mu(\Omega)^{1-p} \sum_{n=1}^\infty \left(\int_\Omega \|f_n(t)\|_X d\mu(t)\right)^p = \mu(\Omega)^{1-p} \|\overline{m}(\Omega)\|_p < \infty.$$

However, unlike the case of $l_1(X)$, the condition is not necessary for $l_p(X)$, $0 < p < 1$, because $(*)$ is an equality only for $p = 1$.

EXAMPLE 2. Let (Ω, Σ, μ) be non-atomic and $0 < p < 1$. Then there is a partition $\{A_i\}$ of Ω with $\bigcup_{i=1}^\infty A_i = \Omega$ and $\mu(A_i) = \dfrac{c}{n^{1/p}}$ for a certain constant $c > 0$. Fix $0 \neq x \in X$ and set

$$\delta_n(t) := \begin{cases} 1 & t \in A_n, \\ 0 & \text{else}, \end{cases}$$

$f_n(t) := \delta_n(t)x$. Then $f = (f_n)_{n\in N}\colon \Omega \to l_p(X)$ is μ-measurable and $\|f(t)\|_p = (\|x\|_X)^p$ for all $t \in \Omega$, but

$$m(\Omega) := (P)-\int_{\Omega} f(t)\,d\mu(t) = \left(\frac{c}{n^{1/p}}\,x\right)_{n\in N} \notin l_p(X).$$

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