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On the distribution of some sequences of integers in residue classes

1. Denote $\Delta(n) = \Omega(n) - \omega(n)$, $n = 1, 2, 3, \ldots$, where $\omega(n)$ is the number of different prime factors and $\Omega(n)$ is the number of all prime factors of n. Let r to be an integer, $r \geqslant 0$. Denote by Q_r the set of all positive integers n such that $\Delta(n) = r$. If $x \geqslant 1$ and if l and h are positive integers, $h \geqslant 1$ we define the function

$$Q_r(x; l, h) = \sum_{\substack{n \leqslant x \\ n \equiv l (\operatorname{mod} h)}} 1.$$

In the case $l = 0$, $h = 1$ the above function has been investigated by A. Rényi [7], H. Delange [2], [3] [4] and A. S. Fajnleib [5].

The purpose of this note is to investigate the distribution of integers belonging to Q_r in residue classes $(\operatorname{mod} h)$, by the use of some estimate of the function $Q_r(x; l, h)$. E. Cohen and R. L. Robinson [1] proved that integers belonging to the set Q_0 are equipartitioned $(\operatorname{mod} h)$ if and only if h is of the form $h = p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$ with $a_i \geqslant 2$, $i = 1, 2, \ldots, s$. We shall prove in the following that integers belonging to the set Q_r $(r \geqslant 1)$ are not equipartitioned $(\operatorname{mod} h)$ for any $h > 1$. We shall prove also that if h is of the form $h = p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$ with $a_i \geqslant 2$, $i = 1, 2, \ldots, s$, then integers $a \in Q_r$ $(r \geqslant 0)$ such that (a, h) is square free, are equipartitioned in residue classes $(\operatorname{mod} h)$. At the end of this note we shall state that for any fixed $i \geqslant 0$, each sufficiently large natural number is a sum of tho positive integers a and b such that $\Delta(a) = i$ and $\Delta(b) = 0$. We shall also give an asymptotic formula for the number of such representations.

2. If a residue class $(\operatorname{mod} h)$ is defined by a residue $l (\operatorname{mod} h)$ such that $\Delta\big((l, h)\big) > r$, then no integer belonging to Q_r is contained in such a residue class $(\operatorname{mod} h)$. Hence if $\Delta\big((l, h)\big) > r$, then obviously $Q_r(x; l, h) = 0$ for all x.

H. Delange mentioned (see [4], p. 182) that by the use of his methods we could get an asymptotic expansion for $Q_r(x; l, h)$. First we state without proof the following

THEOREM I. *Suppose $d = (l, h)$ is square free, R is the product of all different primes dividing d but not dividing h/d. Then*

$$Q_r(x; l, h) = \lambda_r x + O\big(\sqrt{x}\,(\log\log x)^r\big),$$

where

$$\lambda_r = \frac{6}{\pi^2}\frac{\varphi(R)}{R}\frac{1}{h\prod\limits_{p|h}\left(1-\dfrac{1}{p^2}\right)}\sum_{\substack{m\in\varepsilon_r(p)\\(m,h)|l}}\frac{\varphi(m)(m,h)}{m^2\prod\limits_{\substack{p|m\\p^2\nmid h}}\left(1-\dfrac{(p^2,h)}{p^2}\right)}$$

and $\varepsilon_r(P)$ denotes the set of all integers m of the form $p_1^{a_1}p_2^{a_2}\dots p_s^{a_s}$, where $p_1, p_2, \dots, p_s$ are different primes, $a_i \geqslant 2$ and $a_1+a_2+\dots+a_s-s = r$.

3. Let us denote by r an integer, $r \geqslant 0$. A residue class $l(\mathrm{mod}\,h)$ will be called *admissible* if $\varDelta\big((l, h)\big) \leqslant r$. A residue class $(\mathrm{mod}\,h)$ will be called *admissible* if is defined by an admissible residue.

The integers belonging to Q_r are said to be *equipartitioned* $(\mathrm{mod}\,h)$ if

$$(1) \qquad \lim_{x\to\infty}\frac{Q_r(x; l, h)}{Q_r(x)} = \frac{1}{I_r(h)}$$

for each admissible residue class $(\mathrm{mod}\,h)$. $I_r(h)$ denotes the number of admissible residue classes $(\mathrm{mod}\,h)$ and $Q_r(x) = Q_r(x; 0, 1)$ (cf. [1]).

It is known that numbers belonging to Q_0 are equipartitioned $(\mathrm{mod}\,h)$ if and only if $h = p_1^{a_1}p_2^{a_2}\dots p_s^{a_s}$, $a_i \geqslant 2$. This case $r = 0$ been investigated in [1].

The following theorem is devoted to the case $r \geqslant 1$.

THEOREM II. *The integers belonging to the set Q_r $(r \geqslant 1)$ are not equipartitioned* $(\mathrm{mod}\,h)$ *for any h.*

Proof. Suppose h is not square free. Let p_1 to be a prime such that $p_1^2|h$. Choose two residues $(\mathrm{mod}\,h)$: 1 and p_1. Both are admissible, because of $\varDelta\big((1, h)\big) = \varDelta\big((p_1, h)\big) = 0$. From Theorem I it follows

$$(2) \qquad \lim_{x\to\infty}\frac{Q_r(x, 1, h)}{Q_r(x)} = \frac{\dfrac{1}{h\prod\limits_{p|h}(1-1/p^2)}\displaystyle\sum_{\substack{m\in\varepsilon_r(p)\\(m,h)=1}}\frac{1}{m\prod\limits_{p|m}(1+1/p)}}{\displaystyle\sum_{m\in\varepsilon_r(p)}\frac{1}{m\prod\limits_{n|m}(1+1/p)}}$$

and

$$(3)\qquad \lim_{x\to\infty}\frac{Q_r(x,p_1,h)}{Q_r(x)}=\frac{\dfrac{1}{h\prod\limits_{p|h}(1-1/p^2)}\sum\limits_{\substack{m\in\varepsilon_r(p)\\(m,h)=1}}\dfrac{1}{m\prod\limits_{p|m}(1+1/p)}}{\sum\limits_{m\in\varepsilon_r(p)}\dfrac{1}{m\prod\limits_{p|m}(1+1/p)}}+$$

$$+\frac{\dfrac{1}{h\prod\limits_{p|h}(1-1/p^2)}\sum\limits_{\substack{m\in\varepsilon_r(p)\\(m,h)=p_1}}\dfrac{p_1\varphi(m)}{m^2\prod\limits_{\substack{p|m\\p\neq p_1}}\big(1-(p^2,h)/p_2\big)}}{\sum\limits_{m\in\varepsilon_r(p)}\dfrac{1}{m\prod\limits_{p|m}(1+1/p)}}.$$

In this cases $R=1$.

From (2), (3) we have

$$\lim_{x\to\infty}\frac{Q_r(x;1,h)}{Q_r(x)}\neq\lim_{x\to\infty}\frac{Q_r(x,p_1,h)}{Q_r(x)}.$$

Thus theorem is proved for h that is not square free.

It remains to prove the theorem for h square free. Suppose the integers belonging to Q_r $(r\geqslant 1)$ being equipartitioned $\bmod h$ with some square free h. In this case all the residue classes are admissible. Hence from (1) it follows

$$\lim_{x\to\infty}\frac{Q_r(x;l,h)}{Q(x)}=\frac{1}{h},\qquad l=1,2,\ldots,h.$$

Let p_1 to be a prime divisor of h. Then we have

$$\lim_{x\to\infty}\frac{Q_r(x;l,p_1)}{Q_r(x)}=\frac{1}{p_1},\qquad l=1,2,\ldots,p_1$$

(see [6], § 5).

Particularely for $l=p_1$

$$(4)\qquad \lim_{x\to\infty}\frac{Q_r(x,p_1,p_1)}{Q_r(x)}=\frac{1}{p_1}.$$

From Theorem I and (4) we have

$$(5)\qquad \frac{p_1}{p_1+1}\Bigg(\sum_{\substack{m\in\varepsilon_r(p)\\(m,p_1)=1}}\frac{1}{m\prod\limits_{p|m}(1+1/p)}+p_1\cdot\sum_{\substack{m\in\varepsilon_r(p)\\p_1|m}}\frac{1}{m\prod\limits_{\substack{p|m\\p\neq p_1}}(1+1/p)}\Bigg)$$

$$=\sum_{m\in\varepsilon_r(p)}\frac{1}{m\prod\limits_{p|m}(1+1/p)}.$$

Let us observe

$$(6)\qquad \sum_{m\in e_r(p)} \frac{1}{m\prod\limits_{p|m}(1+1/p)} = \sum_{\substack{m\in e_r(p)\\(m,p_1)=1}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \\ + \frac{p_1}{p_1+1}\left(\frac{1}{p_1^2}\sum_{\substack{m\in e_{r-1}(p)\\(m,p_1)=1}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \right. \\ \left. + \frac{1}{p_1^3}\sum_{\substack{m\in e_{r-2}(p)\\(m,p_1)=1}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \ldots + \frac{1}{p_1^{r+1}}\right)$$

and

$$(7)\qquad \sum_{\substack{m\in e_r(p)\\p_1|m}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} = \frac{1}{p_1^2}\sum_{\substack{m\in e_{r-1}(p)\\(m,p_1)=1}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \\ + \frac{1}{p_1^3}\sum_{\substack{m\in e_{r-2}(p)\\(m,p_1)=1}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \ldots + \frac{1}{p_1^{r+1}}.$$

Substituting (7), (6) to (5) we get

$$\sum_{\substack{m\in e_i(p)\\(m,p_1)=1}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \frac{1}{p_1}\sum_{\substack{m\in e_{i-1}(p)\\(m,p_1)=1}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \ldots + \frac{1}{p_1^i} \\ = \sum_{\substack{m\in e_{i-1}(p)\\(m,p_1)=1}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \frac{1}{p_1}\sum_{\substack{m\in e_{i-2}(p)\\(m,p_1)=1}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \ldots + \frac{1}{p_1^{i-1}}.$$

It is easily to notice that we have a contradiction. We can namely prove by induction that

$$\sum_{\substack{(m,p_1)=1\\m\in e_r(p)}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \frac{1}{p_1}\sum_{\substack{(m,p_1)=1\\m\in e_{r-1}(p)}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \ldots + \frac{1}{p^r} \\ < \sum_{\substack{(m,p_1)=1\\m\in e_{r-1}(p)}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \frac{1}{p_1}\sum_{\substack{(m,p_1)=1\\m\in e_{r-2}(p)}} \frac{1}{m\prod\limits_{p|m}(1+1/p)} + \ldots + \frac{1}{p_1^{r-1}}$$

and the theorem is thus proved.

THEOREM III. *If h is of the form $p_1^{a_1}p_2^{a_2}p_3^{a_3}\ldots p_s^{a_s}$, $a_i' \geqslant 2$, $i = 1, 2, \ldots, s$, then the integers $a\in Q_r$ $(r \geqslant 0)$ such that (a, h) is squre free, are equipartitioned* mod h.

Proof. Denote by $\bar{Q}_r(x)$ the number of $a\in Q_r$, $a\leqslant x$ and such that (a, h) is square free and denote further by $\bar{Q}_r(x; l, h)$ the number of $a\in Q_r$, $a\leqslant x$, $a\equiv l(\operatorname{mod} h)$, (a, h) — square free. The integers $a\in Q_r$ such that (a, h) is square free belong to residue classes $(\operatorname{mod} h)$ defined by residues $l(\operatorname{mod} h)$ such that (l, h) are square free. Hence

$$\bar{Q}_r(x; l, h) = \begin{cases} Q_r(x; l, h), & \text{if } (l, h)-\text{square free}, \\ 0, & \text{otherwise}. \end{cases}$$

Since $a\in Q_r$, therefore $a = mq$, $(m, q) = 1$, $m\in\varepsilon_r(p)$, q square free. If $h = p_1^{a_1}\ldots p_s^{a_s}$, $a_i\geqslant 2$, $i = 1, 2, \ldots, s$, the condition (a, h) being square free is equivalent to $(m, h) = 1$. Hence owing to Theorem I we have

(8)

$$\bar{Q}_r(x; l, h) = \frac{6}{\pi^2}\frac{1}{h\prod\limits_{p|h}(1-1/p^2)}\sum_{\substack{m\in\varepsilon_r(p)\\(m,h)=1}}\frac{1}{m\prod\limits_{p|m}(1+1/p)}+O\big(\sqrt{x}(\log\log x)^r\big),$$

$$\bar{Q}_r(x) = \frac{6}{\pi^2}\sum_{\substack{m\in\varepsilon_r(p)\\(m,h)=1}}\frac{1}{m\prod\limits_{p|m}(1+1/p)}+O\big(\sqrt{x}(\log\log x)^r\big). \tag{9}$$

From (8), (9) it follows

$$\lim_{x\to\infty}\frac{\bar{Q}_r(x; l, h)}{\bar{Q}_r(x)} = \frac{1}{h\prod\limits_{p|h}(1-1/p^2)} = \frac{1}{h\prod\limits_{p^2|h}(1-1/p^2)} = \frac{1}{\varphi_2(h)},$$

where $\varphi_2(h)$ denotes the number of residues $l(\operatorname{mod} h)$ such that (l, h) is square free.

THEOREM IV. *The integers belonging to* Q_r $(r\geqslant 0)$ *relatively prime with* h *are equipartitioned* mod h.

The proof is similar to the proof of Theorem III.

THEOREM V. *Let* i *be a fixed non-negative integer. Then each sufficiently large positive integer is a sum of two positive integers* a *and* b *such that* $\Delta(a) = i$, $\Delta(b) = 0$.

Proof. Let us denote by $T_i(n)$ the number of all representations of a positive integer n in the form of a sum of two positive integers a and b with $\Delta(a) = i$ and $\Delta(b) = 0$. By the use of the method contained in [1] (§ 4, the proof of Corollary 2) it can be proved that

$$T_i(n) \sim \frac{6}{\pi^2}n\prod_p\left(1-\frac{1}{p^2-1}\right)\frac{1}{\prod\limits_{\substack{p\\p^2|n}}\big(1-1/(p^2-1)\big)}\sum_{\substack{m\in\varepsilon_i(p)\\\Delta((m,n))=0}}\frac{\varphi(m)}{m^2\prod\limits_{p|m}(1-2/p^2)}$$

and thus the theorem follows.

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