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Capacity of finite systems of elements in Banach algebras

Abstract. In this paper we give a generalization of P. R. Halmos' results concerning capacity in Banach algebras. The generalization in question is obtained for the case of finite systems of elements of a commutative Banach algebra with unit. The principal results are the following: (i) Sets of quasi-algebraically dependent systems of elements and systems of the capacity zero coincide. (ii) The capacity of elements is equal to the capacity of their joint spectrum.

Introduction. P. R. Halmos in [3] defined the notion of capacity of an element in a complex Banach algebra with unit (see also [2], p. 251). Using this notion he characterized a set of quasi-algebraic elements of this algebra. Namely, he proved that an element is quasi-algebraic if and only if its capacity is equal to zero. Moreover, it was shown that capacity defined in such a way is very close to the well-known potential-theoretic notion of capacity. The main result in Halmos' paper states that the capacity of an element is equal to the capacity of its spectrum. Using this result Halmos obtained some interesting applications.

The main purpose of this paper is to generalize Halmos' results for the case of finite systems of elements of a commutative Banach algebra with unit over the field of complex numbers. The paper is divided into two parts. In the first one we shall define the algebraic dependence of elements and next its analytic version, the so called quasi-algebraic dependence of elements of a Banach algebra. The second part is concerned with the notion of capacity of elements. We shall prove that elements are quasi-algebraically dependent if and only if their capacity is equal to zero. Next, it will be proved that the capacity of elements is equal to the capacity of their joint spectrum. Finally, the relationship between the capacity of the whole system of elements and the capacity of a part of it will be studied.

1. Algebraically and quasi-algebraically dependent systems of elements. We start with the notion of a monic polynomial of n variables. This notion is basic for our further considerations. Let $C[X_1, \ldots, X_n]$, where $n \geqslant 1$, be a ring of polynomials of n formal variables $X_1, \ldots, X_n$ with complex coefficients. A polynomial $w\in C[X_1, \ldots, X_n]$ is called *monic of degree* k $(k \geqslant 1)$ if it is of the form

$$w(X_1, \ldots, X_n) = \sum_{i_1+\cdots+i_n\leqslant k} a_{i_1\ldots i_n} X_1^{i_1}\ldots X_n^{i_n},$$

where $\sum_{i_1+\cdots+i_n=k} a_{i_1\ldots i_n} = 1$.

A set of all monic polynomials of n variables of degree k will be denoted by $P_k^1(n)$.

LEMMA 1. *If* $w\in P_k^1(n)$ *and* $v\in P_l^1(n)$, *then* $wv\in P_{k+l}^1(n)$.

Proof. Straightforward computations.

COROLLARY. *If* $p\in P_m^1(1)$ *and* $w\in P_k^1(n)$, *then* $p\circ w\in P_{km}^1(n)$.

Throughout this paper A will denote a commutative Banach algebra with unit over the field of complex numbers.

DEFINITION. Elements $x_1, \ldots, x_n\in A$ are *algebraically dependent* if there exists a polynomial $w\in P_k^1(n)$, where $k\geqslant 1$, such that $w(x_1, \ldots, x_n) = 0$

Now we shall present some elementary properties of algebraically dependent elements.

(i) If $x_i = 0$ for some $i = 1, \ldots, n$, then elements $x_1, \ldots, x_n$ are algebraically dependent.

(ii) If an element x_i is algebraic (see [3] or [2], p. 251) for some $i = 1, \ldots, n$, then elements $x_1, \ldots, x_n$ are algebraically dependent.

(iii) Let $\mathrm{alg}_n A$ denote the set of all n-tuples of algebraically dependent elements of the algebra A. The following inclusion holds

$$(\mathrm{alg}_n A)\times A \subset \mathrm{alg}_{n+1} A.$$

(iv) If $A\oplus \ldots \oplus A = \overset{n}{\bigoplus} A$ is a direct sum of the algebra A (see [6], p. 15), then the following inclusion holds

$$\mathrm{alg}_1(\overset{n}{\bigoplus} A) = (\mathrm{alg}_1 A)\times \ldots \times (\mathrm{alg}_1 A) \subset \mathrm{alg}_n A.$$

Remark 1. Now, we shall give an example which shows that the above inclusion cannot be replaced by an equality. Let $A = C([0,1])$ (an algebra of all complex-valued continuous functions defined on the interval $[0,1]$) and let $x_1 = (t\mapsto 1+t)$, $x_2 = \big(t\mapsto 1/(1+t)\big)$ be two elements of this algebra.

If we take the first degree monic polynomial of two variables $w(X_1, X_2) = X_1X_2-1$, then for any $t\in[0,1]$ we have $w\big(x_1(t), x_2(t)\big) = 0$. It means that the elements x_1, x_2 are algebraically dependent. However, none of them is algebraic not even quasi-algebraic (see [3] or [2], p. 251). Since we have

$$\sigma(x_1) = \{\lambda\in C\colon (x_1-\lambda)^{-1}\notin A\} = x_1\{[0,1]\} = [1,2],$$

then $\operatorname{cap} x_1 = \operatorname{cap}\sigma(x_1) = \frac{1}{4} > 0$ (see [3], Theorem 3, and [5], p. 84, Corollary 3). Similarly, $\operatorname{cap} x_2 = \operatorname{cap}\sigma(x_2) = \operatorname{cap}[\frac{1}{2}, 1] = \frac{1}{8} > 0$.

It means that (see [3], Theorem 2) elements x_1, x_2 are not quasi-algebraic. So we have proved

$$(\mathrm{alg}_1 A) \times (\mathrm{alg}_1 A) \not\supset \mathrm{alg}_2 A.$$

Obviously, this example one can easy generalize for any finite number of elements and for any algebras.

(v) A system $x, \ldots, x$ is algebraically dependent if and only if x is an algebraic element.

(vi) Let $x, y_1, \ldots, y_n \in A$. Let us suppose that x is algebraic and there exists a monic polynomial w of n variables such that $x = w(y_1, \ldots, y_n)$. Then the elements $y_1, \ldots, y_n$ are algebraically dependent (it follows from the corollary after Lemma 1).

Definition. Elements $x_1, \ldots, x_n \in A$ are *quasi-algebraically dependent* if there exists a sequence $\{w_k\}$ of monic polynomials of n variables, with degree w_k equals to $d(k)$, such that

$$\|w_k(x_1, \ldots, x_n)\|^{1/d(k)} \to 0 \quad \text{as } k \to \infty.$$

Remark 2. It is obvious that "quasi" versions of (i)–(vi) properties of algebraically dependent elements are true.

Remark 3. Clearly, any algebraically dependent elements are quasi-algebraically dependent. The converse is not true. For example, let us take a linear bounded operator T acting on a Banach space X with the countable spectrum. Let $\mathscr{A}_T$ be the maximal abelian subalgebra of $\mathscr{L}(X)$ (the algebra of all linear bounded operators acting on X) which contains the operator T. If we consider this operator as an element of this algebra, then it is quasi-algebraic (see [3] and [5], p. 57, Theorem III.8), but it is not algebraic. Now in view of "quasi" version of (v) we see that the system $T, \ldots, T$ is quasi-algebraically dependent but it is not algebraically dependent (the same argument but in a normal version).

2. Capacity of elements. Now we are going to define the main notion we deal with in this paper. For arbitrary elements $x_1, \ldots, x_n \in A$ we denote

$$\operatorname{cap}_k(x_1, \ldots, x_n) = \inf\{\|w(x_1, \ldots, x_n)\|: w \in \boldsymbol{P}_k^1(n)\} \quad (k = 1, 2, \ldots).$$

Then for any polynomials $w \in \boldsymbol{P}_k^1(n)$ and $v \in \boldsymbol{P}_l^1(n)$ the following inequality holds

$$\operatorname{cap}_{k+l}(x_1, \ldots, x_n) \leqslant \|wv(x_1, \ldots, x_n)\| \leqslant \|w(x_1, \ldots, x_n)\| \cdot \|v(x_1, \ldots, x_n)\|.$$

This implies

$$\operatorname{cap}_{k+l}(x_1, \ldots, x_n) \leqslant \operatorname{cap}_k(x_1, \ldots, x_n) \cdot \operatorname{cap}_l(x_1, \ldots, x_n)$$

and consequently the sequence $\{(\operatorname{cap}_k(x_1, \ldots, x_n))^{1/k}\}$ is convergent (see [1], p. 365–366, or [4], p. 257).

DEFINITION. We denote $\operatorname{cap}(x_1, \ldots, x_n) = \lim_k (\operatorname{cap}_k(x_1, \ldots, x_n))^{1/k}$ and call this number the *capacity of elements* $x_1, \ldots, x_n$. In particular, let us take an algebra $C(\Omega)$ of all complex-valued continuous functions defined on a compact subset Ω of C^n. Let $\pi_i(z_1, \ldots, z_n) = z_i$, where $i = 1, \ldots, n$, be the i-th projection. Now we compute the capacity of these functions

$$\operatorname{cap}(\pi_1, \ldots, \pi_n) = \lim_k (\operatorname{cap}_k(\pi_1, \ldots, \pi_n))^{1/k},$$

where

$$\begin{aligned}\operatorname{cap}_k(\pi_1, \ldots, \pi_n) &= \inf\{\|w(\pi_1, \ldots, \pi_n)\|_\Omega \colon\ w \in \boldsymbol{P}_k^1(n)\} \\ &= \inf\{\max_{z \in \Omega} |w(z_1, \ldots, z_n)| \colon\ w \in \boldsymbol{P}_k^1(n)\}.\end{aligned}$$

DEFINITION. We call $\operatorname{cap}(\pi_1, \ldots, \pi_n)$ the *capacity of set* Ω and we denote it by $\operatorname{cap}\Omega$.

Now we shall prove a theorem characterizing the set of quasi-algebraically dependent elements.

THEOREM 1. *Elements* $x_1, \ldots, x_n$ *of a Banach algebra* A *are quasi-algebraically dependent if and only if their capacity is equal to zero.*

Proof. Let us suppose that $\operatorname{cap}(x_1, \ldots, x_n) = 0$. From a definition of the capacity it follows immediately that there exists a sequence of monic polynomials $\{w_k\}$ of n variables such that $w \in \boldsymbol{P}_k^1(n)$ and $\lim_k \|w_k(x_1, \ldots, x_n)\|^{1/k} = \operatorname{cap}(x_1, \ldots, x_n)$. Since $\operatorname{cap}(x_1, \ldots, x_n) = 0$ we have $\lim_k \|w_k(x_1, \ldots, x_n)\|^{1/k} = 0$. It means that the elements $x_1, \ldots, x_n$ are quasi-algebraically dependent. Now let us assume that the converse is also true. Let $\{w_k\}$ be a sequence of monic polynomials, with degree w_k equals to $d(k)$, such that $\|w_k(x_1, \ldots, x_n)\|^{1/d(k)} \to 0$ as $k \to \infty$. If the sequence $\{d(k)\}$ of degrees has a bounded infinite subsequence, it may be assumed (consider a suitable subsequence) that there is a positive integer N such that $d(k) = N$ for all k. Since $\|w_k(x_1, \ldots, x_n)\| \geqslant \operatorname{cap}_{d(k)}(x_1, \ldots, x_n)$ we obtain the following inequalities

$$0 \leqslant (\operatorname{cap}_N(x_1, \ldots, x_n))^{1/N} \leqslant \|w_k(x_1, \ldots, x_n)\|^{1/N}.$$

But $\|w_k(x_1, \ldots, x_n)\|^{1/N} \to 0$ as $k \to \infty$, hence we have $\operatorname{cap}_N(x_1, \ldots, x_n) = 0$. Consequently, $\operatorname{cap}(x_1, \ldots, x_n) = 0$. If, on the other hand, $d(k) \to \infty$, then, since $\|w_k(x_1, \ldots, x_n)\| \geqslant \operatorname{cap}_{d(k)}(x_1, \ldots, x_n)$ it follows that a subsequence of $\{(\operatorname{cap}_k(x_1, \ldots, x_n))^{1/k}\}$ tends to zero. Since, the entire sequence is always convergent it follows that the limit of the entire sequence must be zero, hence the elements $x_1, \ldots, x_n$ have the capacity equal to zero. This completes the proof.

Now we shall establish a relationship between the capacity of elements and the capacity of their joint spectrum.

THEOREM 2. *The capacity of any n elements $x_1, \ldots, x_n$ of a Banach algebra A is equal to the capacity of their joint spectrum.*

Proof. Let $\sigma(x_1, \ldots, x_n)$ denote the joint spectrum of the elements $x_1, \ldots, x_n$ (see [6], p. 47). For any monic polynomial of n variables w and for any maximal ideal $M \in \mathfrak{M}(A)$ the following equality holds

$$w(x_1, \ldots, x_n)^{\wedge}(M) = w\big(x_1^{\wedge}(M), \ldots, x_n^{\wedge}(M)\big)$$

($x^{\wedge}(M)$ denote the Gelfand transform of an element x; see [6], p. 38) This implies that $\sigma\big(w(x_1, \ldots, x_n)\big) = w\big(\sigma(x_1, \ldots, x_n)\big)$.

Next, we obtain the following equalities

$$\begin{aligned}\|w(x_1, \ldots, x_n)\|_s &= \sup\{|\lambda|:\ \lambda \in \sigma\big(w(x_1, \ldots, x_n)\big)\}\\ &= \sup\{|w(\mu)|:\ \mu \in \sigma(x_1, \ldots, x_n)\} = \|w\|_{\sigma(x_1, \ldots, x_n)}.\end{aligned}$$

For simplicity we denote $\Omega = \sigma(x_1, \ldots, x_n)$. Therefore for every positive integer k and for every polynomial $w \in \boldsymbol{P}_k^1(n)$ we have

$$\operatorname{cap}_k \Omega = \inf\{\|v\|_\Omega:\ v \in \boldsymbol{P}_k^1(n)\} \leqslant \|w(x_1, \ldots, x_n)\|_s \leqslant \|w(x_1, \ldots, x_n)\|.$$

This implies $\operatorname{cap}_k \Omega \leqslant \operatorname{cap}_k(x_1, \ldots, x_n)$. Consequently, we obtain the inequality

$$(*) \qquad \operatorname{cap} \Omega \leqslant \operatorname{cap}(x_1, \ldots, x_n).$$

We take a sequence of polynomials $\{v_k\}$, such that $v_k \in \boldsymbol{P}_k^1(n)$ for every $k = 1, 2, \ldots$ and $\lim\limits_k (\|v_k\|_\Omega)^{1/k} = \operatorname{cap} \Omega$. If for some integer k the equality $\|v_k\|_\Omega = 0$ holds, then $\lim\limits_N \big\|\big(v_k(x_1, \ldots, x_n)\big)^N\big\|^{1/N} = 0$ and all the more $\lim\limits_N \big\|\big(v_k(x_1, \ldots, x_n)\big)^N\big\|^{1/kN} = 0$. Thus, the elements $x_1, \ldots, x_n$ are quasi-algebraically dependent. Applying Theorem 1 we obtain the equality $\operatorname{cap}(x_1, \ldots, x_n) = 0$, in view of $(*)$ it implies $\operatorname{cap} \Omega = 0$. So, the required equality holds. Now, we consider the case when $\|v_k\|_\Omega > 0$ for every positive integer k. Using the definition of the spectral radius in terms of the norm we obtain an integer $N(k)$, such that the following inequality holds

$$\big\|\big(v_k(x_1, \ldots, x_n)\big)^{N(k)}\big\|^{1/N(k)} \leqslant 2\,\|v_k\|_\Omega.$$

This implies

$$\big\|\big(v_k(x_1, \ldots, x_n)\big)^{N(k)}\big\|^{1/kN(k)} \leqslant 2^{1/k}(\|v_k\|_\Omega)^{1/k}.$$

The right term tends to $\operatorname{cap} \Omega$ as $k \to \infty$, and therefore

$$\limsup_k \big\|\big(v_k(x_1, \ldots, x_n)\big)^{N(k)}\big\|^{1/kN(k)} \leqslant \operatorname{cap} \Omega.$$

On the other hand, by definition,

$$\left(\operatorname{cap}_{kN(k)}(x_1, \ldots, x_n)\right)^{1/kN(k)} \leqslant \left\| \left(v_k(x_1, \ldots, x_n)\right)^{N(k)} \right\|^{1/kN(k)},$$

and therefore

$$\operatorname{cap}(x_1, \ldots, x_n) \leqslant \limsup_k \left\| \left(v_k(x_1, \ldots, x_n)\right)^{N(k)} \right\|^{1/kN(k)}.$$

Finally, $\operatorname{cap}(x_1, \ldots, x_n) \leqslant \operatorname{cap}\Omega$ and this completes the proof.

Now we are in a position to prove the following theorem.

THEOREM 3. *For any elements $x_1, \ldots, x_n$ of a Banach algebra A and any sequence $1 \leqslant i_1 < \ldots < i_m \leqslant n$ of indices the following inequality holds*

$$\operatorname{cap}(x_1, \ldots, x_n) \leqslant \operatorname{cap}(x_{i_1}, \ldots, x_{i_m}).$$

Proof. Let $j_1, \ldots, j_{n-m}$ be a sequence of integers having the following properties:

(i) $j_t \neq i_s$ for $t = 1, \ldots, n-m$ and $s = 1, \ldots, m$;

(ii) $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_{n-m}\} = \{1, \ldots, n\}$.

Let p be a monic polynomial of $n-m$ variables, such that $p(x_{j_1} \ldots, x_{jn-m}) \neq 0$. Let the degree of p be equal to l. Let $\{v_k\}$ be a sequence of monic polynomials of m variables, such that

$$\lim_k \|v_k(x_{i_1}, \ldots, x_{i_m})\|^{1/k} = \operatorname{cap}(x_{i_1}, \ldots, x_{i_m}).$$

Since

$$w_k(X_1, \ldots, X_n) = v_k(X_{i_1}, \ldots, X_{i_m}) \cdot p(X_{j_1}, \ldots, X_{j_{n-m}}), \qquad k = 1, 2, \ldots$$

are monic polynomials of n variables we have

$$\operatorname{cap}_{k+l}(x_1, \ldots, x_n) \leqslant \|w_k(x_1, \ldots, x_n)\| \leqslant \|v_k(x_{i_1}, \ldots, x_{i_m})\| \cdot \|p(x_{j_1}, \ldots, x_{j_{n-m}})\|.$$

Consequently

$$\left(\operatorname{cap}_{k+l}(x_1, \ldots, x_n)\right)^{1/(k+l)} \leqslant \|v_k(x_{i_1}, \ldots, x_{i_m})\|^{1/(k+l)} \cdot \|p(x_{j_1}, \ldots, x_{j_{n-m}})\|^{1/(k+l)}.$$

The right-hand side term tends to $\operatorname{cap}(x_{i_1}, \ldots, x_{i_m})$ as $k \to \infty$, and therefore

$$\operatorname{cap}(x_1, \ldots, x_n) \leqslant \operatorname{cap}(x_{i_1}, \ldots, x_{i_m}).$$

The proof is completed.

Remark 4. The example given in Remark 1 shows that the inequality in Theorem 3 may be essentially strong. So, we have in that case $\operatorname{cap}(x_1, x_2) = 0$ while $\operatorname{cap} x_1 = \frac{1}{4}$ and $\operatorname{cap} x_2 = \frac{1}{8}$.

COROLLARY 1. *For any system of elements of a Banach algebra A and for any sequence $1 \leqslant i_1 < \ldots < i_m \leqslant n$ of integers the following inequality holds*

$$\operatorname{cap}\sigma(x_1, \ldots, x_n) \leqslant \operatorname{cap}\sigma(x_{i_1}, \ldots, x_{i_m}).$$

COROLLARY 2. *For any element $x \in A$ the following equality holds*

$$\operatorname{cap} x = \operatorname{cap}(x, \ldots, x).$$

Proof. Let $\{v_k\}$ be a sequence of monic polynomials, such that $v_k \in \boldsymbol{P}_k^1(n)$ and $\lim_k \|v_k(x, \ldots, x)\|^{1/k} = \operatorname{cap}(x, \ldots, x)$. Then $p_k(X) = v_k(X, \ldots, X)$ is the monic polynomial of one variable with degree k. Hence we have $\operatorname{cap}_k x \leqslant \|p_k(x)\| = \|v_k(x, \ldots, x)\|$.

This implies $\operatorname{cap} x \leqslant \operatorname{cap}(x, \ldots, x)$. The proof is completed.

COROLLARY 3. *For any element $x \in A$ the following equality holds*

$$\operatorname{cap} \sigma(x) = \operatorname{cap} \sigma(x, \ldots, x).$$

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