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Almost-bounded sets and some functions

1. Introduction. In 1973, P. Th. Lambrinos [2] introduced the concept of bounded sets in a topological space. Quite recently, in [3] and [4], he has also defined almost-bounded sets and nearly-bounded sets as generalizations of bounded sets and investigated their properties. The relations among these boundedness concepts are similar to those among compactness, almost-compactness and near-compactness. The main purpose of this note is to show the following two results: 1) The θ-continuous image of an almost-bounded set is almost-bounded; 2) The inverse image of an almost-bounded set under an almost-closed open surjection (not necessarily continuous) with nearly-bounded point inverses is almost-bounded.

Throughout the present note X and Y will always denote topological spaces on which no separation axioms are assumed. Let A be a subset of a topological space X. The closure of A in X and the interior of A in X are denoted by $\mathrm{Cl}_X(A)$ and $\mathrm{Int}_X(A)$ respectively. A subset A of X is said to be *regularly open* if $\mathrm{Int}_X(\mathrm{Cl}_X(A)) = A$, and *regularly closed* if $\mathrm{Cl}_X(\mathrm{Int}_X(A)) = A$.

2. Definitions and remarks. The following definitions of boundedness and its generalizations are due to P. Th. Lambrinos [3]. A family $\mathscr{F} \subset 2^X$ is called an *ideal* on X if the family $\mathscr{F}^c = \{X - F \mid F \in \mathscr{F}\}$ is a filter on X. A subfamily $\mathscr{F}^*$ of an ideal $\mathscr{F}$ is called a *base* (resp. *subbase*) of $\mathscr{F}$ if $\mathscr{F}^{*c} = \{X - F \mid F \in \mathscr{F}^*\}$ is a base (resp. subbase) of $\mathscr{F}^c$. An ideal $\mathscr{F}$ is said to be *local* on a subset A of X if for each point $x \in A$, there exists a member $F \in \mathscr{F}$ such that F is an open set containing x.

Definition 1. A subset A of X is said to be *bounded* (resp. *almost-bounded*, *nearly-bounded*) in X [3] if A belongs to every ideal $\mathscr{F}$ on X having the following properties: 1) $\mathscr{F}$ is local on X; 2) $\mathscr{F}$ has a base (resp. base, subbase) consisting of open (resp. closed, regularly open) sets.

Remark 1. It is known that boundedness $\Rightarrow$ near-boundedness $\Rightarrow$ almost-boundedness, but none of these implications is reversible [3].

DEFINITION 2. A function $f\colon X \to Y$ is said to have *nearly-bounded* (resp. *bounded*) *point inverses* if for each point $y \in Y$, $f^{-1}(y)$ is nearly-bounded (resp. bounded) in X.

We shall recall some definitions of functions which are weaker than continuous functions.

DEFINITION 3. A function $f\colon X \to Y$ is said to be *almost-continuous* (resp. *θ-continuous*, *weakly-continuous*) [10] (resp. [1], [5]) if for each point $x \in X$ and each neighborhood V of $f(x)$ in Y, there exists a neighborhood U of x in X such that

$$f(U) \subset \mathrm{Int}_Y(\mathrm{Cl}_Y(V)) \quad (\text{resp. } f(\mathrm{Cl}_X(U)) \subset \mathrm{Cl}_Y(V),\ f(U) \subset \mathrm{Cl}_Y(V)).$$

Remark 2. It is known that continuity $\Rightarrow$ almost-continuity $\Rightarrow$ θ-continuity $\Rightarrow$ weak-continuity [8], [10].

DEFINITION 4. A function $f\colon X \to Y$ is said to be *almost-open* (resp. *almost-closed*) [10] if for each regularly open (resp. regularly closed) set A of $X, f(A)$ is open (resp. closed) in Y.

Remark 3. Every open (resp. closed) function is almost-open (resp. almost-closed), but the converse does not hold [10].

3. The θ-continuous image of an almost-bounded set. The following lemmas, due to P. Th. Lambrinos, are very usefull in the sequel.

LEMMA 1 (Lambrinos [2]). *A subset A of X is bounded in X if and only if for any open cover $\mathscr{U}$ of X, there exists a finite subfamily $\mathscr{U}_0$ of $\mathscr{U}$ such that $A \subset \bigcup \{U \mid U \in \mathscr{U}_0\}$.*

LEMMA 2 (Lambrinos [3]). *A subset A of X is almost-bounded (resp. nearly-bounded) in X if and only if for any open cover $\mathscr{U}$ of X, there exists a finite subfamily $\mathscr{U}_0$ of $\mathscr{U}$ such that*

$$A \subset \bigcup \{\mathrm{Cl}_X(U) \mid U \in \mathscr{U}_0\} \quad (\textit{resp. } A \subset \bigcup \{\mathrm{Int}_X(\mathrm{Cl}_X(U)) \mid U \in \mathscr{U}_0\}).$$

P. Th. Lambrinos showed that the continuous image of an almost-bounded set is almost-bounded [3], Theorem 3.1. The word "continuous" in this result can be replaced by "θ-continuous", as the following theorem shows.

THEOREM 1. *The θ-continuous image of an almost-bounded set is almost-bounded.*

Proof. Suppose that $f\colon X \to Y$ is a θ-continuous function and A is an almost-bounded set in X. We shall show that $f(A)$ is an almost-bounded set in Y. For this purpose let $\{V\alpha \mid \alpha \in \mathscr{A}\}$ be any open cover of Y. Then for each point $x \in X$, there exists an element $\alpha(x) \in \mathscr{A}$ such that $f(x) \in V\alpha(x)$. Since f is θ-continuous, there exists an open neighborhood $U\alpha(x)$ of x in X such that $f(\mathrm{Cl}_X(U\alpha(x))) \subset \mathrm{Cl}_Y(V\alpha(x))$. The family $\{U\alpha(x) \mid x \in X\}$ is an open cover of X. Since A is almost-bounded in X, by Lemma 2, there exists a finite subfamily $\{\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)\}$ of $\mathscr{A}$ such that $A \subset \bigcup\{\mathrm{Cl}_X(U\alpha(x_j)) \mid j = 1, \ldots, n\}$. Therefore, we have

$$f(A) \subset \bigcup_{j=1}^{n} f\big(\mathrm{Cl}_X(U\alpha(x_j))\big) \subset \bigcup_{j=1}^{n} \mathrm{Cl}_Y(V\alpha(x_j)).$$

By Lemma 2, we observe that $f(A)$ is almost-bounded in Y.

THEOREM 2. *The θ-continuous almost-open image of a nearly-bounded set is nearly-bounded.*

Proof. Suppose that $f\colon X \to Y$ is a θ-continuous almost-open function and A is a nearly-bounded set in X. We shall show that $f(A)$ is a nearly-bounded set in Y. For this purpose let $\{V\alpha \mid \alpha \in \mathscr{A}\}$ be any open cover of Y. Then the family $\{\mathrm{Int}_Y(\mathrm{Cl}_Y(V\alpha)) \mid \alpha \in \mathscr{A}\}$ is a regularly open cover of Y. Since f is θ-continuous almost-open, it is almost-continuous [6], Theorem 4. Since the inverse image of a regularly open set under an almost-continuous and almost-open function is regularly open [7], Lemma 1, the family $\{f^{-1}(\mathrm{Int}_Y(\mathrm{Cl}_Y(V\alpha))) \mid \alpha \in \mathscr{A}\}$ is a regularly open cover of X. Since A is nearly-bounded in X, by Lemma 2, there exists a finite subfamily $\mathscr{A}_0$ of $\mathscr{A}$ such that $A \subset \bigcup\{f^{-1}(\mathrm{Int}_Y(\mathrm{Cl}_Y(V\alpha))) \mid \alpha \in \mathscr{A}_0\}$. Thus we have

$$f(A) \subset \bigcup\{\mathrm{Int}_Y(\mathrm{Cl}_Y(V\alpha)) \mid \alpha \in \mathscr{A}_0\}.$$

By Lemma 2, we observe that $f(A)$ is nearly-bounded in Y.

COROLLARY 1 (Lambrinos [3]). *Let $f\colon X \to Y$ be a continuous (resp. continuous open) function. If A is an almost-bounded (resp. nearly-bounded) set in X, then $f(A)$ is almost-bounded (resp. nearly-bounded) in Y.*

Proof. This is an immediate consequence of Theorem 1 and Theorem 2.

THEOREM 3. *The almost-continuous (resp. weakly-continuous) image of a bounded set is nearly-bounded (resp. almost-bounded).*

Proof. This is proven similarly to Theorem 1.

4. The inverse image of an almost-bounded set.

LEMMA 3 (Sikorski [9]). *A function $f\colon X \to Y$ is open if and only if $f^{-1}(\mathrm{Cl}_Y(B)) \subset \mathrm{Cl}_X(f^{-1}(B))$ for every subset B of Y.*

THEOREM 4. *The inverse image of an almost-bounded set under an open and almost-closed surjection with nearly-bounded point inverses is almost-bounded.*

Proof. Suppose that $f\colon X \to Y$ is an open and almost-closed surjection with nearly-bounded point inverses. Let B be an almost-bounded set in Y and we will show that $f^{-1}(B)$ is an almost-bounded set in X. Let $\{U\alpha \mid \alpha \in \mathscr{A}\}$ be any open cover of X. Since f has nearly-bounded point inverses, for each point $y \in Y$, there exists a finite subset $\mathscr{A}(y)$ of $\mathscr{A}$ such that $f^{-1}(y) \subset \bigcup\{\mathrm{Int}_X(\mathrm{Cl}_X(U\alpha)) \mid \alpha \in \mathscr{A}(y)\}$. Let us put Uy

$= \mathrm{Int}_X[\bigcup\{\mathrm{Cl}_X(U\alpha) \mid \alpha \in \mathscr{A}(y)\}]$, then Uy is a regularly open set containing $f^{-1}(y)$. Moreover, put $Vy = Y - f(X - Uy)$, then we obtain $f^{-1}(Vy) \subset Uy$ and Vy is an open neighborhood of y in Y because f is almost-closed. The family $\{Vy \mid y \in Y\}$ is an open cover of Y. Since B is almost-bounded in Y, by Lemma 2, there exist a finite number of points $y_1, y_2, \dots, y_n$ in Y such that $B \subset \bigcup\{\mathrm{Cl}_Y(Vy_j) \mid j = 1, 2, \dots, n\}$. Since f is open, by using Lemma 3, we obtain

$$f^{-1}(B) \subset \bigcup_{j=1}^{n} f^{-1}\big(\mathrm{Cl}_Y(Vy_j)\big) \subset \bigcup_{j=1}^{n} \mathrm{Cl}_X\big(f^{-1}(Vy_j)\big) \subset \bigcup_{j=1}^{n} \mathrm{Cl}_X(Uy_j)$$
$$= \bigcup_{j=1}^{n} \bigcup_{\alpha \in \mathscr{A}(y_j)} \mathrm{Cl}_X(U\alpha).$$

By Lemma 2, we observe that $f^{-1}(B)$ is almost-bounded in X.

Remark 4. In Theorem 4, the assumption "open" can be replaced by the following condition: $f^{-1}\big(\mathrm{Cl}_Y(V)\big) \subset \mathrm{Cl}_X\big(f^{-1}(V)\big)$ for every open set V of Y.

COROLLARY 2. *Let f: $X \to Y$ be a perfect (closed continuous surjection with compact point inverses) open function. If A is an almost-bounded set in X (resp. Y), then $f(A)$ (resp. $f^{-1}(A)$) is almost-bounded in Y (resp. X).*

Proof. This is an immediate consequence of Theorem 1 and Theorem 4.

THEOREM 5. *The inverse image of a bounded set under an almost-closed surjection with nearly-bounded point inverses is almost-bounded.*

Proof. Suppose that f: $X \to Y$ is an almost-closed surjection with nearly-bounded point inverses. Let B be a bounded set in Y and we will show that $f^{-1}(B)$ is almost-bounded in X. Let $\{U\alpha \mid \alpha \in \mathscr{A}\}$ be any open cover of X. Similarly to the proof of Theorem 4, for each point $y \in Y$, there exist a finite subfamily $\mathscr{A}(y)$ of $\mathscr{A}$ and an open neighborhood Vy of y in Y such that $f^{-1}(Vy) \subset \bigcup\{\mathrm{Cl}_X(U\alpha) \mid \alpha \in \mathscr{A}(y)\}$ because f is almost-closed and $f^{-1}(y)$ is nearly-bounded. The family $\{Vy \mid y \in Y\}$ is an open cover of Y. Since B is a bounded set in Y, by Lemma 1, there exist a finite number of point $y_1, y_2, \dots, y_n$ in Y such that $B \subset \bigcup\{Vy_j \mid j = 1, 2, \dots, n\}$. Therefore, we obtain

$$f^{-1}(B) \subset \bigcup_{j=1}^{n} f^{-1}(Vy_j) \subset \bigcup_{j=1}^{n} \bigcup_{\alpha \in \mathscr{A}(y_j)} \mathrm{Cl}_X(U\alpha).$$

By Lemma 2, we observe that $f^{-1}(B)$ is almost-bounded in X.

THEOREM 6. *The inverse image of a bounded set under a closed surjection with nearly-bounded (resp. bounded) point inverses is nearly-bounded (resp. bounded).*

Proof. This is proven similarly to Theorem 4.

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