Almost-bounded sets and some functions

1. Introduction. In 1973, P. Th. Lambrinos [2] introduced the concept of bounded sets in a topological space. Quite recently, in [3] and [4], he has also defined almost-bounded sets and nearly-bounded sets as generalizations of bounded sets and investigated their properties. The relations among these boundedness concepts are similar to those among compactness, almost-compactness and near-compactness. The main purpose of this note is to show the following two results: 1) The $\theta$-continuous image of an almost-bounded set is almost-bounded; 2) The inverse image of an almost-bounded set under an almost-closed open surjection (not necessarily continuous) with nearly-bounded point inverses is almost-bounded.

Throughout the present note $X$ and $Y$ will always denote topological spaces on which no separation axioms are assumed. Let $A$ be a subset of a topological space $X$. The closure of $A$ in $X$ and the interior of $A$ in $X$ are denoted by $\text{Cl}_X(A)$ and $\text{Int}_X(A)$ respectively. A subset $A$ of $X$ is said to be regularly open if $\text{Int}_X(\text{Cl}_X(A)) = A$, and regularly closed if $\text{Cl}_X(\text{Int}_X(A)) = A$.

2. Definitions and remarks. The following definitions of boundedness and its generalizations are due to P. Th. Lambrinos [3]. A family $\mathcal{F} \subseteq 2^X$ is called an ideal on $X$ if the family $\mathcal{F}^c = \{X - F \mid F \in \mathcal{F}\}$ is a filter on $X$. A subfamily $\mathcal{F}^*$ of an ideal $\mathcal{F}$ is called a base (resp. subbase) of $\mathcal{F}$ if $\mathcal{F}^c = \{X - F \mid F \in \mathcal{F}^*\}$ is a base (resp. subbase) of $\mathcal{F}^c$. An ideal $\mathcal{F}$ is said to be local on a subset $A$ of $X$ if for each point $x \in A$, there exists a member $F \in \mathcal{F}$ such that $F$ is an open set containing $x$.

DEFINITION 1. A subset $A$ of $X$ is said to be bounded (resp. almost-bounded, nearly-bounded) in $X$ [3] if $A$ belongs to every ideal $\mathcal{F}$ on $X$ having the following properties: 1) $\mathcal{F}$ is local on $X$; 2) $\mathcal{F}$ has a base (resp. base, subbase) consisting of open (resp. closed, regularly open) sets.

Remark 1. It is known that boundedness $\Rightarrow$ near-boundedness $\Rightarrow$ almost-boundedness, but none of these implications is reversible [3].
Definition 2. A function \( f: X \to Y \) is said to have nearly-bounded (resp. bounded) point inverses if for each point \( y \in Y \), \( f^{-1}(y) \) is nearly-bounded (resp. bounded) in \( X \).

We shall recall some definitions of functions which are weaker than continuous functions.

Definition 3. A function \( f: X \to Y \) is said to be almost-continuous (resp. \( \theta \)-continuous, weakly-continuous) [10] (resp. [1], [5]) if for each point \( x \in X \) and each neighborhood \( V \) of \( f(x) \) in \( Y \), there exists a neighborhood \( U \) of \( x \) in \( X \) such that
\[
f(U) \subseteq \text{Int}_Y(\text{Cl}_Y(V)) \quad \text{(resp. } f(\text{Cl}_X(U)) \subseteq \text{Cl}_Y(V), f(U) \subseteq \text{Cl}_Y(V)).\]

Remark 2. It is known that continuity \( \Rightarrow \) almost-continuity \( \Rightarrow \) \( \theta \)-continuity \( \Rightarrow \) weak-continuity [8], [10].

Definition 4. A function \( f: X \to Y \) is said to be almost-open (resp. almost-closed) [10] if for each regularly open (resp. regularly closed) set \( A \) of \( X \), \( f(A) \) is open (resp. closed) in \( Y \).

Remark 3. Every open (resp. closed) function is almost-open (resp. almost-closed), but the converse does not hold [10].

3. The \( \theta \)-continuous image of an almost-bounded set. The following lemmas, due to P. Th. Lambrinos, are very useful in the sequel.

Lemma 1 (Lambrinos [2]). A subset \( A \) of \( X \) is bounded in \( X \) if and only if for any open cover \( \mathcal{U} \) of \( X \), there exists a finite subfamily \( \mathcal{U}_0 \) of \( \mathcal{U} \) such that \( A \subseteq \bigcup \{ U \mid U \in \mathcal{U}_0 \} \).

Lemma 2 (Lambrinos [3]). A subset \( A \) of \( X \) is almost-bounded (resp. nearly-bounded) in \( X \) if and only if for any open cover \( \mathcal{U} \) of \( X \), there exists a finite subfamily \( \mathcal{U}_0 \) of \( \mathcal{U} \) such that
\[ A \subseteq \bigcup \{ \text{Cl}_X(U) \mid U \in \mathcal{U}_0 \} \quad \text{(resp. } A \subseteq \bigcup \{ \text{Int}_X(\text{Cl}_X(U)) \mid U \in \mathcal{U}_0 \}).\]

P. Th. Lambrinos showed that the continuous image of an almost-bounded set is almost-bounded [3], Theorem 3.1. The word "continuous" in this result can be replaced by "\( \theta \)-continuous", as the following theorem shows.

Theorem 1. The \( \theta \)-continuous image of an almost-bounded set is almost-bounded.

Proof. Suppose that \( f: X \to Y \) is a \( \theta \)-continuous function and \( A \) is an almost-bounded set in \( X \). We shall show that \( f(A) \) is an almost-bounded set in \( Y \). For this purpose let \( \{ V_{a} \mid a \in \mathcal{A} \} \) be any open cover of \( Y \). Then for each point \( a \in X \), there exists an element \( a(x) \in \mathcal{A} \) such that \( f(x) \in V_{a(x)}. \) Since \( f \) is \( \theta \)-continuous, there exists an open neighborhood \( U_{a(x)} \) of \( x \) in \( X \) such that \( f(\text{Cl}_X(U_{a(x)})) \subseteq \text{Cl}_Y(V_{a(x)}). \) The family \( \{ U_{a(x)} \mid x \in X \} \) is an open cover of \( X \). Since \( A \) is almost-bounded in \( X \), by Lemma 2,
there exists a finite subfamily \( \{a(w_1), a(w_2), \ldots, a(w_n)\} \) of \( A \) such that 
\[ A \subseteq \bigcup \{ \text{Cl}_X(Ua(x_j)) \mid j = 1, \ldots, n \}. \]
Therefore, we have
\[ f(A) \subseteq \bigcup_{j=1}^n f(\text{Cl}_X(Ua(x_j))) \subseteq \bigcup_{j=1}^n \text{Cl}_Y(Va(w_j)). \]
By Lemma 2, we observe that \( f(A) \) is almost-bounded in \( Y \).

**Theorem 2.** The \( \theta \)-continuous almost-open image of a nearly-bounded set is nearly-bounded.

**Proof.** Suppose that \( f: X \to Y \) is a \( \theta \)-continuous almost-open function and \( A \) is a nearly-bounded set in \( X \). We shall show that \( f(A) \) is a nearly-bounded set in \( Y \). For this purpose let \( \{V\alpha \mid \alpha \in A\} \) be any open cover of \( Y \). Then the family \( \{\text{Int}_Y(\text{Cl}_Y(V\alpha)) \mid \alpha \in A\} \) is a regularly open cover of \( Y \). Since \( f \) is \( \theta \)-continuous almost-open, it is almost-continuous [6], Theorem 4. Since the inverse image of a regularly open set under an almost-continuous and almost-open function is regularly open [7], Lemma 1, the family \( \{f^{-1}(\text{Int}_Y(\text{Cl}_Y(V\alpha))) \mid \alpha \in A\} \) is a regularly open cover of \( X \). Since \( A \) is nearly-bounded in \( X \), by Lemma 2, there exists a finite subfamily \( A_0 \) of \( A \) such that
\[ A \subseteq \bigcup \{f^{-1}(\text{Int}_Y(\text{Cl}_Y(V\alpha))) \mid \alpha \in A_0\}. \]
Thus we have
\[ f(A) \subseteq \bigcup \{\text{Int}_Y(\text{Cl}_Y(V\alpha)) \mid \alpha \in A_0\}. \]

By Lemma 2, we observe that \( f(A) \) is nearly-bounded in \( Y \).

**Corollary 1 (Lambrinos [3]).** Let \( f: X \to Y \) be a continuous (resp. continuous open) function. If \( A \) is an almost-bounded (resp. nearly-bounded) set in \( X \), then \( f(A) \) is almost-bounded (resp. nearly-bounded) in \( Y \).

**Proof.** This is an immediate consequence of Theorem 1 and Theorem 2.

**Theorem 3.** The almost-continuous (resp. weakly-continuous) image of a bounded set is nearly-bounded (resp. almost-bounded).

**Proof.** This is proven similarly to Theorem 1.

**4. The inverse image of an almost-bounded set.**

**Lemma 3 (Sikorski [9]).** A function \( f: X \to Y \) is open if and only if 
\[ f^{-1}(\text{Cl}_Y(B)) \subseteq \text{Cl}_X(f^{-1}(B)) \]
for every subset \( B \) of \( Y \).

**Theorem 4.** The inverse image of an almost-bounded set under an open and almost-closed surjection with nearly-bounded point inverses is almost-bounded.

**Proof.** Suppose that \( f: X \to Y \) is an open and almost-closed surjection with nearly-bounded point inverses. Let \( B \) be an almost-bounded set in \( Y \) and we will show that \( f^{-1}(B) \) is an almost-bounded set in \( X \). Let \( \{U\alpha \mid \alpha \in A\} \) be any open cover of \( X \). Since \( f \) has nearly-bounded point inverses, for each point \( y \in Y \), there exists a finite subset \( A(y) \) of \( A \) such that 
\[ f^{-1}(y) \subseteq \bigcup \{\text{Int}_X(\text{Cl}_X(U\alpha)) \mid \alpha \in A(y)\}. \]
Let us put \( Uy \)}
= \text{Int}_X \left[ \bigcup \{ \text{Cl}_X(Ua) \mid a \in \mathcal{A}(y) \} \right]$, then $Uy$ is a regularly open set containing $f^{-1}(y)$. Moreover, put $Vy = Y - f(X - Uy)$, then we obtain $f^{-1}(Vy) \subseteq Uy$ and $Vy$ is an open neighborhood of $y$ in $Y$ because $f$ is almost-closed. The family $\{ Vy \mid y \in Y \}$ is an open cover of $Y$. Since $B$ is almost-bounded in $Y$, by Lemma 2, there exist a finite number of points $y_1, y_2, \ldots, y_n$ in $Y$ such that $B \subseteq \bigcup \{ \text{Cl}_Y(Vy_j) \mid j = 1, 2, \ldots, n \}$. Since $f$ is open, by using Lemma 3, we obtain

$$f^{-1}(B) = \bigcup_{j=1}^n f^{-1}(\text{Cl}_Y(Vy_j)) \subseteq \bigcup_{j=1}^n \text{Cl}_X(f^{-1}(Vy_j)) \subseteq \bigcup_{j=1}^n \text{Cl}_X(Uy_j)$$

$$= \bigcup_{j=1}^n \bigcup_{a \in \mathcal{A}(y_j)} \text{Cl}_X(Ua).$$

By Lemma 2, we observe that $f^{-1}(B)$ is almost-bounded in $X$.

Remark 4. In Theorem 4, the assumption "open" can be replaced by the following condition: $f^{-1}(\text{Cl}_Y(V)) \subseteq \text{Cl}_X(f^{-1}(V))$ for every open set $V$ of $Y$.

Corollary 2. Let $f : X \to Y$ be a perfect (closed continuous surjection with compact point inverses) open function. If $A$ is an almost-bounded set in $X$ (resp. $Y$), then $f(A)$ (resp. $f^{-1}(A)$) is almost-bounded in $Y$ (resp. $X$).

Proof. This is an immediate consequence of Theorem 1 and Theorem 4.

Theorem 5. The inverse image of a bounded set under an almost-closed surjection with nearly-bounded point inverses is almost-bounded.

Proof. Suppose that $f : X \to Y$ is an almost-closed surjection with nearly-bounded point inverses. Let $B$ be a bounded set in $Y$ and we will show that $f^{-1}(B)$ is almost-bounded in $X$. Let $\{ Ua \mid a \in \mathcal{A} \}$ be any open cover of $X$. Similarly to the proof of Theorem 4, for each point $y \in Y$, there exist a finite subfamily $\mathcal{A}(y)$ of $\mathcal{A}$ and an open neighborhood $Vy$ of $y$ in $Y$ such that $f^{-1}(Vy) \subseteq \bigcup \{ \text{Cl}_X(Ua) \mid a \in \mathcal{A}(y) \}$ because $f$ is almost-closed and $f^{-1}(y)$ is nearly-bounded. The family $\{ Vy \mid y \in Y \}$ is an open cover of $Y$. Since $B$ is a bounded set in $Y$, by Lemma 1, there exist a finite number of point $y_1, y_2, \ldots, y_n$ in $Y$ such that $B \subseteq \bigcup \{ Vy_j \mid j = 1, 2, \ldots, n \}$. Therefore, we obtain

$$f^{-1}(B) = \bigcup_{j=1}^n f^{-1}(Vy_j) \subseteq \bigcup_{j=1}^n \bigcup_{a \in \mathcal{A}(y_j)} \text{Cl}_X(Ua).$$

By Lemma 2, we observe that $f^{-1}(B)$ is almost-bounded in $X$.

Theorem 6. The inverse image of a bounded set under a closed surjection with nearly-bounded (resp. bounded) point inverses is nearly-bounded (resp. bounded).

Proof. This is proven similarly to Theorem 4.
References


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