ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XIX (1977)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XIX (1977)

T. W. KÖRNER (Trinity Hall, Cambridge)

# Integration and pointwise convergence

**1. Introduction.** One method of constructing $L^1([0,1])$ is to make use of Osgood's Theorem.

THEOREM 1. *Let $f_1, f_2, \ldots$ be continuous functions from $[0,1]$ to $\mathbf{R}$. Suppose*

(i) *There exists an $M$ such that $|f_i(x)| \leqslant M$ for all $x \in [0,1]$, $i \geqslant 1$;*

(ii) *$f_n(x) \to 0$ as $n \to \infty$ for all $x \in [0,1]$.*

*Then $\int_0^1 f_n(x)\,dx \to 0$ as $n \to \infty$.*

In this paper we shall give a new proof of this result.

**2. A proof of Osgood's Theorem.** Osgood's Theorem is a simple consequence of the following theorem, well known in the theory of uniform algebras.

THEOREM 2. *Under the conditions of Theorem 1, given $\varepsilon > 0$, we can find $\lambda_1, \lambda_2, \ldots, \lambda_n \geqslant 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that*

$$\Big\| \sum_{i=1}^{n} \lambda_i f_i \Big\|_{C([0,1])} \leqslant \varepsilon.$$

That is to say that pointwise convergence implies uniform convergence for some sequence of convex combinations.

Proof of Theorem 1 using Theorem 2. Suppose $\int_0^1 f_n(x)\,dx \not\to 0$. Then we can find a $\delta > 0$ such that $\left|\int_0^1 f_n(x)\,dx\right| \geqslant \delta$ for infinitely many $n$. Thus either $\int_0^1 f_n(x)\,dx \geqslant \delta$ infinitely often or $\int_0^1 f_n(x)\,dx \leqslant -\delta$ infinitely often (or both events occur). Thus without loss of generality we may assume that there exist $n(j) \to \infty$ with $\int_0^1 f_{n(j)}(x)\,dx \geqslant \delta$.

But by Theorem 2 we can find $\lambda_1, \lambda_2, \ldots, \lambda_k$ with $\Big\|\sum_{j=1}^{k} \lambda_j f_{n(j)}\Big\| \leqslant \delta/2$

and so

$$\delta/2 \geqslant \int \sum_{j=1}^{k} \lambda_j f_{n(j)}(x)\,dx \geqslant \delta$$

which is absurd. Thus THEOREM 1 follows.

Unfortunately the standard proof of Theorem 2 uses Osgood's theorem followed by the theorem of Hahn and Banach. (For the proof and an extension by Björk and, independently, Kaufman to solve a long open problem in harmonic analysis, see [4], Chapter XIII.) The original proofs by Zalcwasser [5] and Gillespie and Hurwitz [3] do not use Osgood's theorem but rely on transfinite induction. (For further references for Theorem 2, see [1], p. 462. For further references for Theorem 1 together with another, very elegant, elementary proof, see [2].)

We give a new proof of Theorem 2 without using the axiom of choice.

**3. A proof of Theorem 2.** In this section $f_1, f_2, \ldots$ will be functions satisfying the conditions of Theorem 1. Let us establish some notation. If $g_1, g_2, \ldots \in C([0,1])$, we write $\Sigma(g_1, g_2, \ldots) = \{\sum_{i=1}^{n} \lambda_i g_i \colon \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_j \geqslant 0$ $[1 \leqslant j \leqslant n],\ n \geqslant 1\}$. We call a closed set $X \subseteq [0,1]$ a $Q(\delta)$ set for some $1 > \delta > 0$ if, given $g_i \in \Sigma(f_i, f_{i+1}, \ldots)$, we can find a $g \in \Sigma(g_1, g_2, \ldots)$ with $\|g\|_{C(X)} \leqslant \delta$. We shall need the following result.

LEMMA 1. *If $X_1$ and $X_2$ are $Q(\delta)$ sets, then so is $X_1 \cup X_2$.*

Proof. Let $g_i \in \Sigma(f_i, f_{i+1}, \ldots)$ be given $[i \geqslant 1]$. Then, since $X_1$ is $Q(\delta)$ and $g_{i+j-1} \in \Sigma(f_i, f_{i+1}, \ldots)$, we can find $h_j \in \Sigma(g_j, g_{j+1}, \ldots)$ such that $\|h_j\|_{C(X_1)} \leqslant \delta$ $[j \geqslant 1]$. But $h_j \in \Sigma(f_j, f_{j+1}, \ldots)$ and $X_2$ is $Q(\delta)$, so we can find $h \in \Sigma(h_1, h_2, \ldots)$ with $\|h\|_{C(X_2)} \leqslant \delta$.

Since $\|h_j\|_{C(X_1)} \leqslant \delta$ for all $j \geqslant 1$, it follows that $\|h\|_{C(X_1 \cup X_2)} \leqslant \delta$ and by construction $h \in \Sigma(g_1, g_2, \ldots)$. The lemma is thus proved.

Using Lemma 1 we can prove

LEMMA 2. *Suppose $\delta > 0$ and $X$ is a closed set which is not $Q(\delta)$. Then, given any $n(0) \geqslant 1$ and any $\delta > \delta' > 0$, we can find a closed set $X' \subseteq X$ and an $n \geqslant n(0)$ such that*

(i) *$X'$ is not $Q(\delta')$,*

(ii) *$|f_n(x)| \geqslant \delta'$ for all $x \in X'$.*

This is the key result, for, once it is established, Theorem 2 follows easily.

Proof of Theorem 2 using Lemma 2. Suppose Theorem 2 is false. Then there exists a $\delta > 0$ such that $[0,1]$ is not a $Q(\delta)$ set. Write $X_0 = [0,1]$ and choose $\delta = \delta_0 > \delta_1 > \delta_2 > \ldots$ such that $\delta_j > \delta/2$ (for example $\delta_j = (1+2^{-j})\delta/2$). Then by repeated use of Lemma 2 we can find closed sets $X_0 \supseteq X_1 \supseteq X_2 \supset \ldots$ and integers $1 \leqslant n(1) < n(2) < \ldots$

such that

(i) $X_m$ is not $Q(\delta_m)$,

(ii) $|f_{n(m)}(x)| \geqslant \delta_m$ whenever $x \in X_m$ for all $m \geqslant 1$.

Since $[0, 1]$ is sequentially compact, we know that $\bigcap_{m=1}^{\infty} X_m \neq \emptyset$. Choose $x \in \bigcap_{m=1}^{\infty} X_m$. Then $|f_{n(m)}(x)| \geqslant \delta_m \geqslant \delta/2$ for all $m$ and so $f_i(x) \nrightarrow 0$ as $i \to \infty$. The lemma follows by reductio ad absurdum.

Proof of Lemma 2. Suppose the result is false. Since $X$ is not $Q(\delta)$, we can find $g_i \in \Sigma(f_i, f_{i+1}, \ldots)$ such that $g \in \Sigma(g_1, g_2, \ldots)$ implies $\|g\|_{C(X)} \geqslant \delta$. Set $\Sigma_0 = \Sigma(g_{n(0)}, g_{n(0)+1}, \ldots)$ and write $h_1 = g_{n_0}$.

Let $X_1 = \{x \in X\colon |h_1(x)| \geqslant \delta'\}$. We know that $h_1 = \sum_{i=n(0)}^{m} \lambda_i f_i$ for some $\sum_{i=n(0)}^{m} \lambda_i = 1$, $\lambda_j \geqslant 0$ $[n(0) \leqslant j \leqslant m]$ and so $X_1 \subset \bigcup_{i=n(0)}^{m} Y_i$, where $Y_j = \{x \in X\colon |f_j(x)| \geqslant \delta'\}$ $[n(0) \leqslant j \leqslant m]$. By hypothesis $Y_j$ is a $Q(\delta')$ set and so $\bigcup_{i=n(0)}^{m} Y_i$ is. Thus $X_1$ is a $Q(\delta')$ set and we can find $h_2 \in \Sigma_0$ such that $\|h_2\|_{C(X_1)} \leqslant \delta'$.

Set $X_2 = \{x \in X\colon |h_2(x)| \geqslant \delta\}$. By the same arguments $X_2$ is a $Q(\delta')$ set and so, by Lemma 1, $X_1 \cup X_2$ is. Thus we can find an $h_3 \in \Sigma_0$ such that $\|h_3\|_{C(X_1 \cup X_2)} \leqslant \delta'$.

Proceeding inductively, we obtain $X_1, X_2, \ldots \in Q(\delta')$ sets and $h_1, h_2, \ldots \in \Sigma_0$ such that

(i) $X_k = \{x \in X\colon |h_k(x)| \geqslant \delta'\}$,

(ii) $|h_k(x)| \leqslant \delta'$ for all $x \in \bigcup_{i=1}^{k-1} X_i$.

It follows at once that

(iii) $|h_k(x)| \leqslant \delta'$ for all $x \notin X_k \setminus \bigcup_{i=1}^{k-1} X_i$.

But by our hypothesis

(iv) $|h_k(x)| \leqslant M$ for all $x$ and so, in particular, for all $x \in X_k \setminus \bigcup_{k=1}^{k-1} X_i$.

Thus, setting $h = m^{-1} \sum_{k=1}^{m} h_k$, we have

(v) $\|h\|_{C(X)} \leqslant \dfrac{(m-1)\delta' + M}{m} < \delta$ provided only that $m \geqslant M(\delta - \delta')^{-1}$.

Since $h \in \Sigma_0 \subseteq \Sigma(g_1, g_2, \ldots)$, this gives the desired contradiction.

**4. Generalizations.** Recall that a set $A$ in a topological vector space $Z$ is said to be *bounded* if, given $U$ a neighbourhood of $0$ in $Z$, we can find $\lambda > 0$ with $\lambda U \supseteq A$. If $X$ is a topological space and $Y$ a topological vector space, we shall write $C(X, Y)$ for the space of continuous functions from $X$ to $Y$. We topologize $C(X, Y)$ by giving each $f_0 \in C(X, Y)$ a neighbourhood basis $\{\{f\colon f(x)-f_0(x)\in U$ for all $x\in X\}\colon U$ a neighbourhood of $0$ in $Y\}$ and call this topology the *uniform topology*. If, as will usually be the case in practice, $X$ is compact, or if $X$ is sequentially compact and $Y$ metrizable, then $C(X, Y)$ is a topological vector space. If $A\subset X, f\in C(X, Y)$ we write $f(A)=\{f(a)\colon a\in A\}$.

Our proof of Theorem 2 carries over, practically word for word, to give a proof of Theorem 2′.

THEOREM 2′. *Let $X$ be a sequentially compact topological space. Let $Y$ be a locally convex topological vector space. Let $f_1, f_2, \ldots \in C(X, Y)$.*

*Suppose*

(i) $\bigcup_{i=1}^{\infty} f_i(X)$ *is bounded,*

(ii) $f_n(x)\to 0$ *as* $n\to\infty$ *for all* $x\in X$.

*Then, given $U$ a neighbourhood of $0$ in $Y$, we can find $\lambda_1, \lambda_2, \ldots, \lambda_m\geqslant 0$ with $\sum_{i=1}^{m}\lambda_i = 1$ such that*

$$\sum_{i=1}^{m}\lambda_i f_i(x)\in U \quad \textit{for all } x\in X.$$

That the condition $X$ sequentially compact cannot be dropped is readily seen from the following standard counter example:

LEMMA 3. *Consider $f_i\colon \boldsymbol{Z}\to\boldsymbol{R}$ given by*

$$f_i(k) = 0 \quad \textit{for } |k|\leqslant i,$$
$$f_i(k) = 1 \quad \textit{otherwise}.$$

*Then $f_1, f_2, \ldots \in C(\boldsymbol{Z}, \boldsymbol{R})$, $|f_i(k)|\leqslant 1$ for all $k\in\boldsymbol{Z}$, $i\geqslant 1$, and $f_i(k)\to 0$ as $i\to\infty$ for all $k\in\boldsymbol{Z}$. However, if $\lambda_1, \lambda_2, \ldots, \lambda_n\geqslant 0$, $\sum_{i=1}^{n}\lambda_i = 1$, then $\sum_{i=1}^{n}\lambda_i f_i(n+2) = 1\not\to 0$.*

The problem of when an analogue of Theorem 2 holds for continuous linear maps $I\colon C(X, Y)\to Z$, where $X$ is a sequentially compact topological space and $Y$ and $Z$ are topological vector spaces appears to be much more difficult. It is easy to construct examples in which any 2 of $X, Y, Z$ are kept fixed and by changing the 3-rd we can make the obvious analogue of Osgood's Theroem true or false.

Even in the case where we wish to find conditions on $Z$ which will make the theorem true for all $X$ and all (locally convex) $Y$, I have not been able to make much progress.

Consider the following possible properties of $Z$.

PROPERTY A. *Let $Z$ be a topological vector space. We say that $Z$ has property A if, given $u_1, u_2, \ldots\in Z$ such that $\{u_1, u_2, \ldots\}$ is bounded and*

*$u_i \not\to 0$ as $i \to \infty$, we can find $n(1) < n(2) < \ldots$ and neighbourhood $W$ of 0 such that, whenever $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ and $\sum_{j=1}^{m} \lambda_j = 1$, we have $\sum_{j=1}^{m} \lambda_j u_{n(j)} \notin W$.*

PROPERTY B. *Let $Z$ be a topological vector space. We say that $Z$ has property* B *if we can find a bounded set $T$ and vectors $u_1, u_2, \ldots \in Z$ such that $u_i \not\to 0$ as $i \to \infty$, yet whenever $|\lambda_1|, \ldots, |\lambda_m| \leq 1$ we have $\sum_{j=1}^{m} \lambda_j u_j \in T$.*

PROPERTY C. *Let $Z$ be a topological vector space. We say that $Z$ has property* C *if we can find a bounded set $T$ and vectors $u_1, u_2, \ldots \in Z$ such that $u_i \not\to 0$ as $i \to \infty$ yet given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n| \leq \delta(\varepsilon), \quad \sum_{i=1}^{m} |\lambda_i| \leq 1 \quad \textit{imply} \quad \sum_{i=1}^{m} \lambda_i u_i \in \varepsilon T.$$

A locally convex topological vector space has property A if and only if it is a Schur space (i.e., its convergent sequences are the same as its weakly convergent sequences). Examples of spaces with property A thus include all locally convex Hausdorff topological vector spaces with their weak topology (and so $\boldsymbol{R}^n$, $\boldsymbol{C}^n$, $l^1(\boldsymbol{Z})$ under their usual norms) and $D([0,1]^n)$ the space of smooth functions on $[0,1]^n$. A locally convex topological vector space has property B if and only if it contains an (isomorphic) copy of $c_0$. Examples are $C^n([0,1])$, $D(\boldsymbol{R}^n)$ and $l^\infty(\boldsymbol{Z})$. Among spaces with property C but not property B are $l^p(\boldsymbol{Z})$ $[1 < p < \infty]$, $M([0,1])$, $L^q([0,1])$ and $L^q(\boldsymbol{R})$ $[1 \leq q < \infty]$.

LEMMA 4. (I) *Let $X$ be a sequentially compact topological space $Y$ a locally convex topological vector space and $Z$ a topological vector space with property* A. *Let $I\colon C(X, Y) \to Z$ be a continuous linear map and let $f_1, f_2, \ldots \in C(X, Y)$. Suppose*

(i) *$\bigcup_{j=1}^{\infty} f_j(X)$ is bounded,*

(ii) *$f_n(x) \to 0$ as $n \to \infty$ for all $x \in X$.*

*Then $If_n \to 0$ as $n \to \infty$.*

(II) *Let $X$ be a normal topological space containing distinct points $x(0), x(1), x(2), \ldots$ such that $x(0)$ has a countable base of neighbourhoods and $x(j) \to x(0)$ as $j \to \infty$. Let $Y$ and $Z$ be locally convex Hausdorff topological vector spaces and let $Z$ have property* B. *Then there exists a continuous linear map $I\colon C(X, Y) \to Z$ and functions $f_1, f_2, \ldots \in C(X, Y)$ such that*

(i) *$\bigcup_{j=1}^{\infty} f_j(X)$ is bounded,*

(ii) *$f_n(x) \to 0$ as $n \to \infty$ for all $x \in X$ yet $If_n \not\to 0$ as $n \to \infty$.*

(III) *Let $Y$ and $Z$ be locally convex Hausdorff topological vector spaces and let $Z$ have property* C. *Then there exists a closed subspace $H$ of $C([0,1], Y$*

*and a continuous linear map* $I: H \to Z$ *together with functions* $f_1, f_2, \ldots \in H$ *such that*

(i) $\bigcup_{i=1}^{\infty} f_i([0,1])$ *is bounded,*

(ii) $f_n(x) \to 0$ *as* $n \to \infty$ *for all* $x \in X$ *yet* $If_n \not\to 0$ *as* $n \to \infty$.

The proof of Lemma 4 (i) is an easy adaptation of our proof of Theorem 1. Note that we can obtain the usual theory of vector valued integrals directly from this result. The proofs of Lemma 4 (ii) and (iii) are more complicated but in view of the unsatisfactory nature of the results themselves we shall not give the proofs here.

I should like to thank Dr. Garling for much useful advice.

## References

[1] N. Dunford and J. T. Schwartz, *Linear operators* (Part I), Interscience, New York 1967.

[2] W. F. Eberlein, *Notes on integration I*, Comm. Pure Appl. Math. 10 (1957), p. 357–360.

[3] D. C. Gillespie and W. A. Hurwitz, *On sequences of continuous functions having continuous limits*, Trans. Amer. Math. Soc. 32 (1930), p. 527–543.

[4] L. A. Lindahl and F. Poulsen (Editors), *Thin sets in harmonic analysis*, Marcel Dekker, New York 1971.

[5] Z. Zalcwasser, *Sur une propriété du champ des fonctions continues*, Studia Math. 2 (1930), p. 63–67.