Weak homomorphisms of general algebras

**Introduction.** In the theory of universal algebras (cf. [2] and [11]) usually there are considered similar algebras of a fixed type and the notions of homomorphism and isomorphism are defined only for them. However, this setting of the type restricts the scope of considerations and is rather unnecessary in these investigations in which, as in the theory of independence, only the set of all algebraic operations is essential. It is purposeful to identify the algebras with the same support and the same algebraic operations, while in the theory of similar algebras indexing of fundamental operations is essential. From that point of view, the ring of integers \((Z; +, \cdot)\) differs essentially from the algebra \((Z; \cdot, +)\) (and they are non-isomorphic even); similarly, the Boolean algebra \(S_B = (B; \cup, \cap, \)'\) differs from the algebra \((B; \cup, \')\) (they have even different types), although the sets of algebraic operations coincide. In the category of (non-indexed) general algebras the appropriate notions are these of weak homomorphism and weak isomorphism, introduced by Goetz and Marczewski (cf. [10] and [14]). The latter notion has been introduced independently (under different names, e.g. “equivalence of algebras”, of [3]) by other authors, too.

In our paper we develop investigations of weak homomorphisms of (non-indexed) general algebras. We apply terminology and notation similar to one employed in Professor Marczewski’s papers (cf. e.g. [13] and [14]). In our considerations the algebra \(\mathfrak{A}\) of all algebraic operations (with operations \(\tilde{f}\) defined by means of a particular kind of superposition, comp. equality (A)) seems to be a better tool than the commonly used algebra \(\mathfrak{A}_{(n)}\) of \(n\)-ary algebraic operations.

We prove some theorems here, which have well-known analogues for indexed algebras (e.g.: the Homomorphism and Isomorphism Theorems). In spite of this similarity the proofs require more refined means than it is in the case of classical theorems; therefore, avoiding appeals to intuitions which are often false, we give complete proofs.

Most of the results of this paper were announced in [9].
1. The algebra of all algebraic operations. Let $\mathfrak{U} = (A; A)$ be an algebra, where $A$ is the set of all algebraic operations of $\mathfrak{U}$, $A \subset O(A)$ (see [13]). Form a new algebra $\overline{\mathfrak{U}} = (A, A)$ assigning to each $f \in A^{(m)}$ an operation $\overline{f}$ on $A$ defined by

$$(\overline{f}(g_1, \ldots, g_m))(x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2}, \ldots, x_{m1}, \ldots, x_{mn_m})$$

$$= f(g_1(x_{11}, \ldots, x_{1n_1}), g_2(x_{21}, \ldots, x_{2n_2}), \ldots, g_m(x_{m1}, \ldots, x_{mn_m}))$$

for every $g_i \in A$ and $g_i \in A^{(n_i)}$ ($i = 1, \ldots, m$; $j_i = 1, \ldots, n_i$). In particular, we get in this way the simple iterations considered in [15]. If $F$ is a family of polyadic quasi-group operations on the set $A$, then $\overline{f}(g_1, \ldots, g_m)$, where $f, g_i \in F$ is a quasi-group operation. The operation $\overline{f}(g_1, \ldots, g_m)$ is $\sum_{i=1}^{m} n_i$-ary.

The set $A$ can be treated as a set of fundamental operations of $\mathfrak{U}$. The algebra $\overline{\mathfrak{U}}$ will be called the algebra of all algebraic operations of $\mathfrak{U}$.

Recall that by the algebra of $n$-ary algebraic operations of $\mathfrak{U}$ one means usually the algebra $\mathfrak{U}^{(n)} = (A^{(n)}; A)$, where the operation $\overline{f} \in A$ is defined by

$$(\overline{f}(g_1, \ldots, g_m))(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$$

for all $g_1, \ldots, g_m \in A^{(n)}$, $f \in A^{(m)}$ and for all $x_1, \ldots, x_n \in A$.

By $A^{(0)}$ we shall denote the class of all constant algebraic operations of $\mathfrak{U}$ and by $C(\emptyset)$ the class of their values.

If all of all algebraic operations of $\mathfrak{U}$ restricted to a subset $B$ of $A$ belong again to $B$, then $B$ is called a subalgebra of algebra $\mathfrak{U}$.

An equivalence relation $\theta$ on $A$ is a congruence relation on the algebra $\mathfrak{U}$ if

$$f(a_1, \ldots, a_n) \theta f(a'_1, \ldots, a'_n)$$

for every $f \in A^{(n)}$ ($n = 1, 2, \ldots$) and all $a_i, a'_i \in A$ such that $a_i \theta a'_i$.

It is easy to check

PROPOSITION 1. Let $B$ be a subalgebra of the algebra $\mathfrak{U} = (A; A)$. Then the relation $\sim_B$ defined on pairs of $n$-ary operations $f$ and $g$ from $A$ by

$$(B) \quad f \sim_B g \iff \bigwedge_{x_1, \ldots, x_n \in B} f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$$

is a congruence relation on the algebra $\overline{\mathfrak{U}}$ of all algebraic operations (as well as on the algebra $\mathfrak{U}^{(n)}$ of $n$-ary operations).

It is worthwhile to add that if an algebra $\mathfrak{U}$ has exactly one constant $c$, then the relation $\sim$ coincides with $\omega_A$, since $f(c, \ldots, c) = c = g(c, \ldots, c)$ for every $f, g \in A^{(n)}$. Obviously, $\sim$ coincides with $\omega_A$ (for the notation see [11]).
Denote by $A/\sim_B$ the quotient set $A/\sim$. An algebra of the form $\mathfrak{B} = (B; A/\sim_B)$, where $B \subseteq A$, will be called (as well as $B$ itself) a subalgebra of the algebra $\mathfrak{A}$. If $C(\mathfrak{O}) \neq \mathfrak{O}$, then $(C(\mathfrak{O}); A|_{C(\mathfrak{O})})$ is the smallest subalgebra of $\mathfrak{A}$.

Let $\theta$ be a congruence relation in the algebra $\mathfrak{A} = (A; A)$. Then one can define an operation $\tilde{f}$ on $A/\theta$ by

$$\tilde{f}([a_1]_\theta, \ldots, [a_n]_\theta) = [f(a_1, \ldots, a_n)]_\theta.$$

Considering these operations as fundamental operations on the set $A/\theta$ we get a new algebra, which will be denoted by $\mathfrak{A}/\theta$. Let us note that the equality $\tilde{f} = \tilde{g}$ is possible even for $f \neq g$. Further,

$$\tilde{f}(\tilde{g}_1, \ldots, \tilde{g}_m) = (\tilde{f}(g_1, \ldots, g_m)),$$

which establishes that the set $\tilde{A} = \{\tilde{f} : f \in A\}$ forms the set of all algebraic operations of the algebra $\mathfrak{A}/\theta$.

Let $\theta$ be a congruence relation on the algebra $\mathfrak{A} = (A; A)$. Define a relation $\bar{\theta}$ on $A$ by putting $\bar{f} \bar{g}$ (for $f, g \in A^{(n)}$) iff

$$(C) \quad f(a_1, \ldots, a_n) \theta g(a_1, \ldots, a_n)$$

for all $a_1, \ldots, a_n \in A$.

One can to verify that

**Proposition 2.** If $\theta$ is a congruence relation on $\mathfrak{A}$ and $f, g \in A^{(n)}$, then $\bar{f} \bar{g}$ if and only if the following condition is satisfied

$$(D) \quad f(a_1, \ldots, a_n) \theta g(a'_1, \ldots, a'_n)$$

for all $a_i, a'_i \in A$ such that $a_i \theta a'_i$, $i = 1, \ldots, n$.

Moreover, $\bar{\theta}$ is a congruence relation on $\mathfrak{A}$ (as well as on $\mathfrak{A}^{(n)}$) and the set of all algebraic operations of the quotient algebra $\mathfrak{A}/\bar{\theta}$ is equal to $A/\bar{\theta}$.

Let us remark that for every $[f] \bar{\theta} \in A/\bar{\theta}$ we can put

$$[f] \bar{\theta}([a_1]_\theta, \ldots, [a_n]_\theta) = [f(a_1, \ldots, a_n)]_\theta$$

for each $[a_1]_\theta, \ldots, [a_n]_\theta \in A/\theta$. This definition is correct by (D).

Now, let $B$ be a subalgebra of $\mathfrak{A} = (A; A)$. Let us denote by $\sim_B$ the congruence relation on the algebra of all algebraic operations of $\mathfrak{A}$, induced (see (C)) by the congruence relation $\sim$ on $\mathfrak{A}$ defined by (B).

From Proposition 1 and 2 we easily conclude

**Proposition 3.** Let $B$ be a subalgebra of the algebra $\mathfrak{A} = (A; A)$. Then $\mathfrak{A}/\sim_B = [\mathfrak{A}]_B$. Therefore $A|_B = \tilde{A}/\sim_B$ and

$$\mathfrak{A}|_B = (A|_B; \tilde{A}|_B) = (A/\sim_B; \tilde{A}/\sim_B).$$
Indeed, let $b_{11}, \ldots, b_{mn}$ be in $B$ and let $g_i \in A^{(n_i)}$ ($i = 1, \ldots, m$). Thus,

\[(f_1, \ldots, g_1) \mapsto (b_{11}, \ldots, b_{mn}) = (f_1(b_{11}, \ldots, b_{1n}), \ldots, g_1(b_{mn}, \ldots, b_{mn})) = (f_1, \ldots, g_1) \mapsto (b_{11}, \ldots, b_{mn}),\]

where we write shortly $[f]$ instead of $[f]_B$ for arbitrary $f \in A$. Hence $[f]_B \sim [g_1], \ldots, [g_m] \sim$ which completes the proof.

2. Weak homomorphisms and weak isomorphisms. A mapping $h: A \to B$ is called a weak homomorphism (see [10]) of an algebra $\mathfrak{A} = (A; A)$ into $\mathfrak{B} = (B; B)$ whenever the following conditions hold:

\[(\mathrm{I}) \quad \bigwedge_{f \in A} \bigvee_{g \in B} f R_h g,\]

\[(\mathrm{II}) \quad \bigwedge_{g \in B} \bigvee_{f \in A} f R_h g\]

are satisfied, where $R_h$ is a relation between operations on $A$ and operations on $B$ (i.e., $R_h \subset O(A) \times O(B)$, where $O(A)$ denotes the set of all operations on $A$) defined by

\[(E) \quad f R_h g \iff \bigwedge_{x_1, \ldots, x_n \in A} h(f(x_1, \ldots, x_n)) = g(h(x_1), \ldots, h(x_n))\]

for $f \in O^{(n)}(A)$ and $g \in O^{(n)}(B)$. (Shortly: $f R_h g \iff h \circ f = g \circ h$.)

This definition has been introduced by A. Goetz and E. Marczewski ([10]), the definition given there employed the symbol $f^*$ instead of $g$, what might suggest existence of one-to-one correspondence. It is easy to see ([10]) that weak homomorphisms are closed under composition (if this composition is defined).

**Theorem 1.** Let $h: A \to B$ be a map from the algebra $\mathfrak{A} = (A; A)$ into $\mathfrak{B} = (B; B)$. If $h$ is a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, then $h(\mathfrak{A}) = (h(A); h(B))$ is a subalgebra of $\mathfrak{B}$ and $h: A \to h(A) \subset B$ is a weak homomorphism of $\mathfrak{A}$ onto $h(\mathfrak{A})$. Conversely: if $h$ is a weak homomorphism of $\mathfrak{A}$ onto $h(\mathfrak{A})$, then $h$ is a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.

**Proof.** It is easy to check that $h(A)$ is a subalgebra of $\mathfrak{B}$. We verify that $h: A \to h(A)$ is a weak homomorphism of $\mathfrak{A}$ onto $h(\mathfrak{A})$. Indeed, if we chose $g \in B$ for $f \in A$ to satisfy $f R_h g$, then also $f R_h g |_{h(A)}$. Now, let $g \in B |_{h(A)}$. Then there exists $g_1 \in B$ such that $g_1 |_{h(A)} = g$. For $g_1$, by the assumption, one can find $f \in A$ such that $f R_h g_1$. Hence $f R_h g$, which proves the first part of our theorem.

Let $f \in A$. Since $h: A \to h(A) \subset B$ is a weak homomorphism of $\mathfrak{A}$ onto $h(\mathfrak{A})$, there exists $g \in B |_{h(A)}$ such that $f R_h g$, and there exists $g_1 \in B$ such that $g_1 |_{h(A)} = g$. Thus $f R_h g_1$. 
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If now \( g_1 \in \mathcal{B} \), then \( g = g_1 \upharpoonright \mathcal{h}(A) \in \mathcal{B} \upharpoonright \mathcal{h}(A) \) and taking into account that \( h : A \rightarrow \mathcal{h}(A) \) is a weak homomorphism of \( \mathfrak{A} \) onto \( \mathcal{h}(\mathfrak{A}) \) we conclude that there exists \( f \in A \) such that \( f \mathcal{R} h g_1 \) and so \( f \mathcal{R} h g_1. \) So we verified that \( h \) is a weak homomorphism of \( \mathfrak{A} \) into \( \mathcal{B}. \) This completes the proof of Theorem 1.

**Theorem 2.** Let \( h : A \rightarrow \mathcal{B}, \mathfrak{A} = (A; A) \) and suppose

\[
(\text{I}') \quad \bigwedge_{f \in A} \bigvee_{g \in \mathcal{O}(\mathcal{B})} f \mathcal{R} h g.
\]

Then \( \mathcal{B} = \{ g \in \mathcal{O}(\mathcal{B}) : \bigvee_{f \in A} f \mathcal{R} h g \} \) forms the set of all algebraic operations of a certain algebra \( \mathcal{B} = (\mathcal{B}; \mathcal{B}) \) and \( h \) is a weak homomorphism of \( \mathfrak{A} \) onto \( \mathcal{B}. \) Furthermore, conditions (I), (I') and

\[
(\text{I''}) \quad \bigwedge_{f \in A} \bigvee_{g \in \mathcal{O}(\mathcal{B})} f \mathcal{R} h g
\]

are equivalent. The algebra \( \mathcal{B} \) is the only algebra on the support \( B \) for which \( h \) is a weak homomorphism of \( \mathfrak{A} \) onto \( \mathcal{B}. \)

(The symbol \( \exists! \) denotes the quantifier: "there exists a unique..."

**Proof.** First we shall prove that \( \mathcal{B} \) contains trivial operations and is closed under composition. Let \( e^{(n)}_k \in \mathcal{O}(\mathcal{B}) \) satisfies the condition \( e^{(n)}_k \mathcal{R} h e^{(n)}_k, \) where \( e^{(n)}_k \) is the trivial operation in \( \mathfrak{A}. \) Since the mapping \( h \) is "onto", so for \( b_i \in \mathcal{B} \) there exists \( a_i \in A \) such that \( h(a_i) = b_i \) \((i = 1, \ldots, n). \) Then we have

\[
e^{(n)}_k(b_1, \ldots, b_n) = h(e^{(n)}_k(a_1, \ldots, a_n)) = h(a_k) = b_k.
\]

Next, let \( g \in \mathcal{B}^{(m)}; g_i \in \mathcal{B}^{(m)} \) \((i = 1, \ldots, m). \) In view of the definition of the set \( \mathcal{B} \) there exist operations \( f \in \mathcal{A}^{(m)} \) and \( f_i \in \mathcal{A}^{(n)} \) such that \( f \mathcal{R} h g, f_i \mathcal{R} h g_i \) \((i = 1, \ldots, m). \) So, we get

\[
(\hat{g}(g_1, \ldots, g_m))(b_1, \ldots, b_n) = g(g_1[h(a_1), \ldots, h(a_n)], \ldots, g_m(h(a_1), \ldots, h(a_n)))
= h(f(f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n))
= \hat{h}(\hat{f}(f_1, \ldots, f_m))(a_1, \ldots, a_n).
\]

Therefore \( \hat{f}(f_1, \ldots, f_m) \mathcal{R} h g \) \((g_1, \ldots, g_m), \) which proves that \( \mathcal{B} \) is closed under composition.

It is easy to see, that \( h \) is a weak homomorphism of \( \mathfrak{A} \) onto \( \mathcal{B}. \) Indeed, we get immediately condition (I) from (I'), and (II) from the definition of \( \mathcal{B}. \)

Obviously, (I') follows from (I''). It remains to show yet that (I) implies (I''). Let \( f \mathcal{R} h g_1 \) and \( f \mathcal{R} h g_2 \) for \( f \in \mathcal{A}^{(m)} \) and \( g_1, g_2 \in \mathcal{O}^{(m)}(\mathcal{B}). \) Therefore we have

\[
g_1(h(a_1), \ldots, h(a_n)) = h(f(a_1, \ldots, a_n)) = g_2(h(a_1), \ldots, h(a_n))
\]

for every \( a_1, \ldots, a_n \in A. \) Since \( h \) maps \( A \) onto \( B, \) we get \( g_1(b_1, \ldots, b_n) = g_2(b_1, \ldots, b_n) \) for every \( b_1, \ldots, b_n \in \mathcal{B}, \) and so \( g_1 = g_2. \)
Let now $\mathfrak{B}' = (B; B')$ and $h: A \rightarrow B$ be a weak homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}'$. Then, by the definition of weak homomorphism, for every $g' \in B'$ there exists $f \in A$ such that $fR_h g'$. Taking into account condition $(I')$ and the definition of the set $B$ we see that there exists $g \in B$ such that $fR_h g$. Condition $(I'')$ implies that $g' = g \in B$. Let finally $g \in B$. From the definition of the set $B$ there exists $f \in A$ such that $fR_h g$. Since $h$ is a weak homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}'$, it follows that there exists $g' \in B'$ such that $fR_h g$. By $(I'')$ we have $g = g' \in B'$; and $\mathfrak{B} = \mathfrak{B}'$, which completes the proof of Theorem 2.

From Theorem 2 we get immediately

**Corollary 1.** If $h: A \rightarrow B$ is a weak homomorphism of $\mathfrak{A} = (A; A)$ onto $\mathfrak{B} = (B; B)$, then conditions $(I)$, $(I')$ and $(I'')$ are equivalent.

**Corollary 2.** Let $\theta$ be a congruence relation on the algebra $\mathfrak{A} = (A; A)$. Then the algebra $\mathfrak{A}/\theta = (A/\theta; A/\theta)$ (see Section 1, Proposition 2) is the only algebra on the support $A/\theta$ such that the natural projection $\eta: A \twoheadrightarrow A/\theta$ is a weak homomorphism of $\mathfrak{A}$ onto an algebra on the support $A/\theta$.

We get also

**Corollary 3.** If $h$ is a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ and $\theta$ is a congruence relation on $\mathfrak{B}$, then the mapping $a \mapsto [h(a)]\theta$ is a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}/\theta$.

In fact, from Corollary 2 we know that the natural projection $\eta: B \twoheadrightarrow B/\theta$ is a weak homomorphism. Therefore the considered mapping is a weak homomorphism because it is a composition of two such homomorphisms.

**Corollary 4.** Let $h_i: A \rightarrow B_i$ ($i = 1, 2$) be weak homomorphisms of an algebra $\mathfrak{A}$ into $B_i$ and let $h_1$ be a surjection and let $h_0: B_1 \rightarrow B_2$ be a map or which the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{h_1} & B_1 \\
| & & | \\
\downarrow{h_2} & & \downarrow{h_0} \\
B_1 & \xrightarrow{h_1} & B_2
\end{array}
$$

is commutative (i.e., $h_0 h_1 = h_2$). Then $h_0$ is also a weak homomorphism of $\mathfrak{B}_1$ into $\mathfrak{B}_2$.

Indeed, from the commutativity of this diagram it follows that $h_0$ is a map onto $h_2(A)$. In view of Theorem 1 we conclude that $h_2(A)$ is a subalgebra of $\mathfrak{B}_2$. By Theorem 2 it remains to verify that for every $g_1 \in B_1$ there exists $g_2 \in B_2$ such that $g_1 R_{h_0} g_2$. In virtue of condition $(II)$ of the definition of weak homomorphism, there exists $f \in A$ such that $f R_{h_1} g_1$. However, using condition $(I)$ for the weak homomorphism $h_2$ we infer that there exists $g_2 \in B_2$ such that $f R_{h_2} g_2$. We shall verify,
that the relation \( g_1 R_{h_0} g_2 \) holds for the chosen \( g_2 \). Let \( b_1, \ldots, b_n \in B_1 \). Thus there exist \( a_1, \ldots, a_n \in A \) for which \( h_1(a_i) = b_i (i = 1, \ldots, n) \) and we have:

\[
\begin{align*}
  h_0(g_1(b_1, \ldots, b_n)) &= h_0(h_1(f(a_1, \ldots, a_n))) = h_2(f(a_1, \ldots, a_n)) \\
  &= g_2((h_0 h_1)(a_1), \ldots, (h_0 h_1)(a_n)) = g_2(h_0(b_1), \ldots, h_0(b_n)).
\end{align*}
\]

Therefore, we proved that \( h_0 : B_1 \otimes h_2(A) \) is a weak homomorphism of the algebra \( B_1 \) onto \( h_2(A) \) and hence, taking once more into account Theorem 1, it is also a weak homomorphism of \( B_1 \) into \( B_2 \), what completes the proof of Corollary 4.

Note that a theorem similar to Corollary 4 is known also for some classes of indexed algebras (e.g. for groups, cf. [18], p. 59).

Now we shall prove the dual statement to Theorem 2.

**Theorem 2*. Let \( h : A \rightarrow B \) be an injective mapping, \( B = (B; B) \) and let

\[
\forall f \in O(A) \exists g \in B \text{ such that } f B h g.
\]

Then \( A \) is the set of all algebraic operations of certain algebra \( A = (A; A) \), and \( h \) is a weak homomorphism of \( A \) into \( B \). Furthermore, conditions \( (\Pi) \), \( (\Pi') \) and

\[
(\Pi'') \forall f \in O(A) \exists g \in B \text{ such that } f B h g.
\]

are equivalent. The algebra \( A \) is the only algebra on the support \( A \) for which \( h \) is a weak homomorphism of \( A \) into \( B \).

**Proof.** First, we shall prove that \( A \) contains trivial operations and is closed under composition. Let \( e^{(n)} \in O(A) \) satisfy the condition \( e^{(n)} R_{h e^{(n)}} \), where \( e^{(n)} \) is the trivial operation in \( B \). In virtue of injectivity of \( h \) we get triviality of \( e^{(n)} \). Let now \( f \in A^{(m)} \) and \( f_i \in A^{(n)} (i = 1, \ldots, m) \). In view of the definition of \( A \) there exist operations \( g \in B^{(m)} \) and \( g_i \in B^{(n)} \) such that \( f R_h g, f_i R_h g_i (i = 1, \ldots, m) \). Then similarly as in the proof of Theorem 2 we have

\[
\begin{align*}
  h((f(f_1, \ldots, f_m))(a_1, \ldots, a_n)) &= (h(g_1, \ldots, g_m))(h(a_1), \ldots, h(a_n)) \\
  &= h(F(a_1, \ldots, a_n)),
\end{align*}
\]

for every \( a_1, \ldots, a_n \in A \), where \( F \in O(A) \) and \( F R_{h g}(g_1, \ldots, g_m) \). Therefore, by the definition of the set \( A \), we have \( F \in A \). In virtue of injectivity of the mapping \( h \) we get \( F = (f(f_1, \ldots, f_m) \), which proves that the set \( A \) is closed under composition.

Similarly as in the proof of Theorem 2 it is easy to verify that \( h \) is a weak homomorphism of \( A \) into \( B \).
Obviously (II'') implies (II'), and (II) follows from (II'). We shall show that (II'') follows from (II). Let $f_1, f_2 \epsilon O^{(n)}(A), g \epsilon B^{(n)}$ and let $f_1 R_h g$ and $f_2 R_h g$. Thus, we get

$$h\left(f_1(a_1, \ldots, a_n)\right) = g(h(a_1), \ldots, h(a_n)) = h\left(f_2(a_1, \ldots, a_n)\right)$$

for every $a_1, \ldots, a_n \epsilon A$. Hence, in virtue of injectivity of $h$, we have $f_1 = f_2$.

Now, let $\mathcal{A} = (A; A')$ and $h: A \rightarrow B$ be a weak homomorphism of $\mathcal{A}$ into $\mathcal{B}$ ($h$ is the same mapping as above). From the definition of weak homomorphism, for every $f ' \epsilon A'$ there exists $g \epsilon B$ such that $f' R_h g$. From (II') and the definition of the set $A$ it follows that there exists $f \epsilon A$ such that $f R_h g$. In virtue of (II'') we conclude that $f' = f \epsilon A$. Conversely, let $f \epsilon A$. Then, by the definition of $A$, there exists $g \epsilon B$ such that $f R_h g$. Taking into account that $h$ is a weak homomorphism of $\mathcal{A}$ into $\mathcal{B}$, we conclude that there exists $f' \epsilon A'$ such that $f' R_h g$, and, by (II''), we have $f = f' \epsilon A'$, what completes the proof of Theorem 2*.

From Theorem 2* we obtain the following dual Corollaries 1*-4*, the proofs of them are analogous to the proofs of Corollaries 1-4 in the same manner as the proof of Theorem 2* is analogous to that of Theorem 2.

**Corollary 1*. If $h: A \rightarrow B$ is an injective weak homomorphism of $\mathcal{A}$ into $\mathcal{B}$, then condition (II), (II') and (II'') are equivalent.

**Corollary 2*. Let $B$ be a subalgebra of the the algebra $\mathcal{A} = (A; A)$. Then the algebra $\mathcal{B} = (B; A|_B)$ is the only algebra on the support $B$, for which the inclusion map $h: B \rightarrow A$ is a weak homomorphism of an algebra on the support $B$ into the algebra $\mathcal{A}$.

**Corollary 3*. If $h$ is a weak homomorphism of $\mathcal{A}$ into $\mathcal{C}$ and $B$ is a subalgebra of $\mathcal{A}$, then $h|_B$ is a weak homomorphism of the algebra $\mathcal{B} = (B; A|_B)$ into $\mathcal{C}$.

**Corollary 4*. Let $h_i: A_i \rightarrow B$ be a weak homomorphism of an algebra $\mathcal{A}_i$ into $\mathcal{B}$ ($i = 1, 2$) and let $h_2$ be an injection, and let $h_0: A_1 \rightarrow A_2$ be a map, for which the diagram

$$\begin{array}{ccc}
A_1 & \xrightarrow{h_1} & B \\
\downarrow{h_0} & & \\
A_2 & \xrightarrow{h_2} & \\
\end{array}$$

is commutative. Then $h_0$ is a weak homomorphism of $\mathcal{A}_1$ into $\mathcal{A}_2$.

From Theorem 2 and 2* one can immediately obtains

**Corollary 5.** If a bijective mapping $h: A \rightarrow B$ is a weak homomorphism of $\mathcal{A}$ onto $\mathcal{B}$, then $|A^{(n)}| = |B^{(n)}|$ for every $n = 0, 1, 2, \ldots$. 


Each bijection \( h: \mathcal{A} \to \mathcal{B} \) induces, in a natural way, a bijection \( h^*: O(\mathcal{A}) \to O(\mathcal{B}) \) defined by

\[
h^*(f)(y_1, \ldots, y_n) = h\left(f(h^{-1}(y_1), \ldots, h^{-1}(y_n))\right).
\]

A bijection \( h: \mathcal{A} \to \mathcal{B} \) is called a weak isomorphism ([10]) of an algebra \( \mathfrak{A} = (\mathcal{A}; \mathcal{A}) \) onto \( \mathfrak{B} = (\mathcal{B}; \mathcal{B}) \) whenever \( h^*(\mathcal{A}) = \mathcal{B} \) for the induced mapping \( h^* \).

From Theorem 2, by a simply verification, we have

**Theorem 3.** A weak isomorphism \( h \) of \( \mathfrak{A} \) onto \( \mathfrak{B} \) is a weak homomorphism and \( \tilde{h}: \mathcal{A} \to \mathcal{B} \) is bijective and \( h^* = \tilde{h} \). Moreover, a weak homomorphism \( h: \mathcal{A} \to \mathcal{B} \) of \( \mathfrak{A} \) into \( \mathfrak{B} \) is a weak isomorphism if and only if \( h \) is a bijection.

Let us remark that the fact that a bijective weak homomorphism is a weak isomorphism was formulated also in Goetz's paper ([10], Theorem 4, p. 167). Theorem 3 shows (taking into account Corollary 5) that the notion of weak isomorphism coincides with the independently introduced notion of equivalence of algebras (see [3], p. 47 and [10], p. 164).

In particular one can consider bijections \( h \) from \( \mathcal{A} \) onto itself. If the induced mapping \( \tilde{h}: O(\mathcal{A}) \to O(\mathcal{A}) \) (for a bijection \( h: \mathcal{A} \to \mathcal{A} \)) is a bijection from \( \mathcal{A} \) onto itself, then \( h \) is called a weak automorphism of the algebra \( \mathfrak{A} \). This notion, introduced by A. Goetz and E. Marczewski (see [14] and [10]), was examined by several authors: J. Dudek and E. Plonka [4], K. Glazek [6], [7], and [8], E. Plonka [17], J. E. Senft [19] and T. Traczyk [20].

From Theorem 3 and Corollary 3* of Theorem 2* we have immediately

**Corollary.** If \( h \) is a weak automorphism of \( \mathfrak{A} \) and \( \mathcal{B} \) is a subalgebra of \( \mathfrak{A} \), then \( h|_{\mathcal{B}} \) is a weak automorphism of the algebra \( \mathfrak{B} = (\mathcal{B}; \mathcal{A}|_{\mathcal{B}}) \).

**3. Some characterization of weak homomorphisms.** Weak homomorphisms can be treated in similar fashion as the weak isomorphisms in [10] (p. 164), namely as pairs of mappings: one of them defined on the support \( \mathcal{A} \) of the algebra \( \mathfrak{A} \) and the second one on the set \( \mathcal{A} \) of all algebraic operations of \( \mathfrak{A} \).

First we shall prove following two lemmas:

**Lemma 1.** Let \( h: \mathcal{A} \to \mathcal{B} \). Then \( h \) is a weak homomorphism of the algebra \( \mathfrak{A} = (\mathcal{A}; \mathcal{A}) \) onto the algebra \( \mathfrak{B} = (\mathcal{B}; \mathcal{B}) \) iff there exists a unique mapping \( \tilde{h}: \mathcal{A} \to \mathcal{B} \) such that \( fR_h \tilde{h}(f) \) for every \( f \in \mathcal{A} \).

Indeed, let \( h \) be a weak homomorphism of \( \mathfrak{A} \) onto \( \mathfrak{B} \). Then, in virtue of Theorem 2, for every \( f \in \mathcal{A} \) there exists exactly one \( g \in \mathcal{B} \) such that \( fR_h g \). Put \( \tilde{h}(f) = g \). We shall show that \( \tilde{h} \) maps \( \mathcal{A} \) onto \( \mathcal{B} \). Take \( g' \in \mathcal{B} \); in virtue of (\( \Pi \)) there exists \( f \in \mathcal{A} \) such that \( fR_h g' \). Taking into account (I‘’) of The-
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orem 2 we get \( g' = \bar{h}(f) \). Conversely, from the assumption condition (I) is satisfied. Since \( h \) maps \( A \) onto \( B \), hence for every \( g \in B \) there exists \( f \in A \) such that \( \bar{h}(f) = g \), which gives condition (II) and so \( h \) is a weak homomorphism.

Lemma 2. Let \( \mathcal{A} = (A; A) \), \( \mathcal{B} = (B; B) \), \( h: A \to B \) and

\[ \varphi: A \xrightarrow{onto} B |_{h(A)} \]

such that \( fR_h \varphi(f) \). Then \( h(\mathcal{A}) = (h(A); B |_{h(A)}) \) is a subalgebra of the algebra \( \mathcal{B} \). Moreover, \( h \) is a weak homomorphism of \( \mathcal{A} \) onto \( h(\mathcal{A}) \) and \( \varphi = \bar{h} \).

Indeed, let \( g \in B |_{h(A)} \) and let \( b_1, \ldots, b_n \in h(A) \). Then there exist \( f \in A^n \) and \( a_1, \ldots, a_n \in A \) such that \( \varphi(f) = g \), \( h(a_i) = b_i \) (\( i = 1, \ldots, n \)). We have \( g(b_1, \ldots, b_n) = \varphi(f)h((a_1), \ldots, h(a_n)) = h(f(a_1, \ldots, a_n)) \in h(A) \). Therefore \( h(A) \) is a subalgebra of \( \mathcal{B} \). In virtue of Theorem 2, the mapping \( h \) is a weak homomorphism of \( \mathcal{A} \) onto \( h(\mathcal{A}) \). By Lemma 1 we have \( \varphi = \bar{h} \).

Using Lemmas 1 and 2, we prove:

Theorem 4. Let \( h: A \to B \). Then \( h \) is a weak homomorphism of the algebra \( \mathcal{A} = (A; A) \) into \( \mathcal{B} = (B; B) \) if and only if there exists a unique mapping \( \bar{h}: A \xrightarrow{onto} B |_{h(A)} \) such that \( fR_h \bar{h}(f) \) for every \( f \in A \).

Proof. Let \( h \) be a weak homomorphism of \( \mathcal{A} \) into \( \mathcal{B} \). Then, by Theorem 1, \( h \) is a weak homomorphism of \( \mathcal{A} \) onto the subalgebra \( h(\mathcal{A}) \) of \( \mathcal{B} \). In view of Lemma 1 we conclude that there exists exactly one mapping \( \bar{h}: A \xrightarrow{onto} B |_{h(A)} \) such that \( fR_h \bar{h}(f) \) for every \( f \in A \).

Now, let \( \varphi: A \xrightarrow{onto} B |_{h(A)} \) be a mapping such that \( fR_h \varphi(f) \) for every \( f \in A \). By Lemma 2, \( h \) is a weak homomorphism of \( \mathcal{A} \) onto \( h(\mathcal{A}) \) and \( \varphi = \bar{h} \).

Taking into account the second part of Theorem 1 we infer that \( h \) is a weak homomorphism of \( \mathcal{A} \) into \( \mathcal{B} \), what completes the proof of Theorem 4.

Corollary 1. Let \( h_1: A \to B \) and \( h_2: B \to C \) be weak homomorphisms of \( \mathcal{A} \) onto \( \mathcal{B} \) and of \( \mathcal{B} \) into \( \mathcal{C} \), respectively. Then

\[ (h_2h_1)^{-1} = \bar{h}_2\bar{h}_1. \]

Indeed, let \( f \in A^n \). Then it is easy to verify that

\[ fR_{h_2h_1}(\bar{h}_2\bar{h}_1)(f), \]

which, in virtue of Theorem 4, gives the required equality.

Also one can check

Corollary 2. Let \( h: A \to B \) be a weak homomorphism of \( \mathcal{A} \) onto \( \mathcal{B} \). Then \( \bar{h}(e_k^{(n)}) = e_k^{(n)} \) (\( k = 1, \ldots, n \)), where \( e_k^{(n)} \) and \( e_k^{(n)} \) are the trivial operations in algebras \( \mathcal{A} \) and \( \mathcal{B} \), respectively.

Corollary 3. Let a mapping \( h: A \to B \) be a weak homomorphism of the algebra \( \mathcal{A} = (A; A) \) onto an algebra \( \mathcal{B} = (B; B) \) and let \( B_1 \) be a subalgebra of the algebra \( \mathcal{B} \). Then the inverse image \( h^{-1}(B_1) \) is a subalgebra of the algebra \( \mathcal{A} \).
**COROLLARY 4.** Let $h_t: A \to B$ ($t \in T$) be a family of weak homomorphisms of $A$ into $B$ such that
\[ h_{t_1}(f) = h_{t_2}(f) \]
for every $f \in A$, and for all $t_1, t_2 \in T$. Then the subset
\[ A_1 = \{ a: h_{t_1}(a) = h_{t_2}(a) \text{ for all } t_1, t_2 \in T \} \]
is a subalgebra of $A$.

It is worthwhile to note that denoting $\tilde{h} = \varphi$ and $\varphi|_{A(n)} = \varphi_n$ we conclude that the mapping $h: A \to B$ is a weak homomorphism iff $\varphi_n: A(n) \to B(n)|_{h(A)}$ and $fR\varphi_n(f)$ for every $f \in A(n)$ and all $n = 1, 2, ...$

Thus, it is easy to see that the definition of weak homomorphism given by S. Fajtlowicz in [5] is weaker than the definition using here. For example, take as $A$ the trivial algebra on the infinite set $A$, and as $B$ the functional complete one on the same set. Then each mapping $h: A \to A$ is a weak homomorphism in the Fajtlowicz' sense and it is not in the sense used here.

One can show the following

**LEMMA 3.** Let $h: A \to B$ be a weak homomorphism of the algebra $A$ into $B$. Then
\[ \tilde{h}(f(f_1, \ldots, f_m)) = (\tilde{h}(f))^\times(\tilde{h}(f_1), \ldots, \tilde{h}(f_m)) \]
for every $f \in A(m)$, $f_1, \ldots, f_m \in A(n)$.

Further, we have

**THEOREM 5 (cf. [10], p. 167).** Let $h: A \to B$ and let $F \subseteq A$ be a set of fundamental operations of the algebra $A = (A; A)$. Then $h$ is a weak homomorphism of $A$ into $B = (B; B)$ if and only if there exists a unique mapping $\psi: F \to B|_{h(A)}$ such that $fR\psi(f)$ for every $f \in F$ and the set $\psi(F)$ can be treated as the set of fundamental operations of the algebra $h(A)$.

**Proof.** Let $\varepsilon_k^{(n)}$ and $\varepsilon_k^{(n)}$ ($k = 1, \ldots, n; n = 1, 2, \ldots$) be the trivial operations in the algebras $A$ and $B$, resp., and put
\[ E(n) = \{ \varepsilon_k^{(n)}: k = 1, \ldots, n \} = A_0^{(n)} \]
\[ A_s^{(n)} = A_s^{(n)} \cup \bigcup_{m=1}^{\infty} \{ \hat{f}(f_1, \ldots, f_m): f_j \in A_s^{(n)}(j = 1, \ldots, m), f \in F^m \} \]
(cf. [13], p. 47). We shall prove by induction with respect to $s$ that $\psi$ can be extended to a mapping $\tilde{h}: A \to B|_{h(A)}$ such that $fR\tilde{h}(f)$ for every $f \in A$. Put $\tilde{h}(\varepsilon_k^{(n)}) = \varepsilon_k^{(n)}$ and $\tilde{h}(f) = \psi(f)$ for $f \in F$. Suppose that $\tilde{h}$ has been defined for all operations from $A_s^{(n)}$. Let $f_0 \in A_{s+1}^{(n)}$ be the form $f_0 = \hat{f}(f_1, \ldots, f_m)$, where $f \in F^m$ and $f_1, \ldots, f_m \in A_s^{(n)}$. Then we put
\[ \tilde{h}(f_0) = (\psi(f))^\times(\tilde{h}(f_1), \ldots, \tilde{h}(f_m)). \]
For arbitrary \( a_1, \ldots, a_n \in A \) we have
\[
\tilde{h}(f_0(a_1, \ldots, a_n)) = \psi(f)(\tilde{h}(f_1)(h(a_1), \ldots, h(a_n)), \ldots, \tilde{h}(f_m)(h(a_1), \ldots, h(a_n)))
\]
\[
= (\psi(f))^{\tilde{h}(f_1)}, \ldots, \tilde{h}(f_m)))(h(q_1), \ldots, h(a_n)).
\]

Therefore \( f_0 \mathrel{\rho_h} \tilde{h}(f_0) \). Remark that, in virtue of condition \((I')\) of Theorem 2, this definition of \( \tilde{h}(f_0) \) does not depend on a manner, in which the operation \( f_0 \) is expressed by operations from \( F \) and \( A_1^{(n)} \). Now we shall show (by induction) that \( \tilde{h} \) is a surjection. Let \( g_0 \in B_s^{(n)} \) and so \( g_0 = \tilde{g}(g_1, \ldots, g_m) \), where \( g = \psi(f) \) for some \( f \in F \) and \( g_1, \ldots, g_m \in B_s^{(n)} \). Suppose that for \( g_i \) there exists \( f_i \in A \) such that \( g_i = \tilde{h}(f_i) \) for every \( i = 1, \ldots, m \). Put \( f_0 = \tilde{f}(f_1, \ldots, f_m) \). Then
\[
g_0 = \tilde{g}(g_1, \ldots, g_m) = (\psi(f))^{\tilde{h}(f_1), \ldots, \tilde{h}(f_m)} = \tilde{h}(f_0).
\]
Hence \( f_0 \mathrel{\rho_h} g_0 \). Taking into account Theorem 4, we conclude that \( h \) is a weak homomorphism of \( \mathfrak{U} \) into \( \mathfrak{B} \).

Conversely, let \( h \) be a weak homomorphism of \( \mathfrak{U} \) into \( \mathfrak{B} \) and let \( \tilde{h} |_{h} = \psi \). It is sufficient to prove that the set \( \psi(F) = \tilde{h}(F) \) can be treated as a set of fundamental operations of the algebra \( h(\mathfrak{U}) \). Let \( g_0 \in B \). Then, by the definition of weak homomorphism, there exists \( f_0 \in A \) such that \( f_0 \mathrel{\rho_h} g_0 \). Since \( F \) is the set of fundamental operations of the algebra \( \mathfrak{U} \), there exist \( f \in F^{(m)} \) and \( f_1, \ldots, f_m \in A^{(m)} \) such that \( f_0 = \tilde{f}(f_1, \ldots, f_m) \). Taking into account Lemma 3 we get
\[
g_0 = \tilde{h}(f_0) = (\tilde{h}(f))^{\tilde{h}(f_1), \ldots, \tilde{h}(f_m)} = \tilde{g}(g_1, \ldots, g_m),
\]
where \( g_i \in B^{(n)} (i = 1, \ldots, m) \) and \( g \in h(F) \). After several steps we get that \( g_0 \) is an algebraic operation with respect to the family \( h(F) \) of fundamental operations in \( h(\mathfrak{U}) \), what completes the proof of Theorem 5.

**COROLLARY** ([10], p. 167). Each homomorphism \( h \) of the indexed algebra \( \mathfrak{U} \) into a similar algebra \( \mathfrak{B} \) is also a weak homomorphism.

**THEOREM 6.** If \( h: A \rightarrow B \) is a weak homomorphism of the algebra \( \mathfrak{U} = (A; A) \) into \( \mathfrak{B} = (B; B) \), then \( \tilde{h}: A \mathop{\rightarrow^{\text{onto}}} B |_{h(A)} \) is a weak homomorphism of \( \mathfrak{U} \) onto \( h(\mathfrak{U}) \) (and \( \tilde{h}_n = \tilde{h} |_{A^{(n)}} \) is also a weak homomorphism of \( \mathfrak{U}^{(n)} \) onto \( (h(\mathfrak{U}))^{(n)} \)).

**Proof.** Since \( A \) is a set of fundamental operations of the algebra \( \mathfrak{U} \), hence in virtue of Theorem 5 it remains only to define \( \psi: A \rightarrow B |_{h(A)} \) such that \( f \mathrel{\rho} \psi(f) \) for every \( f \in A \). Put \( \psi(f) = \tilde{h}(f) \). Let \( b_{11}, \ldots, b_{in_1}, \ldots, b_{m1}, \ldots, b_{mnm} \in h(A) \). Then there exist \( a_{i1}, \ldots, a_{in_1}, \ldots, a_{m1}, \ldots, a_{mnm} \in A \) such that \( h(a_{ij}) = b_{ij} \) \( i = 1, \ldots, m, j = 1, \ldots, n_i \) and for each \( g_1, \ldots, g_m \in A \) we have:
\[
\tilde{h}(\langle g_1, \ldots, g_m \rangle)(b_{11}, \ldots, b_{in_1}, \ldots, b_{m1}, \ldots, b_{mnm})
\]
\[
= h(\langle f_1(a_{11}, \ldots, a_{in_1}), \ldots, f_m(a_{m1}, \ldots, a_{mnm}) \rangle)
\]
\[
= (\tilde{h}(f_1)(\tilde{h}(g_1), \ldots, \tilde{h}(g_m))(b_{11}, \ldots, b_{in_1}1, \ldots, b_{m1}, \ldots, b_{mnm}).
\]
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Thus \( \tilde{h}(\langle f(g_1), \ldots, g_m \rangle) = \psi(\tilde{f})\langle \tilde{h}(g_1), \ldots, \tilde{h}(g_m) \rangle \), i.e., \( fR_\tilde{h}^{}\psi(\tilde{f}) \), what yields that \( \tilde{h} \) is a weak homomorphism of \( \bar{\mathfrak{A}} \) onto \( \tilde{h}(\mathfrak{A}) \).

In a similar way we verify that \( \tilde{h}_n \) is a weak homomorphism of \( \mathfrak{A}^{(n)} \) onto \( (\tilde{h}(\mathfrak{A}))^{(n)} \). Thus Theorem 6 is proved.

4. Weak homomorphism, and weak isomorphism theorem. Recall that in Section 1 we have defined the congruence relation \( \tilde{\theta} \) on \( \mathfrak{A} \) for every congruence relation \( \theta \) on the algebra \( \mathfrak{A} = (A; A) \). Let \( h \) be a weak homomorphism of the algebra \( \mathfrak{A} = (A; A) \) into \( \mathfrak{B} \). Let us define a relation \( \theta_h \) on \( A \) by

\[
(G) \quad a \theta_h a' \iff h(a) = h(a').
\]

LEMMA 4. The relation \( \theta_h \) is a congruence relation on the algebra \( \mathfrak{A} \). Operations \( f, g \in A^{(n)} \) are in the congruence relation \( \theta_h \) iff \( h(f) = h(g) \).

Indeed, similarly as in the case of usual homomorphisms it is easy to verify the first part of Lemma 4. In view of Proposition 2 we infer that \( \tilde{\theta}_h \) is a congruence relation on the algebra \( \bar{\mathfrak{A}} \). From condition (C) of the definition of \( \tilde{\theta} \) we have: \( f \theta_h g \iff h(f(a_1, \ldots, a_n)) = h(g(a_1, \ldots, a_n)) \) for every \( a_1, \ldots, a_n \in A \). Hence, putting \( h(a_i) = b_i \) (\( i = 1, \ldots, n \)), we have:

\[
\tilde{h}(f)(b_1, \ldots, b_n) = \tilde{h}(g)(b_1, \ldots, b_n)
\]

for each \( b_1, \ldots, b_n \in h(A) \).

Now, we have

THEOREM 7 (WEAK HOMOMORPHISM THEOREM). Let \( \mathfrak{A} = (A; A) \) and \( \mathfrak{B} = (B; B) \) be algebras and let \( h: A \rightarrow B \) be a weak homomorphism of \( \mathfrak{A} \) onto \( \mathfrak{B} \). Further let \( \theta_h \) denote the congruence relation induced on \( \mathfrak{A} \) by \( h \). Then the quotient algebra \( \mathfrak{A}/\theta_h \) is weakly isomorphic to \( \mathfrak{B} \), and the weak isomorphism \( a: A/\theta_h \rightarrow B \) is defined by

\[
a([a])_{\theta_h} = h(a).
\]

Proof. By Lemma 4, the relation \( \theta_h \) defined by \( h \) is a congruence relation on the algebra \( \mathfrak{A} \). Hence it follows that the value \( h(a) \) is independent of a choice of a representative from the class \([a]_{\theta_h}\), and so the mapping \( a \) is well defined. Moreover, \( a \) is a bijection.

In virtue of Corollary 2 from Theorem 2, the natural mapping \( \eta \) is a weak homomorphism of \( \mathfrak{A} \) onto \( \mathfrak{B}/\theta_h \). In view of Corollary 4 from the same theorem, we conclude that \( a \) is a weak homomorphism and, in view of Theorem 3, \( a \) is a weak isomorphism, what completes the proof of Theorem 7.

One can prove analogously the following dual theorem:

THEOREM 7*. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be algebras and \( h: B \rightarrow A \) be an injective weak homomorphism of \( \mathfrak{B} \) into \( \mathfrak{A} \). Then the algebra \( h(\mathfrak{B}) \) is weakly isomorphic with the algebra \( \mathfrak{B} \).
Theorem 8 (Weak isomorphism theorem). Let \( \theta_1 \subset \theta_2 \) be two congruence relations on the algebra \( \mathcal{A} \). Then there exists a unique weak homomorphism \( h \) of \( \mathcal{A}/\theta_1 \) onto \( \mathcal{A}/\theta_2 \) such that \( h\eta_1 = \eta_2 \), where \( \eta_1 \) and \( \eta_2 \) are the natural mappings \( \eta_i : A \rightarrow A/\theta_i \) \( (i = 1, 2) \), and \( (\mathcal{A}/\theta_1)/\theta_h \) is weakly isomorphic to \( \mathcal{A}/\theta_2 \).

Proof. In virtue of the known theorem (see, e.g. [2], Theorem 3.4) there exists a unique mapping \( h : A/\theta_1 \rightarrow A/\theta_2 \) such that \( h\eta_1 = \eta_2 \). Taking into account Corollary 4 from Theorem 2 we conclude that \( h \) is a weak homomorphism. From Theorem 7 it follows that \( (\mathcal{A}/\theta_1)/\theta_h \) and \( \mathcal{A}/\theta_2 \) are weakly isomorphic.

This fact is illustrated by the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_1} & A/\theta_1 & \xrightarrow{(A/\theta_1)/\theta_h} & (A/\theta_1)/\theta_h \\
\downarrow{\eta_2} & & \downarrow{h} & & \\
A/\theta_2 & & \\
\end{array}
\]

From Theorem 8 we have the following

Corollary 1. Let \( \theta_1 \subset \theta_2 \) be congruence relations on the algebra \( \mathcal{A} \). Then \( \tilde{\theta}_1 \subset \tilde{\theta}_2 \).

Indeed, from Theorem 8 there exists a weak homomorphism \( h : A/\theta_1 \rightarrow A/\theta_2 \) such that the algebras

\[
\mathcal{A}/\theta_2 = (A/\theta_2; A/\theta_2) \quad \text{and} \quad (\mathcal{A}/\theta_1)/\theta_h = ((A/\theta_1)/\theta_h; (A/\tilde{\theta}_1)/\tilde{\theta}_h)
\]

are weakly isomorphic and the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{} & A/\tilde{\theta}_2 \xleftarrow{(A/\tilde{\theta}_1)/\tilde{\theta}_h} & \\
\end{array}
\]

is commutative, and so we have the thesis.

Obviously, the following dual theorem is also true

Theorem 8*. If \( B_1 \subset B_2 \subset A \) are subalgebras of the algebra \( \mathcal{A} = (A; A) \), then \( B_1 \) is a subalgebra of the algebra \( \mathcal{B}_2 = (B_2; A|_{B_2}) \).

Corollary 1*. If \( B_1 \subset B_2 \subset A \) are subalgebras of \( \mathcal{A} \), then

\[
(A|_{B_2})|_{B_1} = A|_{B_1}
\]

Finally, we shall prove the following

Theorem 9. Let \( \theta \) be a congruence relation on the algebra \( \mathcal{A} = (A; A) \) and let \( \tilde{\theta} \) and \( \tilde{\theta} \) be the induced relations on the set \( A \) of all algebraic oper-
ations of the algebra $\mathfrak{A}$ and on the set of all algebraic operations of the algebra $\mathfrak{A}$, respectively. Then

$$(\overline{A}/\overline{\theta}; \overline{A}/\overline{\theta}) = (\overline{A}/\overline{\theta}; \overline{A}/\overline{\theta}),$$

i.e., the class of all algebraic operations determined by $\overline{A}/\overline{\theta}$ on the support $A/\theta$ coincides with the class of all algebraic operations generated by $A/\theta$.

**Proof.** In view of Corollary 2 from Theorem 2 the natural mapping $\eta: A \rightarrow A/\theta$ is a weak homomorphism of the algebra $\mathfrak{A}$ onto $\mathfrak{A}/\theta = (A/\theta; A/\theta)$. Taking into account Theorem 6, we conclude that the mapping $\eta$ is a weak homomorphism of the algebra $\overline{\mathfrak{A}} = (A; \overline{A})$ onto the algebra $\overline{\mathfrak{A}}/\theta = (A/\theta; A/\theta)$. On the other hand $\overline{\mathfrak{A}}/\theta = (A/\theta; A/\theta)$, where $\theta$ is the congruence relation on the set of all algebraic operations of the algebra $\overline{\mathfrak{A}}$, because $\theta$ is a congruence relation on the algebra $\overline{\mathfrak{A}}$ (Proposition 2). Hence the natural mapping $\mu: A \rightarrow A/\theta$ is a weak homomorphism of $\mathfrak{A}$ onto $\overline{\mathfrak{A}}/\theta$ and $\mu = \eta$. Therefore, by Corollary 2 from Theorem 2 we get the required equality, what completes the proof of Theorem 9.

5. **Examples.** Taking into account the weak homomorphism theorem (Theorem 7) and the fact that a composition of weak homomorphisms is also a weak homomorphism, we conclude that:

(i) The mapping $h: \mathfrak{A} \rightarrow \mathfrak{B}$ is a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ iff $h$ is the composition of the natural projection of $\mathfrak{A}$ onto the quotient algebra $\mathfrak{A}/\theta_h$ and a weak isomorphism of $\mathfrak{A}/\theta_h$ onto a subalgebra of $\mathfrak{B}$.

This is a good tool for description of weak homomorphisms.

**Example 1.** Let $\mathfrak{G}_i = (G_i; \cdot, ^{-1})$, $i = 1, 2$, be groups and in $G_i$ the square of each element belongs to the centre. Then in view of Theorem 1 of [10] we have

(ii) The mapping $h: G_1 \rightarrow G_2$ is a weak homomorphism of $\mathfrak{G}_1$ into $\mathfrak{G}_2$ iff $h$ is a homomorphism or an anti-homomorphism (i.e., $h(x \cdot y) = h(y) \cdot h(x)$) of $\mathfrak{G}_1$ into $\mathfrak{G}_2$.

In particular

(iii) The mapping $h: G_1 \rightarrow G_2$ is a weak homomorphism of an abelian group $\mathfrak{G}_1$ into $\mathfrak{G}_2$ iff $h$ is a homomorphism of $\mathfrak{G}_1$ into $\mathfrak{G}_2$.

Taking into consideration the result of Hanna Neumann ([16]), we infer that (ii) is also true for free groups $G_1$ and $G_2$. However, this is not true for nilpotent groups of order 2 (see [12]).

**Example 2.** Let $\mathfrak{B}_i = (B_i; \cup, \cap, ^{'} , 0, 1)$, $i = 1, 2$, be Boolean algebras. Then taking into account Theorem 1 of [20] we have

(iv) The mapping $h: B_1 \rightarrow B_2$ is a weak homomorphism of $\mathfrak{B}_1$ into $\mathfrak{B}_2$ iff $h$ is a homomorphism of $\mathfrak{B}_1$ into $\mathfrak{B}_2$ or such mapping that $h(x \cup y) = h(x) \cap$
\( h(y), \ h(x \cap y) = h(x) \cup h(y) \) and \( h(x') = (h(x))' \) (and so \( h(1) = 0 \) and \( h(0) = 1 \)).

In particular

(v) If \( h: B_1 \to B_2 \) is a weak homomorphism of \( B_1 \) into \( B_2 \) and \( h(0) = 0 \), then \( h \) is a homomorphism.

Similar results one can get for so-called Post algebras (\cite{20}).

**Example 3.** Let \( B_i = (V_i; x+y, \{\lambda x\}_{\lambda \in K}), \ i = 1, 2, \) be linear spaces over a field \( K \). Then taking into account Theorem 1.1 of \cite{4} we have

(vi) A mapping \( h: V_1 \to V_2 \) is a weak homomorphism of \( B_1 \) into \( B_2 \) iff there exist an endomorphism \( h_0: V_1 \to V_1 \), an isomorphic injection \( g: h(V_1) \to V_2 \) and an automorphism \( \varphi \) of the field \( K \) such that

\[
  h(\lambda x + \mu y) = \varphi(\lambda) g(h_0(x)) + \varphi(\mu) g(h_0(y)).
\]

In particular

(vii) If \( h: V_1 \to V_2 \) is a weak homomorphism of \( B_1 \) into \( B_2 \) and \( K \) is simple, then \( h \) is a homomorphism.

**Example 4.** Finally, let \( \mathfrak{S}_i = (S_i; \circ_i), \ i = 1, 2, \) be commutative semi­
groups, and let \( \mathfrak{S}_i^1 = (S_i^1; \circ_i) \) denote the semigroup, which is formed from \( \mathfrak{S}_i \) by adjoining of the unity \( e_i \) (see \cite{1}, 1.1). A. Iwanik has a proof (his proof is unpublished) that each weak automorphism of a commutative
semigroup is an automorphism. We shall prove (in a little other way) a generalization of Iwanik's result:

(viii) Let \( h: S_1^1 \to S_2 \) be a weak homomorphism of \( \mathfrak{S}_1^1 \) onto an arbitrary commutative semigroup \( \mathfrak{S}_2 \). Then \( h \) is a (usual) homomorphism and \( e_2 = h(e_1) \) is the unity in \( S_2 \).

Indeed, taking into account our assumptions, we have

\[
y = h(x) = h(x \circ_1 e_1) = h(x) h(\circ_1) h(e_1) = y h(\circ_1) h(e_1)
\]

for every \( y \in S_2 \). Thus \( h(e_1) \) is the unit of the operation \( h(\circ_1) \). Using Theorem 5, we know that \( \{h(\circ_1)\} \) forms the set of fundamental operations of the algebra \( (S_2; \circ_2) = (S_2; h(\circ_1)) \). Since the operation \( \circ_1 \) is commutative, hence \( h(\circ_1) \) is too. Therefore

\[
y_1 h(\circ_1) y_2 = y_1^m \circ_2 y_2^m;
\]

moreover,

\[
y_1 \circ_2 y_2 = y_1^k h(\circ_1) y_2^k
\]

for every \( y_1, y_2 \in S_2 \) and for some positive integers \( k \) and \( m \), whereas in

\((**)\) the powers are taken in the sense of operation \( h(\circ_1) \). From \((**)\) we infer that \( h(e_1) \) is idempotent with respect to the operation \( \circ_2 \). By Theorem 5 it is suffice to show that \( h(\circ_1) = \circ_2 \). From \((*)\) we have \( y = y^m \circ_2 h(e_1) \)
for every \( y \in S_2 \) and we get
\[
y_1 h(\circ_1) y_2 = (y_1^m \circ_2 h(e_1))^m \circ_2 y_2^m = y_1^m \circ_2 (h(e_1))^{m+2} \circ_2 y_2^m
\]
\[
= (y_1^m \circ_2 h(e_1))^m \circ_2 (h(e_1))^{m+2} \circ_2 y_2^m = y_1 \circ_2 y_2.
\]
Thus \( h \) is a homomorphism of \( S_1 \) onto \( S_2 \) and so \( h(e_1) \) is a unity of \( S_2 \).

(ix) Let \( S_i, i = 1, 2, \) be two arbitrary semigroups without the unity. Then if \( h: S_1 \rightarrow S_2 \) is a weak isomorphism \( S_1 \) onto \( S_2 \), then the mapping \( h_1: S_1 \rightarrow S_2 \), defined by equalities \( h_1(e_1) = e_2 \) and \( h_1(x) = h(x) \) for \( x \in S_1 \), is a weak isomorphism of \( S_1 \) onto \( S_2 \).

Indeed, taking into account the definition \( h_1 \) and Theorem 3, it suffices to show that \( \circ_1 R_h \tilde{h}_1(\circ_1) \). By the definition \( \tilde{h}_1 \) (see (F)) and Theorem 5 we need to verify that \( \tilde{h}_1(\circ_1) \) is an algebraic operation in \( S_2 \). \( \tilde{h}(\circ_1) \) is algebraic in \( S_2 \) by the assumption about \( h \). Obviously, the operation \( \tilde{h}_1(\circ_1) \) is commutative, and \( x \tilde{h}(\circ_1) y = x \tilde{h}(\circ_1) \) for every \( x, y \in S_2 \). Since
\[
h_1(h_1^{-1}(x) \circ_1 h_1^{-1}(e_2)) = h_1(h_1^{-1}(x) \circ_1 e_1) = h_1(h_1^{-1}(x)) = x.
\]
Hence \( \tilde{h}_1(\circ_1) e_2 = x \) for every \( x \in S_2 \). And so \( \tilde{h}_1(\circ_1) \) is the required extension of \( \tilde{h}(\circ_1) \) on the semigroup \( S_2 \). Thus \( h_1 \) is a weak isomorphism.

By (i), (viii) and (ix) we get

(x) Each weak homomorphism of commutative semigroups into other one is a homomorphism.

References