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On a differential equation with deviating argument

1. In the present paper we are concerned with the differential-functional equation of the form

$$\frac{d\varphi}{dx} = h\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)], u\big), \tag{1}$$

where φ is an unknown function and $h, f, i = 1, \ldots, n$ are known functions and u is a real parameter.

We shall prove, under suitable assumptions, that equation (1) possesses exactly one solution defined in the interval $\langle 0, \infty)$ and fulfilling the initial condition

$$\varphi(0) = \xi \tag{2}$$

and this solution depends continuously on ξ and u. It is some generalization of results obtained in [1], [3], [5], [6].

For the equation

$$\frac{d\varphi}{dx} = h\big((x, \varphi(x), \varphi[f(x)], u\big) \tag{3}$$

the corresponding problem has been investigated by T. Dłotko [4]. For the equation

$$\frac{d\varphi}{dx} = h\left(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)], \frac{d\varphi}{dx}, \frac{d\varphi}{dx}[g_1(x)], \ldots, \frac{d\varphi}{dx}[g_m(x)]\right)$$

for $x \in \langle 0, a\rangle$, $a < \infty$ (under different assumptions) some theorems (Theorems 5 and 7) have been obtained in [7].

2. We assume the hypotheses:

Hypothesis 1.

(i) *Let* $(B, \|\cdot\|)$ *be a Banach space. The function* $h\colon I \times B^{n+1} \times R \to B$, $I = \langle 0, \infty)$, $R = (-\infty, \infty)$ *is continuous in* $I - B^{n+1} \times R$.

(ii) *There exist real functions* $L_0, \ldots, L_n$ *continuous in* I, $L_i(x) \geqslant 0$ *for* $x \in I$, $i = 0, 1, \ldots, n$, *such that for every* $(r_0, \ldots, r_n)$, $(z_0, \ldots, z_n) \in B^{n+1}$, $x \in I$, $u \in R$ *we have*

$$(4) \qquad \|h(x, r_0, \ldots, r_n, u) - h(x, z_0, \ldots, z_n, u)\| \leqslant \sum_{i=0}^{n} L_i(x)\|r_i - z_i\|.$$

(iii) *The real functions* f_i, $i = 0, 1, \ldots, n$, *where* $f_0(x) = x$, *are continuous in* I *and* $f_i(I) \subset I$, $i = 1, \ldots, n$.

(iv) *There exist constants* $\alpha > 1$, K, $\theta \in \langle 0, 1)$ *and* $(y_0, \ldots, y_n) \in B^{n+1}$ *such that for every fixed* $u \in R$, *we have*

$$(5) \qquad \int_0^x \|h(s, y_0, \ldots, y_n, u)\| ds \leqslant K \exp\big(\alpha L(x)\big), \qquad x \in I,$$

where

$$(6) \qquad L(x) = \int_0^x \sum_{i=0}^{n} L_i(s) ds, \qquad x \in I.$$

Moreover,

$$(7) \qquad \sum_{i=0}^{n} L_i(x) \exp\big(\alpha L[f_i(x)]\big) \leqslant \alpha\theta \exp\big(\alpha L(x)\big) \sum_{i=0}^{n} L_i(x), \qquad x \in I.$$

HYPOTHESIS 2. *There exist constant* M *and functions* $N\colon I \to I$, $\omega\colon I \to I$ *such that* $\omega(u) \to 0$ *as* $u \to 0+$ *and*

$$\exp\big(-\alpha L(x)\big) \int_0^x N(s) ds \leqslant M, \qquad x \in I.$$

For every $(z_0, \ldots, z_n) \in B^{n+1}$, $x \in I$, $u_1, u_2 \in R$ *we have*

$$\|h(x, z_0, \ldots, z_n, u_1) - h(x, z_0, \ldots, z_n, u_2)\| \leqslant N(x)\,\omega(|u_1 - u_2|).$$

Remark. Condition (7) is fulfilled, for example, if $f_i(x) \leqslant x$ or $f_i(x) \geqslant x$ and

$$L[f_i(x)] - L(x) \leqslant c < \alpha^{-1} \ln \alpha, \qquad i = 1, \ldots, n.$$

3. Now we shall prove

THEOREM 1. *If Hypothesis* 1 *holds, then for every* $u \in R$ *there exists exactly one function* φ *satisfying equation* (1) *and fulfilling the initial condition* (2), *where* $\xi \in B$, *and such that*

$$\|\varphi(x)\| = O\big(\exp[\alpha L(x)]\big).$$

It is the limit of successive approximations.

Proof. We define G as the space of these functions which are defined and continuous in I and $\varphi(x)\in B$ for $x\in I$ and

$$\|\varphi(x)\| = O\big(\exp[aL(x)]\big), \qquad x\in I. \tag{8}$$

For $\varphi\in G$ we define the norm (cf. also [1])

$$|\varphi| = \sup_{x\in I}\Big(\|\varphi(x)\|\exp\big(-aL(x)\big)\Big).$$

We can easily verify that G is a Banach space. Equation (1) with condition (2) is equivalent with the equation

$$\varphi(x) = \xi + \int_0^x h\big(s, \varphi(s), \varphi[f_1(s)], \ldots, \varphi[f_n(s)], u\big)ds. \tag{9}$$

We define the transform

$$\Phi(x) = \xi + \int_0^x h\big(s, \varphi(s), \ldots, \varphi[f_n(s)], u\big)ds. \tag{10}$$

We shall prove that (10) maps G into itself. Φ is evidently continuous in I and $\Phi(x)\in B$ for $x\in I$. Now, by (4) and (5) we have

$$\|\Phi(x)\| \leqslant \xi + \int_0^x \Big\|h\big(s, \varphi(s), \ldots, \varphi[f_n(s)], u\big) - h(s, y_0, \ldots, y_n, u)\Big\|ds + \int_0^x \|h(s, y_0, \ldots, y_n, u)\|ds$$

$$\leqslant \|\xi\| + \int_0^x \Big(\sum_{i=0}^n L_i(s)\,\|\varphi[f_i(s)] - y_i\|\Big)ds + J,$$

where

$$J = \int_0^x \|h(s, y_0, \ldots, y_n, u)\|ds.$$

Next

$$\|\Phi(x)\| \leqslant \|\xi\| + \max(|\varphi - y_0|, \ldots, |\varphi - y_n|)\int_0^x \Big(\sum_{i=0}^n L_i(s)\exp\big(L[f_i(s)]\big)\Big)ds + J. \tag{11}$$

From (7) and (6) we obtain

$$\int_0^x \Big(\sum_{i=0}^n L_i(s)\exp\big(aL[f_i(s)]\big)\Big)ds \leqslant \theta\int_0^x a\Big(\sum_{i=0}^n L_i(s)\Big)\exp\big(aL(s)\big)ds = \theta\exp\big(aL(s)\big)\Big|_0^x \leqslant \theta\exp\big(aL(x)\big).$$

Consequently, by (11) we have

$$\|\Phi(x)\| \leqslant \|\xi\| + \theta\max(|\varphi - y_0|, \ldots, |\varphi - y_n|)\exp\big(aL(x)\big) + J,$$

whence

$$\|\Phi(x)\|\exp\big(-aL(x)\big) \leqslant \text{const} \quad \text{for } x \geqslant 0.$$

Thus $\Phi \in G$. Now we shall prove that (10) is a contraction map. Actually,

$$(12)\quad \|\Phi(x) - \Psi(x)\|$$
$$\leqslant \int_0^x \Big\|h\big(s, \varphi(s), \ldots, \varphi[f_n(s)], u\big) - h\big(s, \psi(s), \ldots, \psi[f_n(s)], u\big)\Big\| ds$$
$$\leqslant \int_0^x \Big(\sum_{i=0}^n L_i(s)\,\|\varphi[f_i(s)] - \psi[f_i(s)]\|\Big)\, ds$$
$$\leqslant |\varphi - \psi| \int_0^x \Big(\sum_{i=0}^n L_i(s)\exp\big(aL[f_i(s)]\big)\Big)\, ds$$
$$\leqslant \theta|\varphi - \psi| \int_0^x a\Big(\sum_{i=0}^n L_i(s)\Big)\exp[aL(s)]\,ds$$
$$= \theta|\varphi - \psi|\big[\exp\big(aL(x)\big) - 1\big] \leqslant \theta|\varphi - \psi|\exp\big(aL(x)\big).$$

Consequently

$$(13)\qquad |\Phi - \Psi| \leqslant \theta|\varphi - \psi|.$$

On account of the Banach theorem for every fixed $u \in R$ there exists a unique fixed point of transform (10), i.e., unique function $\varphi \in G$ satisfying equation (1) and condition (2) and it is given as the limit of successive approximations. This completes the proof.

Next, we shall prove the following

THEOREM 2. *If Hypotheses* 1 *and* 2 *are fulfilled, then solution of equation* (1) *depends a continuously on* ξ *and* u.

Proof. We assume $a(\xi, u) = \theta < 1$ and

$$F = F(x, \varphi, \xi, u) = \xi + \int_0^x h\big(s, \varphi(s), \ldots, \varphi[f_n(s)], u\big)ds.$$

From (12) and (13) we have

$$|F(x, \varphi, \xi, u) - F(x, \psi, \xi, u)| \leqslant \theta|\varphi - \psi|.$$

The proof that F is continuous with respect to ξ and u (on acconut of Hypothesis 2) does not differ from that given for Theorem 3 in [4] and is therefore omitted. Consequently Theorem 2 follows from [2] (Theorem 7), which completes the proof.

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