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On the degree of approximation of an entire function

1. Let $f(x)$ be a real-valued continuous function defined on $[-1, 1]$, and let

$$E_n(f) = \inf_{p\in\pi_n} \|f-p\|, \qquad n = 0, 1, 2, \ldots,$$

where the norm is the maximum norm on $[-1, 1]$ and π_n denotes the set of all polynomials with real coefficients of degree at most n. Bernstein [1], p. 118, proved that

$$\lim_{n\to\infty} E_n^{1/n}(f) = 0$$

if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function.

Further, let $f(z)$ be an entire function, and

$$M(r) = \max_{|z|=r} |f(z)|,$$

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \log\log M(r)/\log r = \begin{matrix}\varrho\\ \lambda\end{matrix} \qquad (0 \leqslant \lambda \leqslant \varrho \leqslant \infty),$$

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^{\varrho}} = \begin{matrix}T\\ t\end{matrix} \qquad \begin{pmatrix}0 < \varrho < \infty \\ 0 \leqslant t \leqslant T \leqslant \infty\end{pmatrix}.$$

where ϱ, λ and T, t denote the order, lower order and type, lower type, respectively, of an entire function $f(z)$.

Bernstein [1], p. 114, has shown that there exist constants ϱ (positive) α, T (non-negative) such that

$$\limsup_{n\to\infty} n^{1/\varrho} E_n^{1/n}(f) = \alpha$$

if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ϱ and type T.

Recently Varga [11], Theorem 1, has proved that

$$\limsup_{n\to\infty} \{n \log n / \log 1/E_n(f)\} = \varrho,$$

where ϱ is the non-negative real number if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ϱ.

The results of Bernstein and Varga give us the clue that the rate at which $E_n^{1/n}(f)$ tends to zero depends on the order and type of an entire function $f(z)$.

The object of this paper is to study the relationship of growth constants defined above with the rate of growth of $E_n(f)$. Some theorems of this nature have been given in [7]–[10] but we consider some further results which will improve the results of Reddy [7], [8].

2. THEOREM 1. *Let $f(x)$ be a real-valued continuous function defined on $[-1,1]$. If $f(x)$ is the restriction to $[-1,1]$ of an entire function $f(z)$ of order ϱ and lower order λ, there exist two sequences $\{k(n)\}$, $\{\varepsilon(n)\}$ $(n=0,1,2,\ldots)$ such that one has*

$$k(n)=n^{1+O(1)},\qquad \varepsilon(n)\to 0\qquad as\ n\to\infty,$$

$$(*)\qquad E_{k(n)}(f)/E_{k(n)+h}(f)>\big(E_n(f)/E_{n+1}(f)\big)^{h(1-\varepsilon(n))}\qquad (n\geqslant n_0,\ h=1,2,3,\ldots),$$

then

$$\frac{1}{\lambda}=\limsup_{n\to\infty}\{\log E_n(f)/E_{n+1}(f)\}/\log n.$$

We require the following lemma in the proof of Theorem 1.

LEMMA 1. *If $\{a_n\}$ is a sequence of real or complex numbers, then*

$$\limsup_{n\to\infty}\log|1/a_n|/n\log n\leqslant\limsup_{n\to\infty}\log|a_n/a_{n+1}|/\log n.$$

Proof of Lemma 1. See [9], p. 193.

Proof of Theorem 1. First, assume that $f(x)$ has an analytic extension $f(z)$, which is an entire function of order ϱ and lower order λ. Following Bernstein's original proof we have (cf. [5], p. 78, [6], p. 84) for each $n\geqslant 0$,

$$(2.1)\qquad E_n(f)\leqslant\frac{2B(\sigma)}{\sigma^n(\sigma-1)}\qquad\text{for any }\sigma>1,$$

where $B(\sigma)$ is the maximum of the absolute value of $f(z)$ on E_σ, and E_σ with $\sigma>1$ denotes the closed interior of ellipse with foci at ±1 and half major axis $\left(\frac{\sigma^2+1}{2\sigma}\right)$ and half minor axis $\left(\frac{\sigma^2-1}{2\sigma}\right)$. The closed discs $D_1(\sigma)$ and $D_2(\sigma)$ bound the ellipse E_σ in the sense that

$$D_1(\sigma)\equiv\left\{z\mid |z|\leqslant\frac{\sigma^2-1}{2\sigma}\right\}\subset E_\sigma\subset D_2(\sigma)\equiv\left\{z\mid |z|\leqslant\frac{\sigma^2+1}{2\sigma}\right\}.$$

From this inclusion, it follows by definition that

$$(2.2)\qquad M_f\left(\frac{\sigma^2-1}{2\sigma}\right)\leqslant B(\sigma)\leqslant M_f\left(\frac{\sigma^2+1}{2\sigma}\right)\qquad\text{for all }\sigma>1.$$

From this, it can be verified that

$$(2.3)\qquad \begin{matrix}\varrho\\ \lambda\end{matrix} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \log\log M(\sigma)/\log\sigma = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \log\log B(\sigma)/\log\sigma.$$

From (2.1), we have

$$(2.4)\qquad E_n(f) \leqslant \frac{KB(\sigma)}{\sigma^n},$$ where K is some positive constant

since $\sigma > 1$,

$$2/(\sigma-1) < K.$$

Consequently from (2.4), we have for any $\varepsilon > 0$

$$(2.5)\qquad \sum_{n=0}^{\infty} E_n(f)\sigma^n \leqslant \sum_{n=0}^{\infty} \frac{KB(\sigma+\varepsilon)}{(\sigma+\varepsilon)^n}\sigma^n = KB(\sigma+\varepsilon)\sum_{n=0}^{\infty}\left(\frac{\sigma}{\sigma+\varepsilon}\right)^n = KB(\sigma+\varepsilon)\left(\frac{\sigma+\varepsilon}{\varepsilon}\right).$$

We note [11] that

$$(2.6)\qquad B(\sigma) \leqslant |P_0(z)| + 2\sigma\sum_{j=0}^{\infty} E_j(f)\sigma^j.$$

Let us write

$$(2.7)\qquad J(\sigma) = \sum_{j=0}^{\infty} E_j(f)\sigma^j$$

which represents an entire function, then we have from (2.5) and (2.6)

$$(2.8)\qquad B(\sigma) \leqslant K'\sigma J(\sigma) \leqslant K''\sigma(\sigma+\varepsilon)B(\sigma+\varepsilon),$$

where K' and K'' are some positive constants. From (2.2) and (2.8) it follows that

$$(2.9)\qquad \begin{matrix}\varrho\\ \lambda\end{matrix} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log B(\sigma)}{\log\sigma} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \log\log J(\sigma)/\log\sigma.$$

Now, applying Lemma 1 to $\{E_n(f)\}$ we have

$$\limsup_{n\to\infty} \log 1/E_n(f)/n\log n \leqslant \limsup_{n\to\infty} \log\{E_n(f)/E_{n+1}(f)\}/\log n.$$

By known result [7], Theorem 2A, we have

$$(2.10)\qquad \frac{1}{\lambda} \leqslant \limsup_{n\to\infty} \log\{E_n(f)/E_{n+1}(f)\}/\log n.$$

If $\lambda = 0$, the theorem is an obvious consequence of this inequality, while, if $\lambda > 0$ it is sufficient to prove that

$$(2.11)\qquad \frac{1}{\lambda} \geqslant \limsup_{n\to\infty} \log\{E_n(f)/E_{n+1}(f)\}/\log n.$$

Let us put for simplicity

$$E_n(f)/E_{n+1}(f) = \psi(n).$$

The sequence $\psi(n)$ cannot be bounded (in fact, $\psi(n) < k$ for each value of n it follows that $E_n^{1/n}(f) > 1/k$, which is absurd). Therefore $\limsup_{n\to\infty} \psi(n) = \infty$.

Then denoting the sequence of the indices by $\{n_h\}$ for which $\psi(n_h) > \psi(n)$ for each $n < n_h$.

Allowing $\psi(n) \leqslant \psi(n_h)$ for $n_h \leqslant n < n_{h+1}$. One has

$$\limsup_{n\to\infty} \frac{\log\psi(n)}{\log n} = \limsup_{n\to\infty} \frac{\log\psi(n_h)}{\log n_h}. \tag{2.12}$$

If $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is an entire function of lower order λ and $\nu'(r)$ is the rank of the maximum term, then

$$\lambda = \liminf_{r\to\infty} \frac{\log\nu'(r)}{\log r} \qquad [12].$$

Applying this to $J(\sigma)$, we see that

$$\lambda = \liminf_{\sigma\to\infty} \frac{\log\nu(\sigma)}{\log\sigma}.$$

[$\nu(\sigma)$ is the rank of maximum term of $J(\sigma)$.]

Choose λ' with $0 < \lambda' < \lambda$ for each $\sigma \geqslant \sigma_0(\lambda')$; then we get

$$\log\nu(\sigma) > \lambda' \log\sigma. \tag{2.13}$$

Let $\sigma_h = \{\psi(n_n)\}^{1-\varepsilon(n_h)}$ and let us evaluate $\nu(\sigma_h)$. For each value of $n > k(n_h)$, we have according to condition (*)

$$E_{k(n_h)}(f)/E_n(f) > \{\psi(n_h)\}^{(n-k(n_h))(1-\varepsilon(n_h))} = \sigma_h^{n-k(n_h)}.$$

Therefore,

$$E_{k(n_h)}(f)\,\sigma_h^{k(n_h)} > E_n(f)\,\sigma_h^n.$$

Hence $\nu(\sigma_h) \leqslant k(n_h)$.

From (2.13), it can be derived that

$$\log k(n_h) \geqslant \log\nu(\sigma_h) \geqslant \lambda'\{1-\varepsilon(n_h)\}\log\psi(n_h),$$

which leads to

$$\log\psi(n_h)/\log n_h \leqslant \frac{1}{\lambda'\{1-\varepsilon(n_h)\}} \frac{\log k(n_h)}{\log n_h}.$$

As $k(n_h) = n_h^{1+O(1)}$ and $\varepsilon(n_h) \to 0$ for $h\to\infty$. One then obtains

$$\limsup_{h\to\infty} \frac{\log\psi(n_h)}{\log n_h} \leqslant \frac{1}{\lambda'} \quad \text{for each } \lambda' \text{ with } 0 < \lambda' < \lambda. \tag{2.14}$$

Now the desired conclusion follows from (2.10), (2.11), (2.12) and (2.14).

Remark. For $\varepsilon(n) = 1/n$, $k(n) = n$, Theorem 1 gives a result of Reddy [8] (Theorem 3, $k = 1$) as a particular case.

3. Theorem 2. *Let $f(x)$ be a real valued continuous function defined on $[-1, 1]$, which is the restriction of an entire function $f(z)$ of order ϱ and lower order λ. Further, if there exists a sequence $\{\varepsilon(n)\}$ such that one has $\varepsilon(n) \to 0$ as $n \to \infty$,*

$$(**) \qquad \frac{E_m(f)}{E_{m+1}(f)} > \big(E_n(f)/E_{n+1}(f)\big)^{1-\varepsilon(n)} \qquad (n \geqslant n_0, \quad m = n, n+1, \ldots),$$

then

$$(3.1) \qquad \liminf_{n\to\infty} \log\{E_n(f)/E_{n+1}(f)\}/\log n = \frac{1}{\varrho},$$

$$(3.2) \qquad \limsup_{n\to\infty} \log\{E_n(f)/E_{n+1}(f)\}/\log n = \frac{1}{\lambda}.$$

Proof of Theorem 2. We omit the proof since it is based on the same lines as the proof of Theorem 1.

Remarks. (i) If the sequence $E_n(f)/E_{n+1}(f)$ is non-decreasing for $n \geqslant n_0$, condition $(**)$ is satisfied with $\varepsilon(n) = 1/n$, hence our Theorem 2 contains a result of Reddy [8] (Theorem 3, $k = 1$) as a particular case.

(ii) Condition $(*)$ is not sufficient to guarantee the sign of equality in (3.1).

4. Theorem 3. *Let $f(x)$ be a real valued continuous function on $[-1, 1]$; then*

$$(4.1) \qquad \frac{1}{\lambda} = \min_{\{n_h\}} \limsup_{n\to\infty} \log\{E_{n_{h-1}}(f)/E_{n_h}(f)\}/(n_h - n_{h-1})\log n_{h-1},$$

where $\{n_h\}$ is an increasing sequence of positive integers and λ is a non-negative real number, if $f(x)$ is the restriction to $[-1, 1]$ of an entire function $f(z)$ of lower order λ.

Proof of Theorem 3. From (2.2) and (2.8) it follows that

$$\lambda = \liminf_{\sigma\to\infty} \frac{\log\log B(\sigma)}{\log\sigma} = \liminf_{\sigma\to\infty} \frac{\log\log J(\sigma)}{\log\sigma}.$$

If $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is an entire function of lower order λ $(0 \leqslant \lambda \leqslant \infty)$, then

$$\lambda = \max_{\{n_h\}} \liminf_{n\to\infty} (n_h - n_{h-1})\log n_{h-1}/\log|b_{n_{h-1}}|/|b_{n_h}|, \qquad [3], \text{ p. } 310.$$

Applying this to $J(\sigma)$, we get the required result, i.e.

$$\frac{1}{\lambda} = \min_{\{n_h\}} \limsup_{n\to\infty} \log\{E_{n_{h-1}}(f)/E_{n_h}(f)\}/(n_h - n_{h-1})\log n_{h-1}.$$

5. To deal with functions of infinite order, we have the following classification [7]. There exists a positive integer $k \geqslant 2$ for which

$$\lim_{r\to\infty} \frac{\sup}{\inf} \frac{l_{k+1}M(r)}{\log r} = \frac{\varrho_k}{\lambda_k}$$

are finite and positive, where $l_k(x) = \log\log\ldots(k\text{-times})x$ $(k = 1, 2, 3, \ldots)$ and $l_k x > 0$ for all sufficiently large positive x. An entire function with $\varrho_{k-1} = \infty$ and $\varrho_k < \infty$ is called an entire function of index k. Thus, ϱ_k and λ_k extend the definitions of ϱ and λ, which correspond to $k = 1$.

If $0 < \varrho_k < \infty$, then there exists a proximate order $\varrho_k(r)$ (may be named as k-th proximate order of $f(z)$) satisfying the following conditions:

(i) $\varrho_k(r)$ is a non-negative, continuous function of r for $r \geqslant r_0$.

(ii) $\lim_{r\to\infty} \varrho_k(r) = \varrho_k$.

(iii) $\lim_{r\to\infty} r\varrho_k'(r)\log r = 0$, where $\varrho_k'(r)$ is either the right-hand or the left-hand derivative at the points where they are different.

(iv) $\limsup_{r\to\infty} l_k M(r)/_r\varrho_k(r) = T_k^*$ $(0 < T_k^* < \infty)$.

T_k^* will be called *k-th proximate type*. In this section we study the relationship of k-th proximate order $\varrho_k(r)$ and its corresponding T_k with the rate of growth of $E_n^{1/n}(f)$. To state and prove the theorem precisely, we introduce a function $\Phi(x)$ which is defined as an unique solution (when $x > x_0$) of the equation

$$x = r^{\varrho_k(r)}.$$

We prove the following

THEOREM 4. *Let $f(x)$ be a real valued continuous function on $[-1, 1]$ which is the restriction to $[-1, 1]$ of an entire function $f(z)$ of index k with order ϱ_k $(0 < \varrho_k < \infty)$ and k-th proximate order $\varrho_k(r)$; then*

$$\limsup_{n\to\infty} \{\Phi(n) E_n^{1/n}(f)\} \tag{5.1}$$

and if $k > 1$,

$$\limsup_{n\to\infty} \{\Phi(l_{k-1}n) E_n^{1/n}(f)\}, \tag{5.2}$$

are finite.

Before we start with the actual proof, we first consider the following lemmas which will be needed in the proof of the theorem.

LEMMA 2. *The type T^* (proximate type) of an entire function*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with the proximate order $\varrho(r)$ is given by the equation

$$\limsup_{n\to\infty} \{\Phi(n)|a_n|^{1/n}\} = (T^* e\varrho)^{1/\varrho}.$$

Proof of Lemma 2. See [4], Theorem 2′, p. 42.

LEMMA 3. *Let $f(z)$ be an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of k-th proximate order $\varrho_k(r)$ and k-th proximate type T_k^*. Then T_k^* is given by the equation*

$$\limsup_{n\to\infty} \{\Phi(l_{k-1} n)|a_n|^{1/n}\} = (T_k^*)^{1/\varrho_k}.$$

We omit the proof since it can be proved on the same lines as given by Jain [2] for entire Dirichlet Series.

Proof of Theorem 4. (2.2) and (2.8) lead to

$$2^{-\varrho_k} \limsup_{\sigma\to\infty} \frac{l_k M(\sigma)}{\varrho_k(\sigma)} = \limsup_{\sigma\to\infty} \frac{l_k B(\sigma)}{\varrho^{\varrho_k(\sigma)}(\sigma)} = \limsup_{\sigma\to\infty} \frac{l_k J(\sigma)}{\varrho^{\varrho_k(\sigma)}}.$$

On applying Lemma 2 to $J(\sigma)$ we obtain (5.1). Similarly on applying Lemma 3 to $J(\sigma)$ we get (5.2). This completes the proof of the theorem.

Remark. Taking $\varrho_k(r) = \varrho_k$ in Theorem 4 we get the result of Reddy [7], Theorem 3, which in turn includes the result of Bernstein [1] on finiteness.

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