The Denjoy integral
in some approximation problems, IV

1. Preliminaries. Denote by $D^*$ the space of all $2\pi$-periodic functions $f$ integrable in the Denjoy-Perron sense on the interval $<0, 2\pi>$. Consider the points $x$ for which

\[
(D^*) \int_{-h}^{h} (f(x+u) - f(x)) du = o(h) \quad (h \to 0 +),
\]

and write

\[
w(f, x, h)_D = \sup_{0 < t \leq h} \left\{ \frac{1}{2t} \left( D^* \int_{-t}^{t} (f(x+u) - f(x)) du \right) \right\}
\]

for any $f \in D^*$. Evidently, the function $w(f, x; h)_D$ is non-decreasing in $h$ and, by (1), $w(f, x; h)_D \to 0$ ($h \to 0 +$) almost everywhere.

Let $0 < t \leq \pi$ and let

\[
f_x(t) = f(x+t) + f(x-t) - 2f(x), \quad F_x(t) = (D^*) \int_{0}^{t} f_x(u) du.
\]

We have then

\[
|F_x(t)| = \left| (D^*) \int_{-t}^{t} (f(x+u) - f(x)) du \right| \leq 2tw(f, x; t)_D.
\]

In this paper we consider some linear operators defined by $D^*$-integral. We give estimates of the rate of pointwise and mean convergence of these operators. In order to show that our results cannot be essentially improved we introduce the class $D(\Omega)$ of all functions $f \in D^*$ such that

\[
\sup_{0 < h \leq \pi} \frac{w(f, x; h)_D}{\Omega(h)} \leq 1 \quad \text{at a fixed } x,
\]

where $\Omega$ is a non-negative and increasing real-valued function defined on $<0, \pi>$ with $\Omega(0) = 0$. 
For $2\pi$-periodic and Lebesgue-integrable functions, in symbols $f \in L$, the corresponding estimates will be given in terms of the expression

$$w(f, x; h)_L = \sup_{0 < t \leq h} \left\{ \frac{1}{2t} \int_{-t}^{t} |f(x + u) - f(x)| \, du \right\}$$

used in [1]. Clearly, $w(f, x; h)_L \to 0$ ($h \to 0^+$) at every point $x$ for which

$$\int_{-h}^{h} |f(x + u) - f(x)| \, du = o(h) \quad (h \to 0^+).$$

Moreover, $w(f, x; h)_D \leq w(f, x; h)_L$.

In the sequel $M_1, M_2, \ldots$ will denote the suitable positive constants.

2. Pointwise polynomial approximation. Given any $f \in D^*$, we consider trigonometric polynomials

$$U_n(x; f) = \frac{1}{\pi} (D^*) \int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} \cos(k(t-x)) \right\} \, dt,$$

where the factors $\lambda_k^{(n)} (k = 1, 2, \ldots, n; n = 1, 2, \ldots)$ are some real numbers. Write

$$\Delta \lambda_0^{(n)} = 1 \quad (n = 0, 1, 2, \ldots),$$

$$\Delta \lambda_k^{(n)} = \lambda_k^{(n)} - \lambda_{k+1}^{(n+1)} \quad (k = 0, 1, 2, \ldots, n-1), \quad \Delta \lambda_n^{(n)} = \lambda_n^{(n)},$$

$$\Delta^2 \lambda_k^{(n)} = \Delta \lambda_k^{(n)} - \Delta \lambda_{k+1}^{(n)} \quad (k = 0, 1, \ldots, n-1), \quad \Delta^2 \lambda_n^{(n)} = \lambda_n^{(n)},$$

$$\Delta^3 \lambda_k^{(n)} = \Delta^2 \lambda_k^{(n)} - \Delta^2 \lambda_{k+1}^{(n)} \quad (k = 0, 1, \ldots, n-1), \quad \Delta^3 \lambda_n^{(n)} = \lambda_n^{(n)}.$$

We start with the following

**THEOREM 1.** Let $f \in D^*$. Suppose that for all $n$

$$(4) \quad (n + 1) \sum_{k=0}^{n} |\Delta^2 \lambda_k^{(n)}| \leq M_1$$

and

$$(5) \quad (n + 1) \sum_{k=1}^{n} (k + 1) |\Delta^2 \lambda_k^{(n)}| \leq M_2.$$

Then

$$(6) \quad |U_n(x; f) - f(x)| \leq \frac{M_3}{n + 1} \sum_{k=0}^{n} w \left( f, x; \frac{\pi}{k + 1} \right) \quad (n = 0, 1, 2, \ldots).$$

**Proof.** Retain the notation (2) and set

$$\varphi_k(t) = \frac{d}{dt} \left\{ \left( \sin \frac{1}{2} (k + 1) t / \sin \frac{1}{2} t \right)^2 \right\}.$$
Applying twice the Abel transformation, we get

\[
U_n(\omega; f) - f(\omega) = \frac{1}{\pi} \int_0^{\pi} f_x(t) \left\{ \frac{1}{2} \sum_{k=1}^{n} \frac{\lambda_k^{(n)}(t)}{\sin \frac{1}{2} \omega t} \right\} dt
\]

\[
= \frac{1}{2\pi} \left( D^* \right) \int_0^{\pi} f_x(t) \sum_{k=0}^{n} A^2 \lambda_k^{(n)} \left( \frac{\sin \frac{1}{2}(k+1)t}{\sin \frac{1}{2}t} \right)^2 dt.
\]

By partial integration ([3], p. 244),

\[
U_n(\omega; f) - f(\omega) = \frac{1}{2\pi} F_x(\pi) \sum_{k=0}^{n} A^2 \lambda_k^{(n)} \left( \sin \frac{1}{2}(k+1)\pi \right)^2 -
\]

\[
- \frac{1}{2\pi} \int_0^{\pi} F_x(t) \sum_{k=1}^{n} A^2 \lambda_k^{(n)} \varphi_k(t) dt.
\]

It is easy to see that

\[
|\varphi_k(t)| \leq 2\pi(k+1)^2 t^{-1} \quad \text{if } t \in (0, \pi/(n+1)),
\]

and

\[
\left| \sum_{k=1}^{n} A^2 \lambda_k^{(n)} \varphi_k(t) \right| \leq \frac{\pi^3}{t^3} \sum_{k=1}^{n} |A^2 \lambda_k^{(n)}| + \frac{\pi^2}{2t^2} \left| \sum_{k=1}^{n} (k+1) A^2 \lambda_k^{(n)} \sin(k+1)t \right|
\]

if \( t \in (\pi/(n+1), \pi) \). These inequalities and (3) lead to

\[
|U_n(\omega; f) - f(\omega)|
\]

\[
\leq \frac{1}{2\pi} |F_x(\pi)| \sum_{k=0}^{n} |A^2 \lambda_k^{(n)}| + \frac{1}{\pi} \int_0^{\pi} |F_x(t)| \left| \sum_{k=1}^{n} A^2 \lambda_k^{(n)} \varphi_k(t) \right| dt
\]

\[
\leq w(f, \omega; \pi_D) \sum_{k=0}^{n} |A^2 \lambda_k^{(n)}| + \frac{1}{\pi} \int_0^{\pi} t w(f, \omega; t_D) \left| \sum_{k=1}^{n} A^2 \lambda_k^{(n)} \varphi_k(t) \right| dt
\]

\[
\leq w(f, \omega; \pi_D) \sum_{k=0}^{n} |A^2 \lambda_k^{(n)}| + 2 \sum_{k=1}^{n} (k+1)^2 |A^2 \lambda_k^{(n)}| \int_0^{\pi/(n+1)} w(f, \omega; t_D) dt +
\]

\[
+ \frac{\pi^3}{t^3} \sum_{k=1}^{n} |A^2 \lambda_k^{(n)}| \int_{\pi/(n+1)}^{\pi} t^{-2} w(f, \omega; t_D) dt +
\]

\[
+ \frac{\pi}{\pi/(n+1)} \sum_{k=1}^{n} (k+1) A^2 \lambda_k^{(n)} \sin(k+1)t I dt
\]

\[
= I_n + Y_n + Z_n + W_n.
\]
In view of (4),

\[ I_n \leq \frac{M_1}{n+1} w(f, \omega; \pi)_{D} \leq \frac{M_1}{n+1} \sum_{k=0}^{n} w \left( f, \omega; \frac{\pi}{k+1} \right)_{D}, \]

\[ Y_n \leq \frac{2\pi}{n+1} \sum_{k=1}^{n} (k+1)^2 |A^2 \lambda_k^{(n)}| w \left( f, \omega; \frac{\pi}{n+1} \right)_{D} \]

\[ \leq 2\pi M_1 w \left( f, \omega; \frac{\pi}{n+1} \right)_{D} \leq \frac{2\pi M_1}{n+1} \sum_{k=0}^{n} w \left( f, \omega; \frac{\pi}{k+1} \right)_{D}, \]

\[ Z_n = \pi \sum_{k=1}^{n} |A^2 \lambda_k^{(n)}| \int_{1}^{n+1} w \left( f, \omega; \frac{\pi}{t} \right)_{D} dt \leq \frac{\pi M_1}{n+1} \int_{1}^{n+1} w \left( f, \omega; \frac{\pi}{t} \right)_{D} dt \]

\[ \leq \frac{\pi M_1}{n+1} \sum_{k=0}^{n} w \left( f, \omega; \frac{\pi}{k+1} \right)_{D}. \]

Applying the Abel transformation to the sum in \( W_n \) we obtain

\[ \left| \sum_{k=1}^{n} (k+1) A^2 \lambda_k^{(n)} \sin(k+1)t \right| = \left| (n+1) A^2 \lambda_k^{(n)} \sum_{i=1}^{n} \sin(i+1)t + \sum_{k=1}^{n-1} \left\{ (k+1) A^2 \lambda_k^{(n)} - A^2 \lambda_k^{(n+1)} \right\} \sum_{i=1}^{n} \sin(i+1)t \right| \]

\[ \leq \frac{3\pi}{t} \left\{ \sum_{k=1}^{n} (k+1) |A^3 \lambda_k^{(n)}| + \sum_{k=1}^{n-1} \left| A^2 \lambda_k^{(n+1)} \right| \right\} \]

for \( t \in \langle \pi/(n+1), \pi \rangle \). Therefore, by (4) and (5),

\[ W_n \leq \frac{3\pi^2 (M_1 + M_2)}{2(n+1)} \int_{\pi/(n+1)}^{\pi} t^{-2} w(f, \omega; t)_{D} dt \]

\[ = \frac{3\pi(M_1 + M_2)}{2(n+1)} \int_{1}^{n+1} w \left( f, \omega; \frac{\pi}{t} \right)_{D} dt \leq \frac{3\pi(M_1 + M_2)}{2(n+1)} \sum_{k=0}^{n} w \left( f, \omega; \frac{\pi}{k+1} \right)_{D}. \]

Collecting the results we get the desired estimate (6) with \( M_3 = \frac{1}{3} \{2 + 9\pi\} M_1 + 3\pi M_2 \).

Consider now the class \( D(\Omega) \). It follows immediately from (6) that, under the assumptions of Theorem 1,

\[ \sup_{f \in D(\Omega)} |U_n(\omega; f) - f(\omega)| \leq \frac{M_3}{n+1} \sum_{k=0}^{n} \Omega \left( \frac{\pi}{k+1} \right). \]
The last inequality, for some operators $U_n(x; f)$, cannot be improved. This is a consequence of our next result.

**Theorem 2.** Suppose that $\Delta^2 \lambda_k^{(n)} \geq 0$ ($k = 0, 1, \ldots, n$) and that

$$
\frac{1}{n+1} \sum_{k=0}^{n} (k+1)^2 \Delta^2 \lambda_k^{(n)} \geq M_k > 0 \quad (n = 0, 1, 2, \ldots).
$$

Then for all $n \geq 2$ we have

$$
\sup_{f \in D(\Omega)} |U_n(x; f) - f(x)| \geq \frac{M_5}{n+1} \sum_{k=2}^{n} \Omega \left( \frac{\pi}{k+1} \right) (M_5 < M_3).
$$

**Proof.** Let $g(t) = \Omega(|t-x|)$ if $|t-x| \leq \pi$ and $g(t+2\pi) = g(t)$. Then $g \in D^*$. Moreover, for $0 < t \leq h \leq \pi$ we have

$$
(D^*) \int_{-t}^{t} (g(x+u) - g(x)) du = \int_{-t}^{t} \Omega(|u|) du \leq 2t \Omega(h).
$$

Consequently, $w(g, x; h)_{D} \leq \Omega(h)$ and $g \in D(\Omega)$. Next,

$$
\sup_{f \in D(\Omega)} |U_n(x; f) - f(x)| \geq |U_n(x; g) - g(x)| = |U_n(x; g)|
$$

$$
= \frac{1}{\pi} \int_{0}^{\pi} \Omega(t) \sum_{k=0}^{n} \Delta^2 \lambda_k^{(n)} \left( \frac{\sin \frac{1}{2}(k+1)t}{\sin \frac{1}{2}t} \right)^2 dt
$$

$$
\geq \frac{4}{\pi} \sum_{k=0}^{n} \Delta^2 \lambda_k^{(n)} \int_{0}^{\pi} \Omega(t) t^{-2} \left( \sin \frac{1}{2}(k+1)t \right)^2 dt
$$

$$
= \frac{4}{\pi} \sum_{k=0}^{n} \Delta^2 \lambda_k^{(n)} \left\{ \sum_{t=1}^{n+1} \int_{t \pi}^{(t+1)\pi} \Omega(t) t^{-2} \left( \sin \frac{1}{2}(k+1)t \right)^2 dt \right\}
$$

$$
\geq 4\pi^{-3} (n+1)^2 \sum_{k=0}^{n} \Delta^2 \lambda_k^{(n)} \left\{ \sum_{t=1}^{n+1} \Omega \left( \frac{i\pi}{n+1} \right) (i+1)^{-2} \lambda_{i+k}^{(n)} \right\},
$$

where

$$
\lambda_{i+k}^{(n)} = \int_{i\pi}^{(i+1)\pi} \left( \sin \frac{1}{2}(k+1)t \right)^2 dt = \frac{2}{k+1} \int_{i(k+1)\pi}^{(i+1)(k+1)\pi} \sin^2 t dt
$$

$$
\geq \frac{4}{k+1} \int_{0}^{\pi} \sin^2 t dt = \frac{\pi(k+1)^2}{12(n+1)^2}.
$$
Since for all \( n \geq 2 \)

\[
\Omega\left(\frac{i\pi}{n+1}\right)(i + 1)^{-2} \geq \frac{\pi}{5(n+1)} \int_{\frac{-(n+1)i}{n+1}}^{\frac{i\pi}{n+1}} \frac{1}{t^2} \Omega(t) \, dt
\]

(see [1], p. 28), we find that

\[
\sup_{f \in D^*} |U_n(x; f) - f(x)| \geq \frac{1}{15\pi(n + 1)^2} \sum_{k=0}^{n} (k + 1)^2 \lambda_k^{(n)} \int_{\frac{\pi}{n+1}}^{\pi/2} \frac{1}{t^2} \Omega(t) \, dt
\]

\[
= \frac{1}{15\pi(n + 1)^2} \sum_{k=0}^{n} (k + 1)^2 \lambda_k^{(n)} \int_{\frac{\pi}{n+1}}^{\pi/2} \Omega\left(\frac{\pi}{t}\right) \, dt
\]

\[
\geq \frac{M_A}{15\pi^2(n + 1)} \sum_{t=0}^{n} \Omega\left(\frac{\pi}{i+1}\right),
\]

and this completes the proof.

Remark. If a function \( f \) is of class \( L \), then

\[
|U_n(x; f) - f(x)| \leq \frac{3M_I}{n+1} \sum_{k=0}^{n} w\left(f, x; \frac{\pi}{k+1}\right)_L (n = 0, 1, \ldots),
\]

provided that assumption (4) holds. In this case Theorem 2 remains valid for the class \( L(\Omega) \) consisting of all functions \( f \in L \) such that

\[
\sup_{0 < h < \pi} \frac{w(f, x; h)_L}{\Omega(h)} \leq 1.
\]

3. Approximation by Abel means. As well known, the Abel means of Fourier series of \( f \in D^* \) can be written in the integral form

\[
P_r(x; f) = \frac{1}{2\pi} (D^*) \int_{-\pi}^{\pi} f(t) \frac{1 - r^2}{1 - 2r \cos(t - x) + r^2} \, dt \quad (0 < r < 1).
\]

We first give an estimate for the difference \( P_r(x; f) - f(x) \) at the Lebesgue–Denjoy points \( x \) defined by (1).

THEOREM 3. If \( f \in D^* \) and \( n = [1/(1 - r)] - 1 \), then

\[
|P_r(x; f) - f(x)| \leq M_6(1 - r) \sum_{k=0}^{n} w\left(f, x; \frac{\pi}{k+1}\right)_D (0 < r < 1).
\]

The inequality cannot be improved.
Proof. Using the symbols (2) and integrating by parts we obtain

\[ P_r(x; f) - f(x) = \frac{1 - r^2}{2\pi} (D^x f)(\pi) \int_0^\pi f_x(t) \frac{1}{1 - 2r \cos t + r^2} \, dt \]

\[ = \frac{1 - r}{2\pi(1 + r)} F_x(\pi) + \frac{r(1 - r^2)}{\pi} \left( \int_0^{\pi/(n+1)} F_x(t) \frac{\sin t}{((1 - r^2)^2 + 4r \sin^2 \frac{t}{2})^2} \, dt \right) \]

\[ = \frac{1 - r}{2\pi(1 + r)} F_x(\pi) + A(r) + B(r). \]

In view of (3),

\[ \left| \frac{1 - r}{2\pi(1 + r)} F_x(\pi) \right| \leq (1 - r) w(f, x; \pi)_D \leq (1 - r) \sum_{k=0}^n w\left( f, x; \frac{\pi}{k+1} \right)_D, \]

and

\[ |A(r)| \leq \frac{r(1 + r)}{\pi(1 - r)^3} \int_0^{\pi/(n+1)} |F_x(t)| \, t \, dt \leq \frac{2r(1 + r)}{\pi(1 - r)^3} \int_0^{\pi/(n+1)} t^2 w(f, x; t)_D \, dt \]

\[ \leq \frac{4\pi^2}{3(1 - r)^3(n + 1)^3} \sum_{k=0}^n w\left( f, x; \frac{\pi}{k+1} \right)_D \]

\[ \leq \frac{64\pi^2}{3} (1 - r) \sum_{k=0}^n w\left( f, x; \frac{\pi}{k+1} \right)_D \]

whenever \( 0 < r < 1 \). Moreover, \( B(r) = 0 \) if \( 0 < r < \frac{1}{2} \) and

\[ |B(r)| \leq \frac{2r(1 - r^2)}{\pi} \int_0^{\pi/(n+1)} w(f, x; t)_D \frac{t^2}{((1 - r^2)^2 + 4r \sin^2 \frac{t}{2})^2} \, dt \]

\[ \leq \frac{(1 - r^2)\pi^3}{8r} \int_0^{\pi/(n+1)} \frac{1}{\pi(t)} \frac{1}{(1 - r^2)^2 + 4r \sin^2 \frac{t}{2}} \, dt \]

\[ \leq \frac{(1 - r^2)\pi^2}{8r} \int_1^{\pi/(n+1)} w\left( f, x; \frac{\pi}{t} \right)_D \, dt \leq \frac{1}{2} \pi^2 (1 - r) \sum_{k=0}^n w\left( f, x; \frac{\pi}{k+1} \right)_D, \]

provided that \( \frac{1}{2} < r < 1 \). Collecting the results we get the estimate (7) with \( M_6 = (6 + 131\pi^2)/6 \).

In order to show that inequality (7) cannot be essentially improved we consider the class \( D(\Omega) \). By (7),

\[ \sup_{f \in D(\Omega)} |P_r(x; f) - f(x)| \leq M_6 (1 - r) \sum_{k=0}^n \Omega\left( \frac{\pi}{k+1} \right). \]
On the other hand, using the function \( g \) defined in Section 2, we get

\[
\sup_{f \in \mathcal{D}(2)} |P_r(x; f) - f(x)| \geq |P_r(x; g) - g(x)|
\]

\[
= \frac{1 - r^2}{\pi} \int_0^\pi \frac{1}{1-2r \cos t + r^2} \, dt \geq \frac{1 - r^2}{\pi} \int_0^\pi \frac{1}{(1-r)^2 + rt^2} \, dt.
\]

Since \((1-r) \leq 1/(n+1) \leq t\), we have

\[
\sup_{f \in \mathcal{D}(2)} |P_r(x; f) - f(x)| \geq \frac{1-r}{\pi} \int_0^\pi t^{-2} \, \Omega(t) \, dt = \frac{1}{\pi^2} (1-r) \int_1^{\pi/2} \Omega(t) \, dt
\]

\[
\geq \frac{1}{\pi^2} (1-r) \sum_{k=1}^n \Omega\left(\frac{\pi}{k+1}\right)
\]

and this completes the proof.

Remark. In the case of \( f \in L \) we can get the following estimate

\[
|P_r(x; f) - f(x)| \leq 12 (1-r) \sum_{k=0}^n w\left(f, x; \frac{\pi}{k+1}\right)_L
\]

when \( 0 < r < 1 \) and \( n = \lfloor 1/(1-r) \rfloor - 1 \).

Following Džvaršeĭvili [2], let us introduce in the space \( D^\ast \) the norm

\[
\|f\|_{D^\ast} = \sup_{\varphi \in V} \left| \int_0^{2\pi} \varphi(x) f(x) \, dx \right|,
\]

where \( V \) denotes the class of all \( 2\pi \)-periodic functions of bounded variation in \( <0, 2\pi> \) and such that

\[
\sup_{0 \leq x \leq 2\pi} |\varphi(x)| \leq 1 \quad \text{and} \quad \var\varphi(x) \leq 1.
\]

Denote by \( E_v(f)_{D} \) the constants of the best approximation of \( f \in D^\ast \) defined by the formula

\[
E_v(f)_{D} = \inf_{v \leq \varphi \leq D} \|f(\cdot) - t_v(\cdot)\|_D \quad (v = 0, 1, 2, \ldots),
\]

where the infimum is extended over all trigonometric polynomials \( t_v(x) \) of degree not exceeding \( v \).

Arguing as in [4], we easily obtain

**Theorem 4.** Let \( f \in D^\ast \) and let \( n = \lfloor 1/(1-r) \rfloor - 1 \). Then

\[
\|P_r(\cdot, f) - f(\cdot)\|_D \leq 94 (1-r) \sum_{v=0}^n E_v(f)_{D} \quad (0 < r < 1).
\]
References


