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On the extension of measure with values in a topological group

Let $\mathscr{R}$ be a ring of sets and X a topological abelian group.

A set function $\mu\colon \mathscr{R} \to X$ is called a *measure* if and only if for any two sets $A, B \in \mathscr{R}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$ and for every decreasing sequence (A_n) of sets in $\mathscr{R}$ with $\bigcap A_n = \emptyset$, $\lim_n \mu(A_n) = 0$.

The problem of extending μ to a measure on some σ-ring of sets containing $\mathscr{R}$ was considered by many authors under various additional assumptions on X and μ (see [2] and [4] for references).

Finally, the problem of extension was solved by M. Sion in [4]. He proved there that, if X is complete, T_0 topological abelian group and measure μ is exhaustive, i.e. for every sequence (A_n) of pairwise disjoint sets in $\mathscr{R}$, $\lim \mu(A_n) = 0$, then μ can be extended to a measure on some σ-ring of sets containing $\mathscr{R}$ (exhaustivity of μ is, of course, necessary for an extension). Sion's proof of this theorem depends on the generalization of the classical method of Carathéodory outer measure.

In the first paragraph of the paper it is shown that this theorem may be obtained with the help of some topological extension arguments (extension by "continuity" was used also by L. Drewnowski in [2] but his method is less direct than ours).

Second paragraph of the paper contains some remarks on the nature of our extension.

In the third paragraph we prove that our and Sion's method of extension gives the same results.

1. By $\mathscr{R}_\sigma$ ($\mathscr{R}_\delta$) we denote the class of all unions (intersections) of countable subclasses of sets in $\mathscr{R}$. By $\boldsymbol{H}(\mathscr{R})$ we denote the class of all subsets of sets in $\mathscr{R}_\sigma$ (if $\mathscr{R}$ is an algebra of subsets of some set Ω then $\boldsymbol{H}(\mathscr{R})$ is simply the class of all subsets of Ω). We shall consider $\boldsymbol{H}(\mathscr{R})$ as an abelian group with the operation of symmetric difference of sets as the addition. This operation will be written as "+" and this will denote the symmetric difference of sets only for sets in $\boldsymbol{H}(\mathscr{R})$; if $U^*, V^* \subset \boldsymbol{H}(\mathscr{R})$, then $U^* + V^* = \{A + B \mid A \in U^*, B \in V^*\}$. Let $\mathscr{R}$ be a ring of sets and μ a measure defined on $\mathscr{R}$ with values in topological abelian group X. Let $\mathfrak{U}$ be a fixed basis of closed, symmetric neighbourhoods of zero of the group X.

1.1. DEFINITION. Let for every $U \in \mathfrak{U}$ the set $U^* \subset \boldsymbol{H}(\mathscr{R})$ will be defined as follows:

$T \in U^*$ if and only if there exists a set $E \in \mathscr{R}_\sigma$ such that $T \subset E$ and for every $A \in \mathscr{R}$ with $A \subset E$, we have $\mu(A) \in U$.

By $\mathfrak{U}^*$ we shall denote the set $\{U^* | U \in \mathfrak{U}\}$.

1.2. THEOREM. *$\mathfrak{U}^*$ is a base of neighbourhoods of $\emptyset$ for some group topology in $\boldsymbol{H}(\mathscr{R})$.*

Proof. It follows immediately from Definition 1.1 that the set $\mathfrak{U}^*$ has the following properties:

(i) for every $U^* \in \mathfrak{U}^*$, $\emptyset \in U^*$,

(ii) for every U^*, $V^* \in \mathfrak{U}^*$ there is $W^* \in \mathfrak{U}^*$ such that $W^* \subset U^* \cap V^*$.

Thus it suffices to prove ([1]; Chapter 3, § 1, n.2) that $\mathfrak{U}^*$ possesses the property:

(iii) for every $U^* \in \mathfrak{U}^*$ there exists such a $V^* \in \mathfrak{U}^*$ that $V^* + V^* \subset U^*$.

We shall prove this by showing that if $V + V + V \subset U$, then $V^* + V^* \subset U^*$.

Let $T \in V^*$ and $Z \in V^*$, what means that there exist sets $E \in \mathscr{R}_\sigma$ and $Q \in \mathscr{R}_\sigma$ such that $T \subset E$, $Z \subset Q$ and $\mu(A) \in V$ for any set $A \in \mathscr{R}$ with $A \subset E$ or $A \subset Q$. Since $T + Z \subset E \cup Q$ and $E \cup Q \in \mathscr{R}_\sigma$ it suffices to show that for any set $A \in \mathscr{R}$ with $A \subset E \cup Q$, $\mu(A) \in U$.

Let $E = \bigcup A_n$ and $Q = \bigcup B_n$, where (A_n) and (B_n) are increasing sequences of sets in $\mathscr{R}$. Thus for any $A \in \mathscr{R}$ with $A \subset E \cup Q$ we have $\mu(A) = \lim_n \big(\mu\big(A \cap (A_n \cup B_n)\big)\big) = \lim_n \big(\mu(A \cap A_n) + \mu(A \cap B_n)\big) - \mu\big(A \cap (A_n \cap B_n)\big)$ $\in V + V + V \subset U$, what completes the proof.

The group topology in $\boldsymbol{H}(\mathscr{R})$ determined by $\mathfrak{U}^*$ obviously does not depend on the choice the basis $\mathfrak{U}$ of neighbourhoods of zero in X. This topology will be called *μ-topology* of $\boldsymbol{H}(\mathscr{R})$.

1.3. THEOREM. *Operation of union of sets is uniformly continuous in μ-topology of $\boldsymbol{H}(\mathscr{R})$.*

Proof. Analogically as in the proof of (iii) of Theorem 1.2, we can show that for any $U^* \in \mathfrak{U}^*$ and $V \in \mathfrak{U}$, if $V + V + V \subset U$, then for every T, $Z \in \boldsymbol{H}(\mathscr{R})$ with $T, Z \in V^*$, we have $T \cup Z \in U^*$. This means that the operation $(T, Z) \to T \cup Z$ is continuos at the zero of the group $\boldsymbol{H}(\mathscr{R})$.

Since for every $U^* \in \mathfrak{U}^*$ and $A \in \mathfrak{U}^*$ if $B \subset A$, then $B \in U^*$, we obtain our thesis from the formula $(T \cup Z) + (T \cup Z') \subset (T + T'') \cup (Z + Z')$, holding for any sets T, T', Z, $Z' \in \boldsymbol{H}(\mathscr{R})$.

1.4. COROLLARY. *Operations of difference and intersection of sets are uniformly continuous in μ-topology of $\boldsymbol{H}(\mathscr{R})$.*

Proof. $T \setminus Z = (T \cup Z) + Z$ and $T \cap Z = (T \cup Z) + (T + Z)$ for any T, $Z \in H(\mathscr{R})$.

1.5. COROLLARY. *Closure of $\mathscr{R}$ in μ-topology, $\overline{\mathscr{R}}$, is a ring of sets.*

Proof. The family $\mathscr{R}$ is closed with respect to the operations of union and difference of sets and the thesis follows from the fact that this operations are uniformly continuous in μ-topology.

Now we are going to prove the main result on μ-topology of $H(\mathscr{R})$ which will allow us to prove our extension theorem.

1.6. THEOREM. *If the measure μ is exhausitive, then for any increasing sequence (M_n) of sets in $\overline{\mathscr{R}}$, $\mu\text{-}\lim\limits_n M_n = \bigcup M_n$ (μ-lim denotes the limit in μ-topology).*

Proof of the theorem will be produced in four steps:

1° If (A_n) is an increasing sequence of sets in $\mathscr{R}$, then

$$\mu\text{-}\lim_n A_n = \bigcup A_n.$$

We must show that for an arbitrary neighbourhood $U^* \in \mathfrak{U}^*$, there is an index n_0 such that for all indices $k \geqslant n_0$, $(\bigcup A_n) + A_k \in U^*$. What means that there exists a set $E \in \mathscr{R}_\sigma$ such that $(\bigcup A_n) + A_k \subset E$ for all $k \geqslant n_0$ and $\mu(A) \in U$ for any $A \in \mathscr{R}$ and $A \subset E$.

First we shall prove that there is an index n_0 such that for all $n = 1, 2, \ldots$ if $A \in \mathscr{R}$ and $A \subset A_n \setminus A_{n_0}$, then $\mu(A) \in U$. Suppose the contrary. Then for any index n_1 there is an index n_2 and the set $B_1 \in \mathscr{R}$ such that $B_1 \subset A_{n_2} \setminus A_{n_1}$ and $\mu(B_1) \notin U$. Iterating this procedure we obtain a sequence (B_n) of pairwise disjoint sets in $\mathscr{R}$ with $\mu(B_n) \notin U$ for $n = 1, 2, \ldots$ what contradicts exhaustivity of μ.

Taking $E = (\bigcup A_n) \setminus A_{n_0}$ we obtain a set in $\mathscr{R}_\sigma$ such that for all $k \geqslant n_0$, $(\bigcup A_n) + A_k \subset (\bigcup A_n) \setminus A_{n_0} \subset E$ and for any set $A \in \mathscr{R}$ with $A \subset E$, we have $\mu(A) = \lim\limits_n \big(\mu\big(A \cap (A_n \setminus A_{n_0})\big)\big) \in U$.

2° If (G_n) is an increasing sequence of sets in $\mathscr{R}_\sigma$, then $\mu\text{-}\lim\limits_n G_n = \bigcup G_n$.

Let $G_n = \bigcup\limits_k A_k^n$, where (A_k^n) are for $n = 1, 2, \ldots$ increasing sequences of sets in $\mathscr{R}$. We may assume that $A_k^n \subset A_k^{n+1}$ for $n, k = 1, 2, \ldots$ Let U^* be an arbitrary neighbourhood in $\mathfrak{U}^*$. In view of 1° there is an increasing sequence of indices (k_n) such that $G_n \setminus A_{k_n}^n \in U^*$ for all $n = 1, 2, \ldots$ Since $\bigcup G_n = \bigcup A_{k_n}^n$ we have by 1° that $\mu\text{-}\lim\limits_n A_{k_n}^n = \bigcup G_n$. Hence by arbitrariness of the neighbourhood U^*, $\mu\text{-}\lim\limits_n G_n = \bigcup G_n$.

3° If $M \in \overline{\mathscr{R}}$, then for every $U^* \in \mathfrak{U}^*$ there exists a set $G \in \mathscr{R}_\sigma$ such that $M \subset G$ and $G \setminus M \in U^*$.

Since $\overline{\mathscr{R}}$ is the closure of $\mathscr{R}$, there exists a set $A \in \mathscr{R}$ such that $M + A \in U^*$. Thus by the Definition 1.1 there is a set $E \in \mathscr{R}_\sigma$ such that $M + A \subset E$ and $E \in U^*$. Taking $G = A \cup E$ we obtain a set in $\mathscr{R}_\sigma$ such that $M \subset G$ and $G \setminus M \subset E$, what implies $G \setminus M \in U^*$.

4° If (M_n) is an increasing sequence of sets in $\overline{\mathscr{R}}$, then μ-$\lim\limits_n M_n = \bigcup M_n$.

Let U^* be an arbitrary neighbourhood in $\mathfrak{U}^*$. Let us choose a neighbourhood $V^* \in \mathfrak{U}^*$ such that $V^*+V^* \subset U$. In view of the continuity of the operation of union of sets there exists a sequence (V_n^*) of neighbourhoods in $\mathfrak{U}^*$ having the following properties: T, $Z \in V_1^*$ implies $T \cup Z \in V^*$ and $T, Z \in V_{n+1}^*$ implies $T \cup Z \in V_n^*$ for $n = 1, 2, \ldots$

By 3° there exists a sequence (G_n) of sets in $\mathscr{R}_\sigma$ such that $M_n \subset G_n$ and $G_n \setminus M_n \in V_n^*$ for $n = 1, 2, \ldots$ Let for $n = 1, 2, \ldots$ $G_n' = \bigcup\limits_{i=1}^{n} G_i$. For all $n = 1, 2, \ldots$ we have that $(G_n' \setminus M_n) = \bigcup\limits_{i=1}^{n} (G_i \setminus M_n) \subset \bigcup\limits_{i=1}^{n} (G_i \setminus M_i) \in V^*$ By 2° μ-$\lim\limits_n G_n' = \bigcup G_n$. Thus for sufficiently large n, $(\bigcup G_n) + G_n' \in V^*$ and hence

$$(\bigcup G_n)\setminus M_n = (\bigcup G_n) + G_n' + (G_n' \setminus M_n) \in V^* + V^* \subset U^*.$$

But $(\cup M_n) + M_n \subset (\cup G_n) \setminus M_n$ for all $n = 1, 2, \ldots$ and so $(\cup M_n) + M_n \in U^*$ for sufficiently large n, what in view of arbitrariness of $U^* \in \mathfrak{U}^*$, finishes the proof.

Now we are able to prove our extension theorem.

1.7. Theorem. *Let $\mathscr{R}$ be a ring of sets and μ an exhaustive measure defined on $\mathscr{R}$ with values in complete, T_0 topological abelian group. Then μ can be extended to a measure $\bar{\mu}$ on the σ-ring of sets $\overline{\mathscr{R}}$.*

Proof. The measure μ is uniformly continous function on $\mathscr{R}$ in μ-topology of $\boldsymbol{H}(\mathscr{R})$. Indeed, for any sets A, $B \in \mathscr{R}$ we have the formula: $\mu(A) - \mu(B) = \mu(A \setminus B) - \mu(B \setminus A)$. Thus if for an arbitrary neighbourhood $U \in \mathfrak{U}$ we choose a neighbourhood $V \in \mathfrak{U}$ such that $V+V \subset U$, then for any two sets A, $B \in \mathscr{R}$ with $A+B \in V^*$, $\mu(A) - \mu(B) \in U$.

Since the group X is complete and T_0, the function μ can be extended to a continous function $\bar{\mu}$ on $\overline{\mathscr{R}}$ ([1], Chapter 2, § 3, n.2). By Corollary 1.5 and Theorem 1.6, $\overline{\mathscr{R}}$ is a σ-ring of sets.

For any two sets $M, N \in \overline{\mathscr{R}}$ with $M \cap N = \emptyset$, we obtain the formula $\bar{\mu}(M \cup N) = \bar{\mu}(M) + \bar{\mu}(N)$ from the identity $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ by passing to the μ-limit successively with $A \to M$ and $B \to N$ and exploiting continuity of $\bar{\mu}$ and continuity of operations of union and intersection of sets.

The fact that $\bar{\mu}$ is actually a measure on $\overline{\mathscr{R}}$ follows from Theorem 1.6 in view of continuity of μ in μ-topology of $\boldsymbol{H}(\mathscr{R})$.

Closing this section we shall prove some approximation property of the measure $\bar{\mu}$ constructed in Theorem 1.6, which will be used the sequel.

1.8. Lemma. *Let $M \in \overline{\mathscr{R}}$ For an arbitrary neighbourhood $U \in \mathfrak{U}$ there exist sets $F \in \mathscr{R}_\delta$ and $G \in \mathscr{R}_\sigma$ such that $F \subset M \subset G$ and $\bar{\mu}(N) \in U$ for all sets $N \in \overline{\mathscr{R}}$, $N \subset G \setminus F$.*

Proof. Let $V \in \mathfrak{U}$ be such a neighbourhood that $V^* + V^* \subset U^*$. Since M belongs to the closure of $\mathscr{R}$ in μ-topology, there are the sets $A \in \mathscr{R}$ and $E \in \mathscr{R}_\sigma$ such that $M + A \subset E$ and $E \in V^*$. Taking $F = A \setminus E$ and $G = A \cup E$ we obtain sets $F \in \mathscr{R}_\delta$ and $G \in \mathscr{R}_\sigma$ such that $F \subset M \subset G$ and $G \setminus F \subset E$. It remains to prove that for any set $N \in \overline{\mathscr{R}}$ if $N \subset E$, then $\overline{\mu}(N) \in U$. But $\overline{\mu}(N)$ is the μ-limit of $\mu(A)$, where $A \in \mathscr{R}$ and A tends to N in μ-topology. So we can confine ourselves to the sets $A \in \mathscr{R}$ and $A \in N + V^* \subset E + + V^* \subset V^* + V^* \subset U^*$.

2. In this section we give some remarks on the nature of the extension obtained in Theorem 1.7, connected with the notion of measure-theoretic completeness.

2.1. Definition. A set A is called *μ-null* iff $A \in \mathscr{R}$ and every set $B \in \mathscr{R}$, $B \subset A$ satisfies $\mu(B) = 0$. A family of all μ-null sets will be denoted by $\mathscr{N}(\mu)$.

A measure μ is called *complete* iff every subset of μ-null set belongs to $\mathscr{R}$ (and hence is μ-null).

Exactly as in the classical case (i.e. when X is a group of real numbers (see [3]; § 13, Theorem B)), one can prove:

2.2. Theorem. *If $\mathscr{R}$ is a σ-ring of sets, then the family $\tilde{\mathscr{R}}$ of all sets of the form $M = A + T$, where $A \in \mathscr{R}$ and T is a subset of μ-null set, is a σ-ring of sets. And the set function defined by $\tilde{\mu}(M) = \mu(A)$ is a complete measure on $\tilde{\mathscr{R}}$.*

2.3. Definition. Measure $\tilde{\mu}$ described in Theorem 2.2 will be called a *completion of measure* μ and σ-ring $\tilde{\mathscr{R}}$ a *completion of σ-ring $\mathscr{R}$*.

Let μ be an exhaustive measure defined on a ring of sets $\mathscr{R}$ with values in complete, T_0 topological abelian group and let $\overline{\mu}$ be an extension of μ obtained in Theorem 1.7. By $\mathscr{S}$ we denote the σ-ring of sets generated by $\mathscr{R}$ and by ν the restriction of $\overline{\mu}$ to $\mathscr{S}$.

2.4. Theorem. *$\mathscr{N}(\overline{\mu}) = \overline{\{\emptyset\}}$, where $\overline{\{\emptyset\}}$ is the closure of $\{\emptyset\}$ in μ-topology.*

Proof. Suppose $N \in \overline{\{\emptyset\}}$ and let U be an arbitrary neighbourhood in $\mathfrak{U}$. Let us choose $V \in \mathfrak{U}$ such that $V^* + V^* \subset U^*$. Let $M \in \overline{\mathscr{R}}$ and $M \subset N$. In the limit $\overline{\mu}(M) = \mu\text{-}\lim_{A \in \mathscr{R}, A \to M} \mu(A)$ we may confine ourselves to the sets $A \in \mathscr{R}$ and $A \in M + V^*$. But since $N \in \overline{\{\emptyset\}}$ we have that $N \in V^*$, and hence $M \in V^*$. Thus $A \in \mathscr{R}$ and $A \in V^* + V^* \subset U^*$ and consequently $\mu(A) \in U$ and in the limit $\overline{\mu}(M) \in U$. But that means, in view of arbitrariness of $U \in \mathfrak{U}$, that $\overline{\mu}(M) = 0$.

Suppose now $N \in \mathscr{N}(\overline{\mu})$ and let $U^* \in \mathfrak{U}^*$. By Lemma 1.8 there are sets $F \in \mathscr{R}_\delta$ and $G \in \mathscr{R}_\sigma$ such that $F \subset N \subset G$ and for every $M \in \overline{\mathscr{R}}$ with

$M \subset G \setminus F$, $\bar{\mu}(M) \in U$. For any set $A \in \mathscr{R}$ and $A \subset G$ we have $\mu(A) = \bar{\mu}(A \cap \cap (G \setminus F)) + \bar{\mu}(A \cap F)$. But since $A \cap F \subset N \in \mathscr{N}(\bar{\mu})$ we obtain that $\mu(A) = \bar{\mu}(A \cap (G \setminus F)) \in U$. But this means, in view of arbitrariness of $A \in \mathscr{R}$, $A \subset G$, that $G \in U^*$ and consequently $N \in U^*$. In view of arbitrariness of $U^* \in \mathfrak{U}^*$ we obtain that $N \in \overline{\{\emptyset\}}$.

2.5. Corollary. *Measure $\bar{\mu}$ is complete.*

Proof. Let $T \subset N \in \mathscr{N}(\bar{\mu})$. By Theorem 2.4, $N \in U^*$ for any $U^* \in \mathfrak{U}$. Hence $T \in U^*$ for any $U^* \in \mathfrak{U}^*$ what means that $T \in \overline{\{\emptyset\}} \subset \bar{\mathscr{R}}$.

2.6. Corollary. $\tilde{\mathscr{S}} \subset \bar{\mathscr{R}}$ *(where $\tilde{\mathscr{S}}$ is the completion of σ-ring $\mathscr{S}$).*

Proof. In view of Corollary 2.6 it suffices to show that $\mathscr{N}(\nu) \subset \mathscr{N}(\bar{\mu})$. Suppose $N \in \mathscr{N}(\nu)$ and let $U \in \mathfrak{U}$. By Lemma 1.8 there are sets $F \in \mathscr{R}_\delta$ and $G \in \mathscr{R}_\sigma$ such that $F \subset N \subset G$ and for every set $M \in \mathscr{S}$ with $M \subset G \setminus F$, $\nu(M) \in U$. For every set $A \in \mathscr{R}$ and $A \subset G$ we have that $\bar{\mu}(A) = \nu(A \cap \cap (G \setminus F)) + \nu(A \cap F)$. And since $A \cap F \in \mathscr{S}$ and $A \cap F \subset N \in \mathscr{N}(\nu)$ we obtain that $\mu(A) = \nu(A \cap (G \setminus F)) \in U$. This means, by arbitrariness of a set $A \in \mathscr{R}$, $A \subset G$, that $G \in U^*$ and consequently $N \in U^*$. Now, by arbitrariness of the neighbourhood $U \in \mathfrak{U}$, we obtain that $N \in \overline{\{\emptyset\}} = \mathscr{N}(\bar{\mu})$.

2.7. Theorem. *Let zero of the group X be the intersection of a countable family of neighbourhoods of zero. Then $\bar{\mu} = \tilde{\nu}$, where $\tilde{\nu}$ is a completion of ν.*

Proof. In view of Corollary 2.6 it remains to show that $\bar{\mathscr{R}} \subset \tilde{\mathscr{S}}$ and $\bar{\mu}(M) = \tilde{\nu}(M)$ for all $M \in \bar{\mathscr{R}}$.

Let (U_n) be a sequence of neighbourhoods in $\mathfrak{U}$ such that $\bigcap U_n = \{0\}$. Let M be an arbitrary set in $\bar{\mathscr{R}}$. By Lemma 1.8 there are the sets $F_n \in \mathscr{R}_\delta$ and $G_n \in \mathscr{R}_\sigma$ $(n = 1, 2, \ldots)$ such that $F_n \subset M \subset G_n$ and for every set $N \in \bar{\mathscr{R}}$ with $N \subset G_n \setminus F_n$, $\bar{\mu}(N) \in U_n$. Take $F = \bigcup F_n$. Now $M = F + (M \setminus F)$, $F \in \mathscr{S}$ and $M \setminus F \subset \bigcap (G_n \setminus F_n)$. But $\bigcap (G_n \setminus F_n) \in \mathscr{S}$ and for every $N \in \bar{\mathscr{R}}$ with $N \subset \bigcap (G_n \setminus F_n)$, $\bar{\mu}(N) \in \bigcap U_n = \{0\}$. Hence the set $\bigcap (G_n \setminus F_n)$ is $\bar{\mu}$-null and consequently ν-null. Thus $M \in \tilde{\mathscr{S}}$ and $\bar{\mu}(M) = \bar{\mu}(F) = \nu(F) = \tilde{\nu}(M)$.

In general σ-ring $\bar{\mathscr{R}}$ is strictly greater than $\tilde{\mathscr{S}}$, what can be illustrated by the following example:

2.8. Example. Let the set Ω be uncountable and let $\mathscr{R}$ be the smallest algebra of subsets of Ω containing all finite sets. Let X be a group of all real functions defined on Ω with the topology of pointwise convergence. Let for any set $A \in \mathscr{R}$, $\mu(A)$ denotes the characteristic function of A. Then μ is an exhaustive measure with values in complete, T_0, topological abelian group. σ-Ring $\bar{\mathscr{R}}$ is simply the family of all subsets of Ω and $\bar{\mu}$ the characteristic function subsets of Ω. But since the set Ω is uncountable, $\mathscr{S} \neq \bar{\mathscr{R}}$ and consequently $\tilde{\mathscr{S}} \neq \bar{\mathscr{R}}$ $(\tilde{\mathscr{S}} = \mathscr{S}$ for $\mathscr{N}(\nu) = \{\emptyset\})$.

3. M. Sion's extension theorem may be stated as follows:

3.1. THEOREM. *Let $\mathscr{R}$ be a ring of sets and μ an exhaustive measure defined on the ring of sets $\mathscr{R}$ with values in complete, T_0 topological abelian group. Then:*

(1) *For every set $T \in \boldsymbol{H}(\mathscr{R})$ there exists a limit $\eta(T) = \lim_{A \in \mathscr{R}, A \subset T} \mu(A)$ in the sense of ordering of the set $\{A | A \in \mathscr{R},\ A \subset T\}$ by the relation $B \leqslant A$ iff $B \subset A$.*

(2) *For every set $T \in \boldsymbol{H}(\mathscr{R})$ there exists a limit $\xi(T) = \lim_{G \in \mathscr{R}_\sigma G \supset T} \eta(T)$ in the sense of ordering of the set $\{G | G \in \mathscr{R}_\sigma,\ G \supset T\}$ by the relation $Q \leqslant G$ iff $G \subset Q$.*

(3) *A class $\mathscr{R}_0$ of all sets in $\boldsymbol{H}(\mathscr{R})$ such that for every set $T \in \boldsymbol{H}(\mathscr{R})$, $\xi(T) = \xi(T \cap M) + \xi(T \setminus M)$ is a σ-ring of sets containing $\mathscr{R}$.*

(4) *The restriction μ_0 of ξ to $\mathscr{R}_0$ is a measure extending μ and η.*

The aim of this section is to prove the following

3.2. THEOREM. *$\mathscr{R}_0 = \overline{\mathscr{R}}$ and $\mu_0 = \bar{\mu}$, where $\mathscr{R}_0$ and μ_0 has the same meaning as in Theorem* 3.1 *and $\overline{\mathscr{R}}$ and $\bar{\mu}$ as in Theorem* 1.7.

Proof of the theorem will be produced in four steps.

1° Let $T \in \boldsymbol{H}(\mathscr{R})$ and $U \in \mathfrak{U}$. There exists a set $G \in \mathscr{R}_\sigma$ such that $T \subset G$ and for all $M \in \mathscr{R}_0$ with $M \subset G \setminus T$, $\mu_0(M) \in U$.

By the definition of ξ there exists a set $G \in \mathscr{R}_\sigma$ such that $T \subset G$ and for every $Q \in \mathscr{R}_\sigma$ with $T \subset Q \subset G$, $\xi(T) - \eta(Q) \in U$. If $M \in \mathscr{R}_0$ and $M \subset G \setminus T$ we may confine ourselves in the limit $\lim_{Q \in \mathscr{R}_\sigma, Q \supset T \cup M} \eta(Q)$ to the sets $Q \in \mathscr{R}_\sigma$ such that $T \cup M \subset Q \subset G$. Thus we obtain that $\xi(T \cup M) - \xi(T) \in U$. But from the definition of $\mathscr{R}_0$, $\xi(T \cup M) = \xi(T) + \xi(M)$ and hence $\mu_0(M) = \xi(M) \in U$.

2° $\mathscr{R}_0 \subset \overline{\mathscr{R}}$.

Let $M \in \mathscr{R}_0$. Since $\mathscr{R}_\sigma \subset \overline{\mathscr{R}}$, it suffices to show that for any $U \in \mathfrak{U}$ there are sets $G \in \mathscr{R}_\sigma$ and $E \in \mathscr{R}_\sigma$ such that $M \subset G$, $G \setminus M \subset E$ and for all sets $A \in \mathscr{R}$ with $A \subset E$, $\mu(A) \in U$.

Let $U \in \mathfrak{U}$ be such that $V + V \subset U$. By 1° there exist sets $G \in \mathscr{R}_\sigma$ and $E \in \mathscr{R}_\sigma$ such that $M \subset G$, $G \setminus M \subset E$ and for any set $N \in \mathscr{R}_0$ with $N \subset G \setminus M$ or $N \subset E \setminus (G \setminus M)$, $\mu_0(N) \in V$. Now for any set $A \in \mathscr{R}$ with $A \subset E$ we have, by the definition of $\mathscr{R}_0$, that

$$\mu(A) = \mu_0\big(A \cap (G \setminus M)\big) + \mu_0\big(A \setminus (G \setminus M)\big) \in V + V \subset U.$$

($(G \setminus M) \in \mathscr{R}_0$ since $\mathscr{R}_0$ is a σ-ring containing $\mathscr{R}$).

3° $\overline{\mathscr{R}} \subset \mathscr{R}_0$.

Let $M \in \overline{\mathscr{R}}$. By Lemma 1.8, for every $U \in \mathfrak{U}$ there exist sets $F_0 \in \mathscr{R}_\delta$ and $G_0 \in \mathscr{R}_\sigma$ such that $F_0 \subset M \subset G_0$ and every set $A \in \mathscr{R}$, $A \subset G_0 \setminus F_0$ satisfies $\mu(A) \in U$. Thus, by the definition of η, for any set $Q \in \mathscr{R}_\sigma$ and $Q \subset G_0 \setminus F_0$, $\eta(Q) \in U$.

Since η restricted to $\mathscr{R}_\sigma$ is the restriction of a measure, we have that for any sets $G, H \in \mathscr{R}_\sigma$ and $F \in \mathscr{R}_\delta$ such that $F \subset H$, the following identity holds:

$$\eta(G) = \eta\big((G\cap H)\cup(G\setminus F)\big) = \eta(G\cap H)+\eta(G\setminus F)-\eta\big(G\cap(H\setminus F)\big).$$

Thus, if $F_0 \subset F \subset M \subset H \subset G_0$, then

$$\eta(G)-\eta(G\cap H)-\eta(G\setminus F) \in U.$$

Let T be an arbitrary set in $\boldsymbol{H}(\mathscr{R})$. If we shall pass to the limit in the last formula (in the sense of suitable direction) simultaneously with G tending to T, $G \cap H$ tending to $T \cap M$ and $G \setminus F$ tending to $T \setminus M$ we obtain that $\xi(T)-\xi(T\cap M)-\xi(T\setminus M) \in U$, what in view of arbitrariness of a set $T \in \boldsymbol{H}(\mathscr{R})$ and $U \in \mathfrak{U}$ implies that $M \in \mathscr{R}_0$.

4° $\mu_0 = \overline{\mu}$.

It suffices to show that μ_0 is a continous function on $\overline{\mathscr{R}}$ in μ-topology. And it follows from the formula

$$\mu_0(M)-\mu_0(N) = \mu_0(M\setminus N)-\mu_0(N\setminus M) \qquad (\text{for all } M,\ N \in \mathscr{R}_0)$$

that μ_0 is uniformly continuous function on $\overline{\mathscr{R}} = \mathscr{R}_0$. Indeed, if $M, N \in \mathscr{R}_0$ and $M+N \in U^*$ for some $U^* \in \mathfrak{U}^*$, then, from the definition of μ-topology, there is a set $E \in \mathscr{R}_\sigma$ such that $M+N \subset E$ and for every $A \in \mathscr{R}$ with $A \subset E$, $\mu(A) \in U$. Thus for any $G \in \mathscr{R}_\sigma$ if $G \subset E$, then $\eta(G) \in U$. But in the limits $\lim\limits_{G\in\mathscr{R}_\sigma, G\supset M\setminus N} \eta(G)$ and $\lim\limits_{G\in\mathscr{R}_\sigma, G\supset N\setminus M} \eta(G)$ we may confine ourselves to the sets $G \in \mathscr{R}_\sigma$ with $G \subset E$ and thus obtain that

$$\mu_0(M\setminus N)-\mu_0(N\setminus M) \in U+U.$$

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