ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XIX (1976)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XIX (1976)

W. FISCHER* and U. SCHÖLER* (Bonn, Federal Republic of Germany)

A characterization of non-locally bounded Orlicz spaces by power series with finite domain of convergence

In this note it is shown that an Orlicz space $L_\Phi(X, \boldsymbol{A}, \mu)$, where Φ is unbounded and satisfies the so-called condition (Δ_2), is not locally bounded if and only if there exists a power series

$$\sum_{n=1}^{\infty} x_n t^n, \qquad x_n \in L_\Phi(X, \boldsymbol{A}, \mu),$$

which converges only for $t = 0$ and $t = 1$.

W. Żelazko [7] shows that in the complex F-space $S[0, 1]$ of Lebesgue measurable functions on the unit interval $[0, 1]$, with identification of two functions equal almost everywhere and with topology of convergence in measure, there exists for every finite subset D of the complex plane $\boldsymbol{C}$ a power series

$$\sum_{n=1}^{\infty} x_n t^n, \qquad x_n \in S[0, 1],\ t \in \boldsymbol{C}$$

which converges in $S[0, 1]$ exactly on $D \cup \{0\}$. (For a more special result see also [1].)

Żelazko asks for suitable characterizations of F-spaces, in which there are power series with domains of convergence different from discs. In this note we shall show that the non-locally bounded Orlicz spaces [4] $L_\Phi(X, \boldsymbol{A}, \mu)$, where Φ is unbounded and satisfies condition (Δ_2), are characterized by this property.

Let $\Phi(t)$ be a continuous, non-negative, non-decreasing function, defined for $t \geqslant 0$, vanishing only for $t = 0$. We shall assume, moreover, that Φ satisfies condition (Δ_2), i.e. there is a positive constant k such that $\Phi(2t) \leqslant k\Phi(t)$.

* The authors were supported by the Sonderforschungsbereich 72 of the Deutsche Forschungsgemeinschaft.

Let $(X, \boldsymbol{A}, \mu)$ be a totally σ-finite measure space (cf. [2]). By $L_\Phi(X, \boldsymbol{A}, \mu)$ we denote the real or complex linear space of all real resp. complex-valued, μ-measurable functions with $J_\Phi(x) := \int_X \Phi(|x|)\,d\mu < \infty$. As usual we identify functions which differ only on a set of measure zero.

$L_\Phi(X, \boldsymbol{A}, \mu)$ is called *Orlicz space*. It is an F-space with F-norm $\|x\|_\Phi := \inf\{\varepsilon > 0:\ J_\Phi(x/\varepsilon) \leqslant \varepsilon\}$. The sets $N_\Phi(\varepsilon) := \{x \in L_\Phi(X, \boldsymbol{A}, \mu):\ J_\Phi(x) \leqslant \varepsilon\}$ form a basis of neighbourhoods for the topology (cf. [3], [4]).

As $(X, \boldsymbol{A}, \mu)$ is totally σ-finite, X may be expressed as $X = X_0 \cup \bigcup_{i=1}^{\infty} a_i$, where $X_0 \in \boldsymbol{A}$ is the non-atomic part of X and $\{a_i\}_{i\in \boldsymbol{N}}$ the family of atoms (we do not consider the case when the number of atoms is finite). We set $A := \bigcup_{i=1}^{\infty} a_i$.

Our main result is the following

THEOREM. *An Orlicz space $L_\Phi(X, \boldsymbol{A}, \mu)$, where Φ is unbounded and satisfies condition (Δ_2), is not locally bounded if and only if there exists a power series*

$$\sum_{n=1}^{\infty} x_n t^n, \qquad x_n \in L_\Phi(X, \boldsymbol{A}, \mu), \qquad t \in \boldsymbol{R} \text{ resp. } \boldsymbol{C},$$

whose domain of convergence is the set $\{0, 1\}$.

Proof. Sufficiency of the condition follows from the existence of a p-homogeneous norm in a locally bounded F-space (cf. [1], [7]).

Now let $L_\Phi(X, \boldsymbol{A}, \mu)$ be not locally bounded. Then one of the subspaces

$$L_\Phi(A) := \{x \in L_\Phi(X, \boldsymbol{A}, \mu):\ \int_{X_0} \Phi(|x|)\,d\mu = 0\}$$

or

$$L_\Phi(X_0) := \{x \in L_\Phi(X, \boldsymbol{A}, \mu):\ \int_{A} \Phi(|x|)\,d\mu = 0\}$$

is not locally bounded. It suffices to construct a power series with the desired property in that subspace, which is not locally bounded. Therefore we may restrict ourselves without loss of generality to the cases where $(X, \boldsymbol{A}, \mu)$ is either a non-atomic or a purely atomic measure space.

Because we have $\lim_{n\to\infty} \|x_n\|_\Phi = 0$ if and only if $\lim_{n\to\infty} J_\Phi(x_n) = 0$ [3], it suffices to construct a power series $\sum_{n=1}^{\infty} x_n t^n$ with $\lim_{m\to\infty} \int_X \Phi\left(\left|\sum_{n=1}^{m} x_n\right|\right) d\mu = 0$ and

$$\lim_{m\to\infty} \int_X \Phi\left(\left|\sum_{n=1}^{m} x_n t^n\right|\right) d\mu = \infty \qquad \text{for } t \notin \{0, 1\}.$$

A. Let $(X, \boldsymbol{A}, \mu)$ be non-atomic. From [5] we get that for every $n \in \boldsymbol{N}$ there is a positive real number s_n with

$$\frac{\Phi\left(\frac{1}{n^n} s_n\right)}{\Phi(s_n)} \geqslant \frac{1}{2}.$$

Moreover, we can choose the sequence $\{s_n\}_{n\in \boldsymbol{N}}$ in such a way that

$$\sum_{n=1}^{\infty} \frac{1}{n\Phi(s_n)} \leqslant \mu(X).$$

Therefore there exist pairwise disjoint sets $E_n \in \boldsymbol{A}$ with $\mu(E_n) = \frac{1}{n\Phi(s_n)}$.

Now we define for all $n \in \boldsymbol{N}$

$$y_0 := 0,$$
$$y_n := s_n \chi_{E_n},$$
$$x_n := y_n - y_{n-1},$$

and consider the power series

$$\sum_{n=1}^{\infty} x_n t^n. \tag{1}$$

We have $J_\Phi(\sum_{n=1}^{m} x_n) = J_\Phi(y_n) = 1/m$, and so series (1) converges for $t = 1$

For every $t \notin \{0, 1\}$ there exists an $N \in \boldsymbol{N}$, such that for all $n \geqslant N$ $\frac{1}{n^n} \leqslant |(1-t)t^n|$. Then we get for all $m > N$

$$J_\Phi\left(\sum_{n=N}^{m} x_n t^n\right) \geqslant J_\Phi\left(\sum_{n=N}^{m-1} \frac{1}{n^n} y_n\right) \geqslant \sum_{n=N}^{m-1} \frac{\Phi\left(\frac{1}{n^n} s_n\right)}{n\Phi(s_n)} \geqslant \sum_{n=N}^{m-1} \frac{1}{2n},$$

which implies that series (1) diverges for $t \notin \{0, 1\}$.

B. Now let $(X, \boldsymbol{A}, \mu)$ be purely atomic. We first prove the following assertion for non-locally bounded spaces $L_\Phi(X, \boldsymbol{A}, \mu)$:

(L) *For every neighbourhood $N_\Phi(\varepsilon)$ there is a neighbourhood $N_\Phi(\delta)$, such that for every real $\lambda_0 > 0$ and any $n \in \boldsymbol{N}$ there exists a function $x := (\xi_i)_{i\in \boldsymbol{N}} \in N_\Phi(\varepsilon)$ with*

$$x \in S(n) := \{y = (\eta_i)_{i\in \boldsymbol{N}} \in L_\Phi(X, \boldsymbol{A}, \mu) : \eta_i = 0,\ 1 \leqslant i \leqslant n\}$$

and $\lambda_0 x \notin N_\Phi(\delta)$.

Proof. As $L_\Phi(X, \boldsymbol{A}, \mu)$ is not locally bounded, there is a $\delta_1 > 0$ and for every $\lambda > 0$ a function $x \in N_\Phi(\varepsilon)$ with $\lambda x \notin N_\Phi(\delta_1)$. Set $\delta := \frac{1}{2}\delta_1$ and take any $n \in \boldsymbol{N}$, $\lambda_0 > 0$. We can choose M so large that

$$\Phi(M)\mu(a_i) > \varepsilon \quad \text{for all } i \in \{1, \ldots, n\} \tag{2}$$

and $\lambda_1 \leqslant \lambda_0$ so small that

$$\Phi(M \cdot \lambda_1) \leqslant \frac{\delta}{n \cdot \mu(a_i)} \quad \text{for all } i \in \{1, \ldots, n\}. \tag{3}$$

Then for all $x = (\xi_i)_{i \in \boldsymbol{N}} \in N_\Phi(\varepsilon)$ we have $|\xi_i| \leqslant M$, $1 \leqslant i \leqslant n$.

By assumption that $L_\Phi(X, \boldsymbol{A}, \mu)$ is not locally bounded, there is an $x \in N_\Phi(\varepsilon)$ with $\lambda_1 x \notin N_\Phi(\delta_1)$. From (2) and (3) we get $\lambda_1 x \cdot \chi_{\bigcup_{i=1}^{n}\{a_i\}} \in N_\Phi(\delta)$ and therefore $\lambda_1 x \cdot \chi_{X \setminus \bigcup_{i=1}^{n}\{a_i\}} \notin N_\Phi(\delta)$. It is obvious that $x\chi_{X \setminus \bigcup_{i=1}^{n}\{a_i\}}$ satisfies the assertion.

Now we construct the power series. For every $\varepsilon = 1/n$, $n \in \boldsymbol{N}$, we choose $\delta(n) > 0$ so that (L) holds for $\delta_0 := 2\delta(n)$, and take $t(n) \in \boldsymbol{N}$ so large that $t(n) \cdot \delta(n) \geqslant 1$. Furthermore we set $r_0 := 0$, $r_m := \sum_{n=1}^{m} t(n)$ for all $m \in \boldsymbol{N}$, and $\lambda(i) := \frac{1}{r_m^{r_m}}$ for all $i \in \boldsymbol{N}$ with $r_{m-1} < i \leqslant r_m$.

By induction we now define for all $i \in \boldsymbol{N} \cup \{0\}$ functions x_i, $y_i \in L_\Phi(X, \boldsymbol{A}, \mu)$ and natural numbers n_i. Set $n_0 := 0$, $x_0, y_0 = 0$. We assume that x_i, y_i, n_i have been defined already for all $0 \leqslant i < i_0$. There is an $m \in \boldsymbol{N}$ with $r_{m-1} < i_0 \leqslant r_m$. From (L) and the definition of $\delta(m)$ we get that there is a function

$$x_{i_0} \in N_\Phi\left(\frac{1}{m}\right) \cap S(n_{i_0-1})$$

with the property $\lambda(i_0) x_{i_0} \notin N_\Phi(2\delta(m))$.

We choose n_{i_0} so large that for

$$y_{i_0} := x_{i_0} \chi_{\bigcup_{j=n_{i_0-1}+1}^{n_{i_0}} \{a_j\}} \tag{4}$$

we have $\lambda(i_0) y_{i_0} \notin N_\Phi(\delta(m))$. (4) implies that every two functions $y_i(t)$ and $y_j(t)$, $i \neq j$, have disjoint support. Obviously we get $\lim_{n \to \infty} J_\Phi(y_n) = 0$. For every $n \in \boldsymbol{N}$ and all i with $r_{n-1} + 1 \leqslant i \leqslant r_n$ we have

$$\lambda(i) y_i = \frac{1}{r_n^{r_n}} y_i \notin N_\Phi(\delta(n)).$$

This and the construction of the numbers $t(n)$ now implies

$$J_\Phi\Big(\sum_{i=r_{n-1}+1}^{r_n} \lambda(i) y_i\Big) = \sum_{i=r_{n-1}+1}^{r_n} J_\Phi\Big(\frac{1}{r_n^{r_n}} y_i\Big) \geqslant t(n)\cdot\delta(n) \geqslant 1.$$

We set $z_i := y_i - y_{i-1}$ for all $i \in \mathbf{N}$ and consider the power series $\sum_{i=1}^{\infty} z_i t^i$.

Clearly $J_\Phi\big(\sum_{i=1}^{n} z_i\big) = J_\Phi(y_n)$, and so $\lim_{n\to\infty} J_\Phi(y_n) = 0$ implies that the power series converges for $t = 1$.

For every $t \notin \{0, 1\}$ there exists an $N \in \mathbf{N}$ such that for all $i \geqslant N$ we have $\frac{1}{i^i} \leqslant |(1-t)t^i|$. Then we get for every $r_{n-1} \geqslant N$

$$J_\Phi\Big(\Big|\sum_{i=r_{n-1}+1}^{r_n+1} z_i t^i\Big|\Big) \geqslant J_\Phi\Big(\Big|\sum_{i=r_{n-1}+1}^{r_n} (1-t)t^i y_i\Big|\Big) \geqslant \sum_{i=r_{n-1}+1}^{r_n} J_\Phi\Big(\Big|\frac{1}{i^i} y_i\Big|\Big)$$
$$\geqslant \sum_{i=r_{n-1}+1}^{r_n} J\Big(\Big|\frac{1}{r_n^{r_n}} y_i\Big|\Big) \geqslant 1$$

which implies that the series $\sum_{i=1}^{\infty} z_i t^i$ diverges for $t \notin \{0, 1\}$.

Remark. If the function Φ is bounded, the assertion of the theorem still remains true under certain conditions, e.g. when

(a) $(X, \mathbf{A}, \mu)$ is purely atomic (i.e. $X = \bigcup_{i=1}^{\infty} a_i$) and $\inf_{n \in \mathbf{N}} \mu(a_i) = k > 0$,

or

(b) $(X, \mathbf{A}, \mu)$ is not purely atomic.

In case (a) $L_\Phi(X, \mathbf{A}, \mu)$ satisfies condition (L) for sufficiently small $\varepsilon > 0$, and so part B of the theorem's proof may be used without change. In the other case we may apply Żelazko's result [7] to the subspace $L_\Phi(X_0, \mathbf{A}, \mu)$, where X_0 is the non-atomic part of X.

However, for bounded Φ the theorem is not true in general. Let $(X, \mathbf{A}, \mu)$ be purely atomic with $\mu(X) < \infty$. Then the Orlicz space $L_\Phi(X, \mathbf{A}, \mu)$ is the space s of all sequences, which is locally convex and not locally bounded.

Looking at the construction of the power series one easily sees that we may apply Lemma 1 of [7] to get the same result for non-locally bounded Orlicz spaces, which Żelazko has proved for the space $S[0, 1]$.

COROLLARY 1. *A complex Orlicz space $L_\Phi(X, \mathbf{A}, \mu)$, where Φ is unbounded and satisfies condition (Δ_2), is not locally bounded if and only if for every finite subset $F \subset \mathbf{C}$ there exists a power series $\sum_{n=1}^{\infty} x_n t^n$, which converges if $t \in D \cup \{0\}$ and diverges if $t \notin D \cup \{0\}$.*

As in a locally pseudoconvex space convergence of a power series $\sum_{n=1}^{\infty} x_n t^n$ for $t_0 > 0$ implies convergence for $|t| < t_0$, we also get for an unbounded function Φ a result analogous to Theorems III.3.6 and III.3.7 in [6].

COROLLARY 2. *An Orlicz space $L_\Phi(X, A, \mu)$, where Φ is unbounded and satisfies condition (Δ_2), is locally bounded provided it is locally pseudoconvex.*

References

[1] L. Arnold, *Über die Konvergenz einer zufälligen Potenzreihe*, J. Reine Angew. Math. 222 (1966), p. 79–112.

[2] P. R. Halmos, *Measure theory*, Van Nostrand, London–New York 1950.

[3] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. 18 (1959), p. 49–65.

[4] W. Orlicz, *On spaces of Φ-integrable functions*, Proc. Intern. Symp. on Linear Spaces, Hebrew Univ. of Jerusalem, 1960, p. 357–365.

[5] S. Rolewicz, *Some remarks on the spaces N(L) and N(l)*, Studia Math. 18 (1959), p. 1–9.

[6] — *Metric linear spaces*, Monografie Matematyczne, Tom 56, Warszawa 1972.

[7] W. Żelazko, *A power series with a finite domain of convergence*, Comm. Math. 15 (1971), p. 115–117.