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ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
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M. Filar (Kraków)

On a certain boundary problem for the equation $(\Delta+a^2)(\Delta+b^2)u = 0$

1. In this paper we construct the Green function and the solution of the equation

(1) $$(\Delta+a^2)(\Delta+b^2)u(X) = 0$$

($X = (x_1, \ldots, x_n)$ and a, b are positive constants, $a \neq b$) for the domain $E_n^+ = \{X\colon x_i \in (-\infty, \infty)$ for $i = 1, 2, \ldots, n-1, x_n > 0\}$ under the boundary conditions

(2) $$u(x_1, \ldots, x_{n-1}, 0) = f_1(x_1, \ldots, x_{n-1}),$$
$$\Delta u(x_1, \ldots, x_{n-1}, 0) = f_2(x_1, \ldots, x_{n-1}).$$

2. We give now some formulae, which will be needed later. Let X and Y be two different points in n-dimensional Euclidean space E_n ($n \geqslant 2$). Let

$$r = XY = \Big[\sum_{i=1}^{n}(x_i-y_i)^2\Big]^{1/2}$$

be the distance of these points. We now consider the function

(3) $$U(X, Y) = U(r) = -a^{-2\nu}(br)^{-\nu}Y_\nu(br)+b^{-2\nu}(ar)^{-\nu}Y_\nu(ar),$$

where $\nu = (n-2)/2$ and $Y_\nu(z)$ is the ν-order Bessel function of the second kind. Since $U(r)$ is a linear combination of solutions $(ar)^{-\nu}Y_\nu(ar)$ and $(br)^{-\nu}Y_\nu(br)$ of the equations

$$(\Delta+a^2)u(X) = 0 \quad \text{and} \quad (\Delta+b^2)u(X) = 0,$$

respectively and thus $U(r)$ as a function of X (or Y) satisfies equation (1). If we use the formula ([2], p. 111)

$$\frac{d}{dz}[z^{-\nu}Y_\nu(z)] = -z^{-\nu}Y_{\nu+1}(z)$$

and the expansion of $Y_\nu(z)$ into series ([2], p. 110 and 113) we easily obtain

$$(4)\qquad U(r) = \begin{cases} O(r^{-2\nu}) & \text{for } \nu > 0, \\ o(r^{-1}) & \text{for } \nu = 0, \end{cases}$$

$$(5)\qquad \frac{dU(r)}{dr} = a^{-2\nu}b(br)^{-\nu}Y_{\nu+1}(br) - ab^{-2\nu}(ar)^{-\nu}Y_{\nu+1}(ar)$$

$$= \begin{cases} O(r^{-2\nu+1}) & \text{for } \nu > 0, \\ O(1) & \text{for } \nu = 0, \end{cases}$$

$$(6)\qquad \Delta U(r) + (a^2+b^2)U(r) = -a^{2-2\nu}(br)^{-\nu}Y_\nu(br) + b^{2-2\nu}(ar)^{-\nu}Y_\nu(ar)$$

$$= \begin{cases} O(r^{-2\nu}) & \text{for } \nu > 0, \\ o(r^{-1}) & \text{for } \nu = 0, \end{cases}$$

$$(7)\qquad \frac{d}{dr}[\Delta U(r) + (a^2+b^2)U(r)]$$

$$= ba^{2-2\nu}(br)^{-\nu}Y_{\nu+1}(br) - ab^{2-2\nu}(ar)^{-\nu}Y_{\nu+1}(ar)$$

$$= 2^{\nu+1}\Gamma(\nu+1)\pi^{-1}b^{-2\nu}a^{-2\nu}(b^2-a^2)r^{-2\nu-1} + H_\nu(r),$$

where

$$H_\nu(r) = \begin{cases} O(r^{-2\nu+1}) & \text{for } \nu > 0, \\ O(1) & \text{for } \nu = 0 \end{cases}$$

(the symbol $O(f(r))$ $(o(f(r)))$ refers to the case $r \to 0$).

Let D be a boundary domain whose boundary we denote by ∂D. Let ∂D be of the class C_1^σ. Let w, v be functions which are of class C^4 in D and of class C^3 in $D \cup \partial D$. Using Green's formula ([1], p. 230) to the system of two functions v and Δw, Δv and w, $(a^2+b^2)v$ and $(a^2+b^2)w$, respectively, and adding obtained equations, we get

$$(8)\qquad \int_D \{w[\Delta^2 v + (a^2+b^2)\Delta v] - v[\Delta^2 w + (a^2+b^2)\Delta w]\}\, dD$$

$$= \int_{\partial D} \left\{ v\frac{d}{dn}[\Delta w + (a^2+b^2)w] - \frac{dv}{dn}[\Delta w + (a^2+b^2)w] + \Delta w\frac{dv}{dn} - \frac{d\Delta W}{dn}v \right\} dS,$$

where n is the inward normal to ∂D.

THEOREM 1. *Let u be a function of class C^4 in D and of class C^3 in $D \cup \partial D$ satisfying equation* (1) *in D. Then*

$$(9)\qquad \frac{1}{\gamma_n}\int_{\partial D} \left\{ u\frac{d}{dn_Y}[\Delta U + (a^2+b^2)U] - \frac{du}{dn}[\Delta U + (a^2+b^2)U] + \right.$$

$$\left. + \Delta u\frac{dU}{dn_Y} - \frac{d\Delta u}{dn}U \right\} dS_Y = \begin{cases} u(X) & \textit{for } X \in D \setminus \partial D, \\ 0 & \textit{for } X \in C(\overline{D \cup \partial D}), \end{cases}$$

where $\gamma_n = \Omega_n 2^{\nu+1}\Gamma(\nu+1)\pi^{-1}(ab)^{-2\nu}(a^2-b^2)$ *and* Ω_n *is the surface of the n-dimensional unit sphere*; $C(\overline{D\cup\partial D})$ *denotes complement of* $\overline{D\cup\partial D}$ *relative to* E_n.

Proof. Let $X\in C(\overline{D\cup\partial D})$. Applying (8) to the function $u(Y)$ and $U(X, Y)$ as a function of Y we get (9). Let X be interior point of the domain D and K_R denote the ball with the centre X and radius R, $K_R\subset D$. If in formula (8) D is replaced by $D\setminus K_R$ and functions w and v are replaced by $u(Y)$ and $U(X, Y)$, respectively, we get

$$(10)\quad \int\limits_{\partial D}\Big\{U\frac{d}{dn}[\Delta u+(a^2+b^2)u]-\frac{dU}{dn_Y}[\Delta u+(a^2+b^2)u]+ +\Delta U\frac{du}{dn}-\frac{d\Delta U}{dn_Y}u\Big\}dS_Y = \int\limits_{\partial K_R}\Big\{u\frac{d}{dn_Y}[\Delta U+(a^2+b^2)U]- -\frac{du}{dn}[\Delta U+(a^2+b^2)U]+\Delta u\frac{dU}{dn_Y}-\frac{d\Delta u}{dn}U\Big\}dS_Y.$$

Applying the mean value theorem to the surface integrals we obtain

$$I_1 = \int\limits_{\partial K_R} u\frac{d}{dn_Y}[\Delta U+(a^2+b^2)U]dS_Y = \Omega_n R^{n-1}u(Q_1)\frac{d}{dr}[\Delta U(r)+(a^2+b^2)U(r)]|_{r=R},$$

$$I_2 = -\int\limits_{\partial K_R}\frac{du}{dn_Y}[\Delta U+(a^2+b^2)U]dS_Y = \Omega_n R^{n-1}\frac{du(Q_2)}{dn}[\Delta U(r)+(a^2+b^2)U(r)]|_{r=R},$$

$$I_3 = \int\limits_{\partial K_R}\Delta u\frac{dU}{dn_Y}dS_Y = \Omega_n R^{n-1}\Delta u(Q_3)\frac{d}{dr}U(r)|_{r=R},$$

$$I_4 = -\int\limits_{\partial K_R}U\frac{d\Delta u}{dn}dS_Y = \Omega_n R^{n-1}\frac{d\Delta u(Q_4)}{dn}U(R),$$

where $Q_i\in\partial K_R$ $(i=1,2,3,4)$. Taking into account (4), (5), (6) and (7) we obtain

$$I_1 = \Omega_n R^{n-1}u(Q_1)[2^{\nu+1}\Gamma(\nu+1)\pi^{-1}(ab)^{-2\nu}(b^2-a^2)R^{-2\nu-1}+H_\nu(R)] \underset{R\to 0}{\longrightarrow} -\gamma_n u(X),$$

$$I_i \underset{R\to 0}{\longrightarrow} 0 \quad (i=2,3,4),$$

hence by (10) we obtain (9).

3. We now pass to the construction of the Green function of equation (1) and for the set E_n^+. Let X and Y be two different points, $X \in E_n^+$, $Y \in \overline{E_n^+}$. Let us denote by $\bar{X} = (x_1, \ldots, -x_n)$ the symmetric image of the point X with respect to the coordinate hyperplane $y_n = 0$. Let us write

$$r_1 = \bar{X}Y = \Big[\sum_{i=1}^{n-1} (x_i - y_i)^2 + (x_n + y_n)^2\Big]^{1/2}, \qquad \varrho = \Big[\sum_{i=1}^{n-1} (x_i - y_i)^2 + x_n^2\Big]^{1/2}.$$

We are going to prove

THEOREM 2. *The function*

$$G(X, Y) = U(r) - U(r_1), \tag{11}$$

where $U(r)$ is given by (3) *is a Green function with the pole X for equation* (1) *and for E_n^+ with the boundary conditions*

$$G(X, Y)|_{y_n=0} = 0, \quad \Delta_Y G(X, Y)|_{y_n=0} = 0. \tag{12}$$

Proof. $U(r_1)$ as a function of Y is defined and of class C^∞ in E_n^+ and satisfies equation (1) in this set. For $Y \in \partial E_n^+$ we have $r = r_1 = \varrho$ and $G(X, Y)|_{y_n=0} = U(\varrho) - U(\varrho) = 0$. Since

$$\Delta_Y U(r) = U''(r) + r^{-1}(n-1)\, U'(r)$$

$$\text{and so} \quad \Delta_Y G(X, Y)|_{y_n=0} = [\Delta_Y U(r) - \Delta_Y U(r_1)]|_{y_n=0} = 0.$$

4. Let $u(X)$ be a function of class C^4 in E_n^+ satisfying equation (1) and the boundary conditions (2). Applying formally formula (9) to functions $U(r)$ and $-U(r_1)$ and for $D = E_n^+$ and adding obtained equations, we get, in view of (2), (11) and (12),

$$u(X) = \frac{1}{\gamma_n} \int_{E_{n-1}} \Big\{ f_1(Y') \frac{\partial}{\partial y_n} [\Delta_Y G(X, Y) + (a^2+b^2) G(X, Y)] + f_2(Y') \frac{\partial}{\partial y_n} G(X, Y) \Big\}\Big|_{y_n=0} dY', \tag{13}$$

where $Y' = (y_1, \ldots, y_{n-1})$ denotes a point in $(n-1)$-dimensional Euclidean space E_{n-1}. It is easily shown, using formulae (5) and (7), that

$$\begin{aligned} \frac{\partial}{\partial y_n} G(X, Y)|_{y_n=0} &= x_n N_2(\varrho), \\ \frac{\partial}{\partial y_n} [\Delta_Y G(X, Y) + (a^2+b^2) G(X, Y)]|_{y_n=0} &= x_n N_1(\varrho), \end{aligned} \tag{14}$$

where

$$N_1(\varrho) = 2b^{2-2\nu}a^2(a\varrho)^{-\nu-1}Y_{\nu+1}(a\varrho) - 2a^{2-2\nu}b^2(b\varrho)^{-\nu-1}Y_{\nu+1}(b\varrho),$$

$$N_2(\varrho) = -2b^2a^{-2\nu}(b\varrho)^{-\nu-1}Y_{\nu+1}(b\varrho) + 2a^2b^{-2\nu}(a\varrho)^{-\nu-1}Y_{\nu+1}(a\varrho).$$

Substituting expressions (14) into (13) we obtain

$$u(X) = \frac{x_n}{\gamma_n}\int\limits_{E_{n-1}} [f_1(Y')N_1(\varrho) + f_2(Y')N_2(\varrho)]\,dY'. \tag{15}$$

LEMMA 1. *Let $|f_i(Y')|$ $(i = 1, 2)$ be a function which is Lebesgue integrable over E_{n-1}. Then the function $u(X)$ given by formula* (15) *is of class C^∞ in E_n^+ and satisfies equation* (1) *in this set.*

Proof. We shall prove that the integral on the right-hand side of formula (15) as well as the integrals we get by differentiating k-times $(k = 1, 2, 3, \ldots)$ with respect to x_i $(i = 1, 2, \ldots, n)$ the functions under the sign of integral (15) exist and are uniformly convergent for X belonging to the set

$$Q = \{X\colon\ |x_i| \leqslant C,\ i = 1, 2, \ldots, n-1,\ A \leqslant x_n \leqslant B\},$$

where A, B, C are arbitrary positive constants. The functions we obtain by k-times differentiation of the kernels $N_1(\varrho)$ and $N_2(\varrho)$ with respect to x_i $(i = 1, 2, \ldots, n)$ are linear combinations of the functions

$$\Phi = \prod_{i=}^{n-1} (x_i - y_i)^{a_i} x_n^{a_n} \varrho^{-\beta} Y_\beta(h\varrho),$$

where $\sum_{i|1}^{n-1} a_i \leqslant \beta$ and $\beta \geqslant 0$, $a_i \geqslant 0$ $(i = 1, \ldots, n-1)$, $h = a, b$. It is enough to know that the integrals

$$\int\limits_{E_{n-1}} f_i(Y')\,\Phi\,dY' \quad (i = 1, 2) \tag{16}$$

exist and are uniformly convergent for $X \in Q$. For $X \in Q$ we have $\varrho \geqslant A$. Now from the assymptotic properties of Bessel functions ([2], p. 145) we see that $Y_\nu(h\varrho)$ are bounded for $\varrho \geqslant A$. From these results we can obtain the inequalities

$$|\Phi| \leqslant M_{a_1,\ldots,a_n,\beta}$$

for $X \in Q$, $Y \in E_{n-1}$, where $M_{a_1,\ldots,a_n,\beta}$ are positive constants. It follows from this that

$$\Big|\int\limits_{E_{n-1}} f_i(Y')\Phi\,dY'\Big| \leqslant M_{a_1,\ldots,a_n,\beta}\int\limits_{E_{n-1}} |f_i(Y')|\,dY' \quad (i = 1, 2).$$

Hence and by assumption of Lemma 1 integrals (16) exist and are uniformly convergent for $X \in Q$.

The function $u(X)$ defined by formula (15) or (13) is of class C^∞ in the domain E_n^+ and its derivatives may be found by differentiation under the sign of the integral.

We shall prove now that the function defined by formula (13) satisfies equation (1). Taking into consideration the above properties and the fact that the function $G(X, Y)$ as a function of the point of X, $X \neq Y$ satisfies equation (1), we have

$$(\Delta+a^2)(\Delta+b^2)u(X) = \frac{1}{\gamma_n}\int_{E_{n-1}} \Big\{ f_1(Y')\frac{\partial}{\partial y_n}\Delta_Y(\Delta_X+a^2)(\Delta_X+b^2)G(X,Y) + f_1(Y')(a^2+b^2)\frac{\partial}{\partial y_n}(\Delta_X+a^2)(\Delta_X+b^2)G(X,Y) + f_2(Y')\frac{\partial}{\partial y_n}(\Delta_X+a^2)(\Delta_X+b^2)G(X,Y)\Big\}\Big|_{y_n=0} dY' = 0.$$

LEMMA 2. *Let $|f(Y')|$ be a function which is Lebesgue integrable over E_{n-1}. Let $f(Y')$ be continuous at the poit the $X_0' = (x_1^0, \ldots, x_{n-1}^0)$. Then the function*

$$L(X) = \frac{2x_n}{\Omega_n}\int_{E_{n-1}} f(Y')\varrho^{-n}dY'$$

is defined for $X \in E_n^+$ and convergent to $f(X_0')$ when $X \to (X_0', 0+)$.

Proof. If we use the formula

$$\int_{E_{n-1}} (1+t_1^2+\ldots+t_{n-1}^2)^{-n/2}dt_1\ldots dt_{n-1} = \Omega_n/2$$

(Ω_n is the surface of the n-dimensional unit sphere, $n \geqslant 2$) we easily find that [3]

$$\frac{2}{\Omega_n}\int_{E_{n-1}} \varrho^{-n}x_n dY' = 1 \quad \text{for } X \in E_n^+. \tag{17}$$

Multiplying both sides of (17) by $f(X_0')$ we obtain

$$\frac{2}{\Omega_n}\int_{E_{n-1}} \varrho^{-n}x_n f(X_0')dY' = f(X_0').$$

We now present the function $L(X)$ in the form

$$L(X) = \frac{2x_n}{\Omega_n}\int_{E_{n-1}} f(X_0')\varrho^{-n}dY' + C(X),$$

where

$$C(X) = \frac{2x_n}{\Omega_n} \int_{E_{n-1}} [f(Y')-f(X_0')]\varrho^{-n} dY'.$$

It is enough to show that $C(X)$ is convergent to zero if $X \to (X_0', 0+)$. Let $\varepsilon > 0$ be given. It follows from the continuity of the function $f(Y')$ at the point X_0' that there exists the $(n-1)$-dimensional ball $K_R(X_0')$ with the centre X_0' and radius $R > 0$ such that

(18) $$|f(Y')-f(X_0')| < \varepsilon/2 \quad \text{for } Y' \in K_R(X_0').$$

Let

(19) $$C(X) = C_1(X)+C_2(X),$$

where

$$C_1(X) = \frac{2x_n}{\Omega_n} \int_{K_R(X_0')} [f(Y')-f(X_0')]\varrho^{-n} dY',$$

$$C_2(X) = \frac{2x_n}{\Omega_n} \int_{C[K_R(X_0')]} [f(Y')-f(X_0')]\varrho^{-n} dY',$$

where $C[K_R(X_0')]$ denotes complement of $K_R(X_0')$ relative to E_{n-1}. It follows from (17) and (18) that

(20) $$|C_1(X)| < \varepsilon/2 \quad \text{for } X \in E_n^+.$$

Let $X'X_0' < R/2$ and $Y'X_0' > R$. Then $X'Y' \geqslant Y'X_0' - X_0'X' \geqslant R/2$ and $C[K_R(X_0')] \subset C[K_{R/2}(X')]$. Hence

(21) $$|C_2(X)| \leqslant C_3(X)+C_4(X),$$

where

$$C_3(X) = \frac{2x_n}{\Omega_n} |f(X_0')| \int_{C[K_{R/2}(X')]} \varrho^{-n} dY',$$

$$C_4(X) = \frac{2x_n}{\Omega_n} \int_{C[K_{R/2}(X')]} |f(Y')|\varrho^{-n} dY'.$$

Introducing the spherical coordinates in $C_3(X)$

$$y_1 = x_1 + r\cos\varphi_1, \quad y_2 = x_2 + r\sin\varphi_1\cos\varphi_2, \quad \ldots,$$
$$+y_{n-1} = x_{n-1} r\sin\varphi_1 \ldots \sin\varphi_{n-2},$$

where $R/2 \leqslant r < \infty$, $0 \leqslant \varphi_i \leqslant \pi$ $(i = 1, \ldots, n-3)$, $0 \leqslant \varphi_{n-2} \leqslant 2\pi$, $|J| = r^{n-2}\psi(\varphi_1, \ldots, \varphi_{n-3})$, we get

$$C_3(X) = Mx_n \int\limits_{R/2}^{\infty} r^{n-2}(r^2+x_n^2)^{-n/2}\,dr \leqslant Mx_n \int\limits_{R/2}^{\infty} \frac{dr}{r^2+x_n^2}$$
$$= M\left[\frac{\pi}{2} - \arctan\frac{R}{2x_n}\right],$$

where M is a positive constant.

Since the last integral is convergent to zero for $x_n \to 0+$, there exists a number $\eta > 0$ such that the conditions $0 < x_n < \eta$ and $X'X_0' < R/2$ imply

(22) $$C_3(X) < \varepsilon/4.$$

Let us now take into consideration the integral $C_4(X)$. For $Y'X' < R/2$ we have $\varrho \geqslant R/2$ and thus

$$C_4(X) \leqslant \frac{2x_n}{\Omega_n} M_1 \int\limits_{C[K_{R/2}(X')]} |f(Y')|\,dY' \leqslant \frac{2x_n}{\Omega_n} M_1 \int\limits_{E_{n-1}} |f(Y')|\,dY',$$

where M_1 is a positive constant.

From this result and by assumption of Lemma 2 it follows that there exists a number $\eta_1 > 0$ such that

(23) $$C_4(X) < \varepsilon/4 \quad \text{for } X'X_0' < R/2 \text{ and } 0 < x_n < \eta_1.$$

It follows from (19), (20), (21), (22) and (23) that

$$|C(X)| < \varepsilon \quad \text{for } X'X_0' < R/2 \text{ and } 0 < x_n < \min(\eta, \eta_1).$$

LEMMA 3. *Let* $|f_i(Y')|$ $(i = 1, 2)$ *be a function which is Lebesgue integrable over* E_{n-1}. *Let* $f_i(Y')$ *be continuous at the point* $X_0' = (x_1^0, \ldots, x_{n-1}^0)$. *Then the function* $u(X)$ *defined by* (15) *satisfies the following boundary conditions:*

(24) $$\lim u(X) = f_1(X_0') \quad as \quad X \to (X_0', 0+),$$

(25) $$\lim \Delta u(X) = f_2(X_0') \quad as \quad X \to (X_0', 0+).$$

Proof. We shall now prove that the function $u(X)$ defined by (15) satisfies condition (24). It is easily shown by using the expansion of $Y_{\nu+1}(z)$ into the series and the asymptotic formulae for these functions ([2], p. 145) that the functions $N_1(\varrho)$ and $N_2(\varrho)$ may be written in the form

(26) $$N_1(\varrho) = 2^{\nu+2}(ab)^{-2\nu}\Gamma(\nu+1)\pi^{-1}\varrho^{-2\nu-2}(a^2-b^2)+P(\varrho),$$

where

$$(27)\qquad \begin{aligned} P(\varrho) &= o(\varrho^{-n+1}) \quad \text{as } \varrho \to 0+, \\ N_2(\varrho) &= o(\varrho^{-n+1}) \quad \text{as } \varrho \to 0+. \end{aligned}$$

Now let us write $u(X)$ given by (15) in the form

$$u(X) = J_1(X) + J_2(X) + J_3(X),$$

where

$$J_1(X) = \frac{x_n 2^{\nu+2} \Gamma(\nu+1)(a^2-b^2)}{\gamma_n (ab)^{2\nu} \pi} \int\limits_{E_{n-1}} f_1(Y') \varrho^{-n} dY',$$

$$J_2(X) = \frac{x_n}{\gamma_n} \int\limits_{E_{n-1}} f_1(Y') P(\varrho) dY',$$

$$J_3(X) = \frac{x_n}{\gamma_n} \int\limits_{E_{n-1}} f_2(Y') N_2(\varrho) dY'.$$

In view of Lemma 2 and definition of γ_n we have

$$\lim J_1(X) = f_1(X_0') \quad \text{when } X \to (X_0', 0+).$$

It is enough to show that integrals $J_i(X)$ $(i = 2, 3)$ are convergent to zero if $X \to (X_0', 0+)$. We shall show this only for the integral $J_2(X)$, the proof for the integral $J_3(X)$ being similar. Since $P(\varrho) = o(\varrho^{-n+1})$ as $\varrho \to 0$, thus there exists a number $\varrho_0 > 0$ such that

$$(28)\qquad |P(\varrho)| \leqslant \varrho^{-n+1} \quad \text{for } 0 < \varrho < \varrho_0.$$

It follows by the continuity of the function $f_1(Y')$ at the point X_0' that there exists the $(n-1)$-dimensional ball $K_\delta(X_0')$ with the centre X_0' and radius $\delta > 0$ and a number $M_\delta > 0$ such that

$$(29)\qquad |f_1(Y')| \leqslant N_\delta \quad \text{for } Y' \in K_\delta(X_0').$$

Let

$$\delta_1 = \min\left(\frac{\varrho_0}{2}, \frac{\delta}{2}\right).$$

The integral $J_2(X)$ may be written in the form

$$(30)\qquad J_1(X) = B_1(X) + B_2(X),$$

where

$$B_1(X) = \frac{x_n}{\gamma_n} \int\limits_{K_{\delta_1}(X_0')} f_1(Y') P(\varrho) dY',$$

$$B_2(X) = \frac{x_n}{\gamma_n} \int\limits_{C[K_{\delta_1}(X_0')]} f_1(Y') P(\varrho) dY'.$$

If $X_0'X' < \delta_1/2$ and $Y'X_0' \geqslant \delta_1$, then $Y'X' \geqslant Y'X_0' - X_0'X' \geqslant \delta_1/2$ and $C[K_{\delta_1}(X_0')] \subset C[K_{\delta_1/2}(X')]$. Hence

$$|B_2(X)| \leqslant \frac{x_n}{\gamma_n} \int\limits_{C[K_{\delta_1/2}(X')]} |f_1(Y')|\,|P(\varrho)|\,dY'.$$

For $Y'X' \geqslant \delta_1/2$ we have $\varrho \geqslant \delta_1/2$ and $|P(\varrho)| \leqslant M_2$, where M_2 is a positive constant. Thus

$$|B_2(X)| \leqslant M_2 \frac{x_n}{\gamma_n} \int\limits_{E_{n-1}} |f_1(Y')|\,dY'.$$

Let $\varepsilon > 0$ be given; then by assumption of Lemma 3 there exists a number $\zeta > 0$ such that the condition $0 < x_n < \zeta$ implies

$$M_2 \frac{x_n}{\gamma_n} \int\limits_{E_{n-1}} |f_1(Y')|\,dY' < \varepsilon/2.$$

Therefore

$$|B_2(X)| < \varepsilon/2 \quad \text{for } 0 < x_n < \zeta \text{ and } X'X_0' < \delta_1/2. \tag{31}$$

Let us now take into consideration the integral $B_1(X)$. Let $Y'X_0' < \delta_1$. If $X'X_0' < \delta_1/2$ and $0 < x_n < \zeta_1 = \min(\zeta, \delta_1/2)$, then $X'Y' \leqslant X_0'Y' + X_0'X' \leqslant \delta_1 + \delta_1/2 = \frac{3}{2}\delta_1$ and $K_{\delta_1}(X_0') \subset K_{\frac{3}{2}\delta_1}(X')$.

From this result and from formulae (28) and (29) we have

$$|B_1(X)| \leqslant \frac{x_n}{\gamma_n} \int\limits_{K_{\frac{3}{2}\delta_1}(X')} |f_1(Y')|\,|P(\varrho)|\,dY' \leqslant \frac{x_n}{\gamma_n} M_\delta \int\limits_{K_{\frac{3}{2}\delta_1}(X')} \varrho^{-n+1}\,dY'.$$

Introducing the spherical coordinates

$$y_1 = x_1 + r\cos\varphi_1, \quad \ldots, \quad y_{n-1} = x_{n-1} + r\sin\varphi_1 \cdot \ldots \cdot \sin\varphi_{n-2},$$

$|J| = r^{n-2}\psi(\varphi_1, \ldots, \varphi_{n-3})$ in the last integral, we get

$$\int\limits_{K_{\frac{3}{2}\delta_1}(X')} \varrho^{-n+1}\,dY' = x_n N \int\limits_0^{\frac{3}{2}\delta_1} r^{n-2}(r^2 + x_n^2)^{(-n+1)/2}\,dr \leqslant x_n N \int\limits_0^{\frac{3}{2}\delta_1} \frac{dr}{\sqrt{r^2 + x_n^2}},$$

where N is a positive constant.

Since the last integral is convergent to zero for $x_n \to 0+$, there exists a number $\zeta_2 > 0$ such that the conditions

$$0 < x_n < \min(\zeta_1, \zeta_2) \quad \text{and} \quad X'X_0' < \delta_1/2$$

imply

(32) $$|B_1(X)| \leqslant \varepsilon/2.$$

It follows from (30), (31) and (32) that

$$|J_2(X)| < \varepsilon \quad \text{for } 0 < x_n < \zeta_3 \text{ and } X'X_0' < \delta_1/2,$$

where $\zeta_3 = \min(\zeta_1, \zeta_2)$.

So we have proved that the function defined by (15) satisfies the boundary condition (24).

We shall now prove that the function $u(X)$ satisfies condition (25). In view of formulae (14) we have

$$\begin{aligned}\frac{\partial}{\partial y_n}[\Delta_X G(X, Y)]|_{y_n=0} &= \frac{\partial}{\partial y_n}[\Delta_Y G(X, Y)]|_{y_n=0}\\ &= 2b^4 a^{-2\nu}(b\varrho)^{-\nu-1} Y_{\nu+1}(b\varrho)x_n - 2a^4 b^{-2\nu}(a\varrho)^{-\nu-1} Y_{\nu+1}(a\varrho)x_n\\ &= [2^{\nu+2}(ab)^{-2\nu}\Gamma(\nu+1)\pi^{-1}\varrho^{-2\nu-2}(a^2-b^2)+W(\varrho)]x_n,\end{aligned}$$

where

$$W(\varrho) = o(\varrho^{-n+1}) \quad \text{when } \varrho \to 0.$$

$$\begin{aligned}&\frac{\partial}{\partial y_n}[\Delta_X\Delta_Y G(X, Y)+(a^2+b^2)\Delta_X G(X, Y)]|_{y_n=0}\\ &= \frac{\partial}{\partial y_n}[\Delta_Y^2 G(X, Y)+(a^2+b^2)\Delta_Y G(X,Y)]|_{y_n=0}\\ &= -2a^4 b^{2-2\nu}(a\varrho)^{-\nu-1} Y_{\nu+1}(a\varrho)x_n + 2b^4 a^{2-2\nu}(b\varrho)^{-\nu-1} Y_{\nu+1}(b\varrho)x_n\\ &= -(ba)^2 N_2(\varrho)x_n.\end{aligned}$$

We have then

$$\Delta u(X) = A_1(X)+A_2(X)+A_3(X),$$

where

$$A_1(X) = \frac{x_n}{\gamma_n}\int_{E_{n-1}} f_1(Y')\big(-a^2b^2N_2(\varrho)\big)\,dY',$$

$$A_2(X) = \frac{2^{2\nu}\Gamma(\nu+1)\,(a^2-b^2)x_n}{(ab)^\nu\pi\gamma_n}\int_{E_{n-1}} f_2(Y')\,\varrho^{-2\nu-2}\,dY',$$

$$A_3(X) = \frac{x_n}{\gamma_n}\int_{E_{n-1}} f_2(Y')\,W(\varrho)\,dY'.$$

In view of Lemma 2 and definition of γ_n we have

$$\lim A_2(X) = f_2(X_0') \quad \text{when } X \to (X_0', 0+).$$

It is enough to show that

$$(33) \qquad A_i(X) \to 0 \quad (i = 1, 3) \quad \text{if } X \to (X_0', 0+).$$

Now the proof that (33) is satisfied is similar to that of the proof for the integral $J_2(X)$.

From Lemmas 1 and 3 we get

THEOREM 3. *By the assumptions of Lemma* 3 *the function* $u(X)$ *defined by formula* (15) *is the solution of equation* (1) *in the set* E_n^+ *with the boundary conditions* (24) *and* (25).

References

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