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On absolute convergence of multiple Fourier series of functions p-integrable with mixed powers

1. In this paper there is investigated absolute convergence of Fourier series of a function of several variables belonging to the space L^p with mixed powers. These spaces were considered e.g. by A. Benedek and R. Panzone [3], and by J. Albrycht [2].

Let $H = \{k_1, \ldots, k_s\}$ be a system of s integers, where $k_1 < k_2 < \ldots < k_s$, and $H \subset E = \{1, 2, \ldots, n\}$. Let f be a function defined in R^n; then the differences Δ^H and F^H are defined as follows. Let $x = (x_1, \ldots, x_n)$, $h = (h_1, \ldots, h_n)$. If $H = \{k\}$, then we write

$$\Delta^H(f; x; h) = f(x_1, \ldots, x_k+h_k, \ldots, x_n) - f(x_1, \ldots, x_k, \ldots, x_n),$$
$$F^H(f; x; h) = f(x_1, \ldots, x_k+h_k, \ldots, x_n) - f(x_1, \ldots, x_k-h_k, \ldots, x_n).$$

If $H = \{k_1, \ldots, k_s\}$, $s > 1$, we define Δ^H and F^H by induction:

$$\Delta^H(f; x; h) = \Delta^{(k_s)}[\Delta^{H-(k_s)}(f; x; h)],$$
$$F^H(f; x; h) = F^{(k_s)}[F^{H-(k_s)}(f; x; h)].$$

The space L^p with mixed powers is defined in the following manner. Let $Q = (a, b) \times \ldots \times (a, b) \subset R^n$, where (a, b) is an interval on the real line, and let $p = (p_1, \ldots, p_n)$, where $1 \leqslant p_i < \infty$ for $i = 1, 2, \ldots, n$. A measurable function $f(x) = f(x_1, \ldots, x_n)$ in Q belongs to $L^p = L^p(Q)$, if (see [2])

$$\|f\|_p = \Big[\int_a^b \Big(\ldots\Big[\int_a^b \Big(\int_a^b |f(x_1, \ldots, x_n)|^{p_1} dx_1\Big)^{p_2/p_1} dx_2\Big]^{p_3/p_2} \ldots\Big)^{p_n/p_{n-1}} dx_n\Big]^{1/p_n} < \infty.$$

Let $f \in L^p(Q)$ be periodic with period $b-a$ in each variable. Then the integral modulus of continuity of f in $L^p(Q)$ corresponding to the set of indices H is defined as

$$\omega_p^H(f; h) = \sup_{|\delta_i| \leqslant h_i} \|\Delta^H(f; x; \delta)\|_p.$$

This modulus possesses the following properties [1]:

1° $\omega_p^H(\lambda f; h) = |\lambda| \omega_p^H(f; h)$.

2° $\omega_p^H(f+g;h) \leqslant \omega_p^H(f;h)+\omega_p^H(g,h)$.

3° If $0 < h_i' < h_i''$ for $i \in H$, then

$$\omega_p^H(f;h') \leqslant \omega_p^H(f;h''),$$

where $h' = (h_1', \ldots, h_n')$, $h'' = (h_1'', \ldots, h_n'')$.

4° If $m_1, \ldots, m_n$ are non-negative integers and $H = \{k_1, \ldots, k_s\}$, then

$$\omega_p^H(f;mh) \leqslant m_{k_1}\cdot\ldots\cdot m_{k_s}\omega_p^H(f;h),$$

where $h = (h_1, \ldots, h_n)$, $mh = (m_1 h_1, \ldots, m_n h_n)$.

5° If $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ and $H = \{k_1, k_2, \ldots, k_s\}$, then

$$\omega_p^H(f;\lambda h) \leqslant (\lambda_{k_1}+1)\ldots(\lambda_{k_s}+1)\,\omega_p^H(f;h).$$

Properties 1°–3° are obvious. 5° follows from 4° as in the case of functions of one variable. Property 4° is proved by means of the well-known Minkowski inequality and some properties of integrals of periodic functions. There will be applied also the Lipschitz class $\mathrm{Lip}_p^H \alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i > 0$, consisting of $(b-a)$-periodic functions $f \in L^p(Q)$ such that

$$\omega_p^H(f;h) = 0(h_{k_1}^{\alpha_{k_1}}, \ldots, h_{k_s}^{\alpha_{k_s}}) \quad \text{as } h_{k_1}, \ldots, h_{k_s} \to 0.$$

If $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, $0 < \alpha_i < \beta_i$ for $i \in H$, then

$$\mathrm{Lip}_p^H \beta \subset \mathrm{Lip}_p^H \alpha.$$

2. Now we consider the coefficients $a_{m_1,\ldots,m_n}^A(f)$ of multiple Fourier series of an integrable function f periodic with period 2 in each variable, taking $Q = (0,2)\times \ldots \times(0,2) \subset R^n$. Let $A \subset E = \{1, 2, \ldots, n\}$, $A' = E \setminus A$, and let $m_1, \ldots, m_n$ be non-negative integers. Then we write [4]

$$a_{m_1\ldots m_n}^A(f) = \int_0^2 \ldots \int_0^2 f(x_1, \ldots, x_n) \prod_{i \in A} \cos m_i \pi x_i \cdot \prod_{j \in A'} \sin m_j x_j \, dx_1 \ldots dx_n,$$

$$\varrho_{m_1\ldots m_n}^{(\gamma)}(f) = \sum_{A \subset E} |a_{m_1\ldots m_n}^A(f)|^\gamma$$

for $\gamma > 0$, where A runs over all subsets of the set E. If $g \in L^p(Q)$, $p = (p_1, \ldots, p_n)$, $1 < p_i \leqslant 2$, $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_n$, $q = (q_1, \ldots, q_n)$, where $1/p_i + 1/q_i = 1$, $i = 1, 2, \ldots, n$, then the following Hausdorff–Young inequality holds:

$$\Big[\sum_{m_n=0}^{\infty}\Big(\cdots\Big[\sum_{m_2=0}^{\infty}\Big(\sum_{m_1=0}^{\infty} \varrho_{m_1\ldots m_n}^{(q_1)}(g)\Big)^{q_2/q_1}\Big]^{q_3/q_2}\cdots\Big)^{q_n/q_{n-1}}\Big]^{1/q_n}$$
$$\leqslant \Big[\int_0^2\Big(\cdots\Big[\int_0^2\Big(\int_0^2 |g(x_1, \ldots, x_n)|^{p_1} dx_1\Big)^{p_2/p_1} dx_2\Big]^{p_3/p_2}\cdots\Big)^{p_n/p_{n-1}} dx_n\Big]^{1/p_n}.$$

This inequality is obtained applying the Riesz–Thorin theorem in $L^p(Q)$ in a similar manner as in [3]. There will be investigated convergence of the series

$$(1) \qquad \sum_{m_1,\ldots,m_n=0}^{\infty} (m_1+1)^{\beta_1}(m_2+1)^{\beta_2}\ldots(m_n+1)^{\beta_n}\varrho_{m_1\ldots m_n}(f),$$

where $\beta_1, \ldots, \beta_n \geqslant 0$, $1 \leqslant \gamma < 2$, and

$$\varrho^{(\gamma)}_{m_1\ldots m_n}(f) = \sum_{A\subset E} |a^A_{m_1\ldots m_n}(f)|^{\gamma}.$$

If series (1) is convergent for $\beta_1 = \beta_2 = \ldots = \beta_n = 0$ and $\gamma = 1$, then the Fourier series of the function f, i.e. the series

$$(2) \qquad f(x_1, \ldots, x_n) \sim \sum_{m_1,\ldots,m_n=0} \lambda_{m_1\ldots m_n} \sum_{A\subset E} a^A_{m_1\ldots m_n}(f) \prod_{i\in A} \cos m_i\pi x_i \prod_{j\in A'} \sin m_i \pi x_j,$$

where $\lambda_{m_1\ldots m_n} = 2^{-\mu(\bar{H})}$, $H = \{i: m_i > 0\}$, and $\mu(\bar{H})$ is the number of elements of the set $\bar{H}$, is absolutely convergent. Let us write (as in [4])

$$A_{i,r_i} = \{(m_1, \ldots, m_n): 2^{r_i-1} \leqslant m_i < 2^{r_i}\} \quad \text{for } r_i \geqslant 1,$$
$$A_{i,0} = \{(m_1, \ldots, m_n): m_i = 0\},$$
$$A_{r_1\ldots r_n} = \bigcap_{i=1}^{\infty} A_{i,ri} \quad \text{for } r_i \geqslant 0.$$

Then

$$\sum_{\substack{m_1,\ldots,m_n=0\\(m_1,\ldots,m_n)\neq 0}}^{\infty} (m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}\varrho^{\gamma}_{m_1\ldots m_n}(f)$$
$$= \sum_{0\neq H\subset E} \sum_{r_{k_s}=1}^{\infty} \cdots \sum_{r_{k_1}=0}^{\infty} \sum_{(m_1\ldots m_n)\in A_{r_1\ldots r_n}} (m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}\varrho^{(\gamma)}_{m_1\ldots m_n}(f),$$

where $H = \{k_1, \ldots, k_s\} = \{i: r_i > 0\}$.

It is easily seen that for two different systems $(r_1, \ldots, r_n)$ (i.e. system possessing different integers at one place at least), the corresponding sets $A_{r_1,\ldots,r_n}$ are disjoint and the set

$$\bigcup_{r_1,\ldots,r_n=0}^{\infty} A_{r_1\ldots r_n}$$

is identical with the set of all systems $(m_1, \ldots, m_n)$ of non-negative integers. In the following, the symbol $\sum_{(m_1,\ldots,m_n)\in A_{r_1\ldots r_n}}$ means summation over all systems $(m_1, \ldots, m_n) \in A_{r_1\ldots r_n}$.

3. The following lemma will be needed:

LEMMA 1. *If* $q_i \geqslant \gamma$, $\gamma \geqslant 1$, $\beta_i \geqslant 0$, $i = 1, 2, \ldots, n$, *then*

$$\sum_{m_1,\ldots,m_n \in A_{r_1\ldots r_n}} (m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n} \varrho^{(\gamma)}_{m_1\ldots m_n}(f)$$
$$\leqslant 2^{c_1(r)}\Big\{\sum_{m_n=2^{r_n-1}}^{2^{r_n}-1}\Big[\ldots\Big(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}\varrho^{(q_1)}_{m_1\ldots m_n}(f)\Big)^{q_2/q_1}\ldots\Big]^{q_n/q_{n-1}}\Big\}^{\gamma/q_n},$$
$$c_1(r) = \sum_{k=1}^{n}\beta_k r_k + \sum_{s=1}^{n-1}(r_s-1)\frac{q_n-1}{q_s} + (r_n-1)\frac{q_n-\gamma}{q_n} + 2^n - 1.$$

In the proof of this lemma we shall apply the following inequalities:

(a) *Hölder inequality with mixed powers for series.*

If

$$a_{m_1\ldots m_n} \geqslant 0, \quad b_{m_1\ldots m_n} \geqslant 0, \quad p = (p_1, \ldots, p_n),$$
$$q = (q_1, \ldots, q_n), \quad 1/p_i + 1/q_i = 1, \quad i = 1, 2, \ldots, n,$$

then

$$\sum_{m_n=0}^{\infty}\ldots\sum_{m_2=0}^{\infty}\sum_{m_1=0}^{\infty}|a_{m_1\ldots m_n}\cdot b_{m_1\ldots m_n}| \leqslant \|a_{m_1\ldots m_n}\|_p\cdot\|b_{m_1\ldots m_n}\|_q.$$

Hölder inequality with mixed powers for integrals.

If $f \in L^p$, $g \in L^q$, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$,

$$1/p_i + 1/q_i = 1, \quad i = 1, 2, \ldots, n,$$

then

$$\|f(x_1, \ldots, x_n)\cdot g(x_1, \ldots, x_n)\|_1 \leqslant \|f(x_1, \ldots, x_n)\|_p\cdot\|g(x_1, \ldots, x_n)\|_q.$$

(b) If $q > \gamma > 0$, $a_1, a_2, \ldots, a_N > 0$ (Lemma 1 in [4]), then

$$2^{-N+1}\Big(\sum_{i=1}^{N} a_i^q\Big)^{\gamma/q} \leqslant \sum_{i=1}^{N} a_i^q \leqslant 2^{N-1}\Big(\sum_{i=1}^{N} a_i^q\Big)^{\gamma/q},$$

(c) $\sum_{m=1}^{\infty} c_m^{\gamma} \leqslant \big(\sum_{m=1}^{\infty} c_m\big)^{\gamma}$ for $\gamma \geqslant 1$, $c_m \geqslant 0$.

Proof of Lemma 1. We apply in series (1), Hölder inequality (a) with

$$a_{m_1\ldots m_n} = (m_1+1)^{\beta_1}(m_2+1)^{\beta_2}\ldots(m_n+1)^{\beta_n},$$
$$b_{m_1\ldots m_n} = \varrho^{(\gamma)}_{m_1\ldots m_n}(f),$$

and $k = (k_1, \ldots, k_n)$, $l = (l_1, \ldots, l_n)$, $1/k_i + 1/l_i = 1$, $i = 1, 2, \ldots, n$, obtaining

$$\sum_{m_1 \ldots m_n \in A_{r_1 \ldots r_n}} (m_1+1)^{\beta_1}(m_2+1)^{\beta_2} \ldots (m_n+1)^{\beta_n} \cdot \varrho^{(\gamma)}_{m_1 \ldots m_n}(f)$$

$$\leqslant \Big(\sum_{m_n = 2^{r_n-1}}^{2^{r_n}-1} \Big[\ldots \Big(\sum_{m_2 = 2^{r_2-1}}^{2^{r_2}-1} \Big[\sum_{m_1 = 2^{r_1-1}}^{2^{r_1}-1} (m_1+1)^{k_1\beta_1}(m_2+1)^{k_1\beta_2} \times$$

$$\times \ldots \times (m_n+1)^{k_1\beta_n} \Big]^{k_2/k_1} \Big)^{k_3/k_2} \ldots \Big]^{k_n/k_{n-1}} \Big)^{1/k_n} \times$$

$$\times \Big[\sum_{m_n = 2^{r_n-1}}^{2^{r_n}-1} \Big(\ldots \Big[\sum_{m_2 = 2^{r_2-1}}^{2^{r_2}-1} \Big(\sum_{m_1 = 2^{r_1-1}}^{2^{r_1}-1} |\varrho^{(\gamma)}_{m_1 \ldots m_n}(f)|^{l_1} \Big)^{l_2/l_1} \Big]^{l_3/l_2} \ldots \Big)^{l_n/l_{n-1}} \Big]^{1/l_n} = A \cdot B.$$

We estimate each of the factors A, B, separately.

In order to estimate B, we apply the right-hand side inequality (b) with $N = 2^n$, $a_i = a^A_{m_1 \ldots m_n}$, $q = l_1$, $l_1 \geqslant \gamma$, to $\varrho_{m_1 \ldots m_n}(f)$ and next we make use of inequality (c) with

$$c_m = \varrho^{(l_1)}_{m_1 \ldots m_n}(f)$$

thus obtaining

$$\sum_{m_1 = 2^{r_1-1}}^{2^{r_1}-1} |\varrho^{(\gamma)}_{m_1 \ldots m_n}(f)|^{l_1} \leqslant 2^{(2^n-1)l_1} \Big(\sum_{m_1 = 2^{r_1-1}}^{2^{r_1}-1} |\varrho^{(l_1)}_{m_1 \ldots m_n}(f)| \Big)^{\gamma} \quad \text{for } l_1 \geqslant \gamma \geqslant 1.$$

Hence

$$\sum_{m_2 = 2^{r_2-1}}^{2^{r_2}-1} \Big[\sum_{m_1 = 2^{r_1-1}}^{2^{r_1}-1} |\varrho^{(\gamma)}_{m_1 \ldots m_n}(f)|^{l_1} \Big]^{l_2/l_1}$$

$$\leqslant \sum_{m_2 = 2^{r_2-1}}^{2^{r_2}-1} \Big[2^{(2^n-1)l_1} \Big(\sum_{m_1 = 2^{r_1-1}}^{2^{r_1}-1} |\varrho^{(l_1)}_{m_1 \ldots m_n}(f)| \Big)^{\gamma} \Big]^{l_2/l_1}$$

$$= 2^{(2^n-1)l_2} \sum_{m_2 = 2^{r_2-1}}^{2^{r_2}-1} \Big[\sum_{m_1 = 2^{r_1-1}}^{2^{r_1}-1} |\varrho^{(l_1)}_{m_1 \ldots m_n}(f)|^{l_2/l_1} \Big]^{\gamma}.$$

By (c) with

$$c_m = \sum_{m_1 = 2^{r_1-1}}^{2^{r_1}-1} |\varrho^{(l_1)}_{m_1 \ldots m_n}(f)|^{l_2/l_1}$$

we have

$$\sum_{m_2=2^{r_2-1}}^{2^{r_2}-1}\Big[\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}|\varrho_{m_1\ldots m_n}^{(\gamma)}(f)|^{l_1}\Big]^{l_2/l_1}$$
$$\leqslant 2^{(2^n-1)l_2}\Big[\sum_{m_2=2^{r_2-1}}^{2^{r_2}-1}\Big(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}|\varrho_{m_1\ldots m_n}^{(l_1)}(f)|\Big)^{l_2/l_1}\Big]^{\gamma}.$$

Applying inequality (c) with respective c_m successively, we get

$$B=\Big[\sum_{m_n=2^{r_1-1}}^{2^{r_n}-1}\Big(\ldots\Big[\sum_{m_2=2^{r_2-1}}^{2^{r_2}-1}\Big(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}|\varrho_{m_1\ldots m_n}^{(\gamma)}(f)|^{l_1}\Big)^{l_2/l_1}\Big]^{l_3/l_2}\ldots\Big)^{l_n/l_{n-1}}\Big]^{1/l_n}$$
$$\leqslant 2^{2^n-1}\Big\{\sum_{m_n=2^{r_n-1}}^{2^{r_n}-1}\Big(\ldots\Big[\sum_{m_2=2^{r_2-1}}^{2^{r_2}-1}\Big(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}|\varrho_{m_1\ldots m_n}^{(l_1)}(f)|\Big)^{l_2/l_1}\Big]^{l_3/l_2}\ldots\Big)^{\gamma l_n/l_{n-1}}\Big\}^{1/l_n}.$$

We put $l_i=q_i$ for $i=1,2,\ldots,n-1$ and $\gamma l_n=q_n$. But $l_i>1$ for $i=1,2,\ldots,n$ and $l_i\geqslant\gamma$ for $i=1,2,\ldots,n-1$, hence $q_i\geqslant\gamma$ for $i=1,2,\ldots,n-1$ and $q_n>\gamma$, $\gamma\geqslant 1$. Thus

$$B=\Big[\sum_{m_n=2^{r_n-1}}^{2^{r_n}-1}\Big(\ldots\Big[\sum_{m_2=2^{r_2-1}}^{2^{r_2}-1}\Big(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}|\varrho_{m_1\ldots m_n}^{(\gamma)}(f)|^{l_1}\Big)^{l_2/l_1}\Big]^{l_3/l_2}\ldots\Big)^{l_n/l_{n-1}}\Big]^{1/l_n}$$
$$\leqslant 2^{2^n-1}\Big\{\sum_{m_n=2^{r_n-1}}^{2^{r_n}-1}\Big(\ldots\Big[\sum_{m_2=2^{r_2-1}}^{2^{r_2}-1}\Big(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}|\varrho_{m_1\ldots m_n}^{(q_1)}(f)|\Big)^{q_2/q_1}\Big]^{q_3/q_2}\ldots\Big)^{q_n/q_{n-1}}\Big\}^{\gamma/q_n}$$

Now, we estimate A. Since $1/k_i+1/l_i=1$, $i=1,2,\ldots,n$, and $l_i=q_i$ for $i=1,2,\ldots,n-1$, $\gamma l_n=q_n$, we get $k_i=q_i/(q_i-1)$ for $i=1,2,\ldots,n-1$ and $k_n=q_n/(q_n-\gamma)$. Hence

$$A=\Big[\sum_{m_n=2^{r_1-1}}^{2^{r_n}-1}\Big(\ldots\Big[\sum_{m_2=2^{r_2-1}}^{2^{r_2}-1}\Big(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}(m_1+1)^{k_1\beta_1}(m_2+1)^{k_1\beta_2}\times\ldots$$
$$\ldots\times(m_n+1)^{k_1\beta_n}\Big)^{k_2/k_1}\Big]^{k_3/k_2}\ldots\Big)^{k_n/k_{n-1}}\Big]^{1/k_n}$$
$$\leqslant 2^{\beta_1r_1+\beta_2r_2+\ldots+\beta_nr_n+(r_1-1)(q_1-1)/q_1+(r_2-1)\frac{(q_2-1)}{q_2}+\ldots+(r_{n-1}-1)(q_{n-1}-1)/q_{n-1}}$$
$$\times 2^{(r_n-1)(q_n-\gamma)/q_n}=2^{c_2(r)},$$

where

$$c_2(r)=\sum_{k=1}^{n}\beta_kr_k+\sum_{s=1}^{n-1}(r_s-1)\frac{q_s-1}{q_s}+(r_n-1)\frac{q_n-\gamma}{q_n}.$$

From these estimations of B and A follows that

$$\sum_{m_1\ldots m_n\in A_{r_1\ldots r_n}} (m_1+1)^{\beta_1}(m_2+1)^{\beta_2}\ldots(m_n+1)^{\beta_n}\varrho^{(\gamma)}_{m_1\ldots m_n}(f)$$

$$\leqslant 2^{c_2(r)+2^n-1}\Bigg\{\sum_{m_n=2^{r_n-1}}^{2^{r_n}-1}\Bigg(\cdots\Bigg[\sum_{m_2=2^{r_2-1}}^{2^{r_2}-1}\Bigg(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}\varrho^{(q_1)}_{m_1\ldots m_n}(f)\Bigg)^{q_2/q_1}\Bigg]^{q_3/q_2}\cdots\Bigg)^{q_n/q_{n-1}}\Bigg\}^{\gamma/q_n}$$

$$= 2^{c_1(r)}\Bigg\{\sum_{m_n=2^{r_n-1}}^{2^{r_n}-1}\Bigg[\cdots\Bigg(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}\varrho^{(q_1)}_{m_1\ldots m_n}(f)\Bigg)^{q_2/q_1}\cdots\Bigg]^{q_n/q_{n-1}}\Bigg\}^{\gamma/q_n}.$$

THEOREM 1. *If* $f\in L^p$, $p=(p_1,\ldots,p_n)$, $\beta_1,\ldots,\beta_n\geqslant 0$ *and* $\gamma\geqslant 1$, *then*

$$\sum_{\substack{m_1\ldots m_n=0\\(m_1,\ldots,m_n)\neq 0}}^{\infty}(m_1+1)^{\beta_1}(m_2+1)^{\beta_2}\ldots(m_n+1)^{\beta_n}\varrho^{(\gamma)}_{m_1\ldots m_n}(f)$$

$$\leqslant\sum_{0\neq H\subset E}\sum_{r_{k_s}=0}^{\infty}\cdots\sum_{r_{k_2}=1}^{\infty}\sum_{r_{k_1}=1}^{\infty}2^{c(r)}[\omega_p^H(f;\,2^{-r})]^{\gamma},$$

where $H=\{k_1,k_2,\ldots,k_s\}$, $2^{-r}=(2^{-r_1},\ldots,2^{-r_n})$ *and*

$$c_r = c_1(r)-\tfrac{1}{2}\gamma\mu(H)$$

$$=\sum_{k=1}^{n}\beta_k r_k+\sum_{s=1}^{n-1}(r_s-1)\frac{q_s-1}{q_s}+(r_n-1)\frac{q_n-\gamma}{q_n}-\frac{\gamma}{2}\mu(H)+2^n-1.$$

Proof. By inequality (5.3) in [4],

$$\varrho^{(q_1)}_{m_1\ldots m_n}(f)\leqslant 2^{-q_1\mu(H)/2}\sum_{A\subset E}|a^A_{m_1\ldots m_n}[F^H(f)]|^{q_1}$$

$$=2^{-q_1\mu(H)/2}\varrho^{(q_1)}_{m_1\ldots m_n}[F^H(f)].$$

Applying Lemma 1 and the Hausdorff–Young inequality we thus obtain for $g=F^H(f)$

$$\sum_{\substack{m_1,\ldots,m_n=0\\(m_1,\ldots,m_n)\neq 0}}^{\infty}(m_1+1)^{\beta_1}(m_2+1)^{\beta_2}\ldots(m_n+1)^{\beta_n}\varrho^{(\gamma)}_{m_1\ldots m_n}(f)$$

$$=\sum_{0\neq H\subset E}\sum_{r_{k_s}=1}^{\infty}\cdots\sum_{r_{k_2}=1}^{\infty}\sum_{r_{k_1}=1}^{\infty}\sum_{m_1\ldots m_n\in A_{r_1\ldots r_n}}(m_1+1)^{\beta_1}(m_2+1)^{\beta_2}\times$$

$$\times\ldots\times(m_n+1)^{\beta_n}\varrho^{(\gamma)}_{m_1\ldots m_n}(f)$$

$$\leqslant\sum_{0\neq H\subset E}\sum_{r_{k_s}=1}^{\infty}\cdots\sum_{r_{k_2}=1}^{\infty}\sum_{r_{k_1}=1}^{\infty}2^{c_1(r)}\times$$

$$\times\Big\{\sum_{m_n=2^{r_2-1}}^{2^{r_n}-1}\Big(\cdots\Big[\sum_{m_2=2^{r_2-1}}^{2^{r_2}-1}\Big(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}\varrho_{m_1\ldots m_n}^{(q_1)}(f)\Big)^{q_2/q_1}\Big]^{q_3/q_2}\cdots\Big)^{q_n/q_{n-1}}\Big\}^{\gamma/q_n}$$

$$\leqslant \sum_{0\neq H\subset E}\sum_{r_{k_s}=1}^{\infty}\cdots\sum_{r_{k_2}=1}^{\infty}\sum_{r_{k_1}=1}^{\infty}2^{c_1(r)-\frac{\gamma}{2}\mu(H)}\times$$

$$\times\Big\{\sum_{m_n=2^{r_n-1}}^{2^{r_n}-1}\Big[\cdots\Big(\sum_{m_1=2^{r_1-1}}^{2^{r_1}-1}\varrho_{m_1\ldots m_n}^{(q_1)}[F^H(f)]\Big)^{q_2/q_1}\cdots\Big]^{q_n/n-1}\Big\}^{\gamma/q_n}$$

$$\leqslant \sum_{0\neq H\subset E}\sum_{r_{k_s}=1}^{\infty}\cdots\sum_{r_{k_1}=1}^{\infty}2^{c(r)}\Big\{\sum_{m_n=0}\Big[\cdots\Big(\sum_{m_1=0}\varrho_{m_1\ldots m_n}^{(q_1)}[F^H(f)]\Big)^{q_2/q_1}\cdots\Big]^{q_n/q_{n-1}}\Big\}^{\gamma/q_n}$$

$$\leqslant \sum_{0\neq H\subset E}\sum_{r_{k_s}=1}^{\infty}\cdots\sum_{r_{k_1}=1}^{\infty}2^{c(r)}\Big\{\int_0^2\Big[\cdots\Big(\int_0^2|F^H(f)|^{p_1}dx_1\Big)^{p_2/p_1}\cdots\Big]^{p_n/p_{n-1}}dx_n\Big\}^{\gamma/p_n}.$$

By inequality (3.1) in [4],

$$\|F^H(f)\|_p\leqslant \omega_p^H(f;\ 2h_1,\ldots,2h_n)$$

for $p=(p_1,\ldots,p_n)$, $h_i=2^{r_i-1}$, $i=1,2,\ldots,n$. Hence

$$\|F^H(f)\|_p=\Big\{\int_0^2\Big(\cdots\Big[\int_0^2\Big(\int_0^2|F^H(f)|^{p_1}dx_1\Big)^{p_2/p_1}dx_2\Big]^{p_3/p_2}\cdots\Big)^{p_n/p_{n-1}}dx_n\Big\}^{1/p_n}$$
$$\leqslant \omega_p^H(f;2^{-r_1},2^{-r_2},\ldots,2^{-r_n})=\omega_p^H(f;2^{-r})$$

and thus

$$\sum_{\substack{m_1\ldots m_n=0\\(m_1,\ldots,m_n)\neq 0}}^{\infty}(m_1+1)^{\beta_1}(m_2+1)^{\beta_2}\ldots(m_n+1)^{\beta_n}\varrho_{m_1\ldots m_n}^{(\gamma)}(f)$$
$$\leqslant \sum_{0\neq H\subset E}\sum_{r_{k_s}=1}^{\infty}\cdots\sum_{r_{k_2}=1}^{\infty}\sum_{r_{k_1}=1}^{\infty}2^{c(r)}[\omega_p^H(f;2^{-r})]^{\gamma}.$$

THEOREM 2. *If $f\in L^p$, $p=(p_1,\ldots,p_n)$, $\beta_1,\ldots,\beta_n\geqslant 0$, $f\in \mathrm{Lip}_p^H\alpha$, $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_i>0$ for $i\in H$, $H=\{k_1,k_2,\ldots,k_s\}$, then series (1) is convergent for*

$$\gamma\alpha_i>\beta_i+1/p_i,\quad i\in H,\ i<n,$$
$$\gamma\alpha_n>\beta_n+1-\gamma(1-1/p_n).$$

Proof. It follows from Theorem 1 that

$$\sum_{m_n=0}^{\infty}\cdots\sum_{m_2=0}^{\infty}\sum_{m_1=0}^{\infty}(m_1+1)^{\beta_1}(m_2+1)^{\beta_2}\dots(m_n+1)^{\beta_n}\varrho_{m_1\dots m_n}^{(\gamma)}(f)$$

$$=\sum_{0\neq H\subset E}\sum_{r_{k_s}=1}^{\infty}\cdots\sum_{r_{k_2}=1}^{\infty}\sum_{r_{k_1}=1}^{\infty}\sum_{m_1\dots m_n\in A_{r_1\dots r_n}}(m_1+1)^{\beta_1}(m_2+1)^{\beta_2}\times\dots$$

$$\dots\times(m_n+1)^{\beta_n}\varrho_{m_1\dots m_n}^{(\gamma)}(f)$$

$$\leqslant\sum_{0\neq H\subset E}\sum_{r_{k_s}=1}^{\infty}\cdots\sum_{r_{k_1}=1}^{\infty}2^{c(r)}[\omega_p^H(2^{-r_1},\dots,2^{-r_n})]^{\gamma}.$$

But $f\in \mathrm{Lip}_p^H\alpha$, i.e.

$$\omega_p^H(h_1,\dots,h_n)\leqslant K_H h_1^{\alpha_1}h_2^{\alpha_2}\dots h_n^{\alpha_n}.$$

Hence

$$\omega_p^H(2^{-r_1},\dots,2^{-r_n})\leqslant K_H 2^{-\alpha_{k_1}r_{k_1}}\dots2^{-\alpha_{k_s}r_{k_s}}=K_H 2^{-\sum_{j=1}^{s}\alpha_{k_j}r_{k_j}}$$

for $H=\{k_1,k_2,\dots,k_s\}$, where $h_i=2^{-r_i}$, $i\in H$.

Let $H=\{k_1,k_2,\dots,k_s\}=H_1$ if $k_s<n$, then $H'=\{l_1,l_2,\dots,l_{n-s}\}=H_1'$ and $l_{n-s}=n$. Now, let $H=\{k_1,k_2,\dots,k_s\}=H_2$ if $k_s=n$, then $H'=\{l_1,l_2,\dots,l_{n-s}\}=H_2'$ and $l_{n-s}<n$.

We denote $c(r)=c_H(r)$ for $H=\{k_1,k_2,\dots,k_s\}$. Then

$$c_{H_1}(r)=\sum_{j=1}^{s}\left[\beta_{k_j}r_{k_j}+(r_{k_j}-1)\frac{q_{k_j}-1}{q_{k_j}}\right]-\sum_{j=1}^{n-s-1}\frac{q_{l_j}-1}{q_{l_j}}-\frac{q_n-\gamma}{q_n}-\frac{\gamma\cdot s}{2}+2^n-1,$$

$$c_{H_2}(r)=\sum_{j=1}^{s-1}\left[\beta_{k_j}r_{k_j}+(r_{k_j}-1)\frac{q_{k_j}-1}{q_{k_j}}\right]+\beta_n r_n+(r_n-1)\frac{q_n-\gamma}{q_n}-$$

$$-\sum_{j=1}^{n-s}\frac{q_{l_j}-1}{q_{l_j}}-\frac{\gamma\cdot s}{2}+2^n-1.$$

Let $H=\{k_1,k_2,\dots,k_s\}=H_1$, i.e. $k_s<n$. Then the series

$$\sum_{r_{k_1}=1}^{\infty}\sum_{r_{k_2}=1}^{\infty}\cdots\sum_{r_{k_s}=1}^{\infty}2^{c_{H_1}(r)}[\omega_p^H(f;2^{-r})]^{\gamma}$$

$$\leqslant c\Big\{\sum_{r_{k_1}=1}^{\infty}2^{(\beta_{k_1}+(q_{k_1}-1)/q_{k_1}-\gamma\alpha_{k_1})r_{k_1}}\cdot\sum_{r_{k_2}=1}^{\infty}2^{(\beta_{k_2}+(q_{k_2}-1)/q_{k_2}-\gamma\alpha_{k_2})r_{k_2}}\times$$

$$\times\dots\times\sum_{r_{k_s}=1}^{\infty}2^{(\beta_{k_s}+(q_{k_s}-1)/q_{k_s}-\gamma\alpha_{k_s})r_{k_s}}\Big\},$$

where

$$c = K_H^\gamma 2^{\left(-\sum_{s=1}^{n-1} \frac{q_s-1}{q_s} - \frac{q_n-\gamma}{q_n} - \frac{\gamma \cdot s}{2} + 2^n - 1\right)}.$$

This series is convergent if $\beta_{k_j} + (q_{k_j} - 1)/q_{k_j} - \gamma a_{k_j} < 0$ for $j = 1, 2, \ldots, s$, i.e.

$$\gamma a_i > \beta_i + 1/p_i \quad \text{for } i \in H \text{ and } i < n.$$

Now, let $H = \{k_1, k_2, \ldots, k_s\} = H_2$, i.e. $k_s = n$. Then the series

$$\sum_{r_{k_1}=1}^{\infty} \sum_{r_{k_2}=1}^{\infty} \ldots \sum_{r_{k_s}=1}^{\infty} 2^{cH_2(r)} [\omega_p^H(f; 2^{-r})]^\gamma$$

$$\leqslant c\Big\{ \sum_{r_{k_1}=1}^{\infty} 2^{(\beta_{k_1} + (q_{k_s}-1)/q_{k_1} - \gamma a_{k_1}) r_{k_1}} \times \sum_{r_{k_2}=1}^{\infty} 2^{(\beta_{k_2} + (q_{k_2}-1)/q_{k_2} - \gamma a_{k_2}) r_{k_2}} \ldots$$

$$\ldots \sum_{r_{k_s}=1}^{\infty} 2^{(\beta_{k_s} + (q_{k_s}-\gamma)/q_{k_s} - \gamma a_{k_s}) r_{k_s}} \Big\} < \infty$$

if

$$\beta_{k_j} + \frac{q_{k_j} - 1}{q_{k_j}} - \gamma a_{k_j} < 0,$$

i.e. $\gamma a_{k_j} > \beta_{k_j} + 1/q_{k_j}$, $j = 1, 2, \ldots, s-1$, and

$$\gamma a_{k_s} > \beta_{k_s} + \frac{q_{k_s} - \gamma}{q_{k_s}} = \beta_{k_s} + 1 - \gamma\left(1 - \frac{1}{p_{k_s}}\right).$$

Since $k_s = n$, series (1) is convergent if

$$\gamma a_i > \beta_i + \frac{1}{p_i}, \quad i \in H, \ i < n,$$

and

$$\gamma a_n > \beta_n + 1 - \gamma(1 - 1/p_n).$$

If $\gamma = 1$ and $\beta_i = 0$ for $i \in H$, then we get the condition $a_i > 1/p_i$ as sufficient in order that series (2) be absolutely convergent.

In case $p_1 = \ldots = p_n = p$, the condition $a_i > 1/p_i$ is equivalent to condition (6.7) in [4], where

$$a_i > \frac{p(\beta_i + 1 - \gamma) + \gamma}{\gamma p}.$$

If $\beta_i = 0$, $\gamma = 1$ and $p_i = p$, then $a_i > 1/p$.

4. We shall still investigate absolute convergence of Fourier series in exponential form, i.e. with respect to the system of functions $e^{\pi i(m_1 x_1 + \ldots + m_n x_n)}$.

The Fourier series of an integrable, 2-periodic in each variable function f with respect to this system is given by

(3) $$f(x_1, \ldots, x_n) \sim \sum_{m_n=0}^{\infty} \ldots \sum_{m_1=0}^{\infty} a_{m_1 \cdots m_n}(f) e^{\pi i(m_1 x_1 + \ldots + m_n x_n)},$$

where

(4) $$a_{m_1 \cdots m_n}(f) = \int_0^2 \ldots \int_0^2 f(x_1, \ldots, x_n) e^{-\pi i(m_1 x_1 + \ldots + m_n x_n)} dx_1 \ldots dx_n.$$

The above system is evidently orthonormal in

$$Q = (0, 2) \times \ldots \times (0, 2) \qquad (n\text{-times}).$$

There will be investigated convergence of the series

(5) $$\sum_{m_n=0}^{\infty} \ldots \sum_{m_1=0}^{\infty} (|m_1|+1)^{\beta_1} \ldots (|m_n|+1)^{\beta_n} |a_{m_1 \ldots m_n}(f)|^{\gamma}, \qquad \gamma \geqslant 1.$$

This series will be shown to be majorized by series (1). First, we prove two lemmas.

LEMMA 2. *Let*

$$a^A_{m_1 \ldots m_n}(f) = \int_0^2 \ldots \int_0^2 f(x_1, \ldots, x_n) \prod_{i \in A} \cos m_i \pi x_i \prod_{j \notin A} \sin m_j \pi x_j \, dx_1 \ldots dx_n.$$

Then

$$a_{m_1 \ldots m_n}(f) = \sum_{A \subset E} \varepsilon_A a^A_{|m_1|, \ldots, |m_n|}(f),$$

where

$$|\varepsilon_A| = 1, \qquad E = \{1, 2, \ldots, n\}.$$

Proof (by induction). If $n = 1$, $E = \{1\}$, we have

$$a_m(f) = \int_0^2 f(x) e^{-im\pi x} dx = \int_0^2 f(x) \cos m\pi x \, dx - i \int_0^2 f(x) \sin m\pi x \, dx.$$

Hence we have for $m \geqslant 0$,

$$a_m(f) = a^{(1)}_m(f) - i a^{\emptyset}_m(f)$$

and for $m < 0$,

$$a_m(f) = a^{(1)}_{-m}(f) + i a^{\emptyset}_{-m}(f).$$

Thus, we have in each case

$$a_m(f) = \sum_{A\subset E} \varepsilon_A a^A_{|m_1|,\ldots,|m_n|}(f),$$

where

$$\varepsilon_1 = 1, \quad \varepsilon_\emptyset = -i \quad \text{if } m \geqslant 0, \quad \varepsilon_\emptyset = i \quad \text{if } m < 0.$$

Now, let us suppose the lemma to be true for n; then we prove it to be true for $n+1$.

Let $E = \{1, \ldots, n, n+1\}$; then denoting

$$f(x_1, \ldots, x_n, x_{n+1}) = g_{x_{n+1}}(x_1, \ldots, x_n),$$

we get

$$a_{m_1,\ldots,m_n,m_{n+1}}(f) = \int_0^2 a_{m_1\ldots m_n}(g_{x_{n+1}}) e^{-im_{n+1}x_{n+1}\pi} dx_{n+1}.$$

Let $h(x_{n+1}) = a_{m_1,\ldots,m_n}(g_{x_{n+1}})$. Then

$$a_{m_1,\ldots,m_n,m_{n+1}}(f) = \int_0^2 h(x_{n+1}) e^{-im_{n+1}x_{n+1}\pi} dx_{n+1}$$
$$= a_{m_{n+1}}(h) = a^{(n+1)}_{|m_{n+1}|}(h) \mp i a^{\emptyset}_{|m_{n+1}|}(h),$$

where we take the sign $-$ if $m_{n+1} \geqslant 0$, and the sign $+$ if $m_{n+1} < 0$. Then

$$a_{m_{n+1}}(h) = \int_0^2 h(x_{n+1}) \cos m_{n+1}\pi x_{n+1} dx_{n+1} -$$
$$- i \int_0^2 h(x_{n+1}) \sin m_{n+1}\pi x_{n+1} dx_{n+1}$$
$$= \int_0^2 a_{m_1\ldots m_n}(g_{x_{n+1}}) \cos m_{n+1}\pi x_{n+1} dx_{n+1} -$$
$$- i \int_0^2 a_{m_1\ldots m_n}(g_{x_{n+1}}) \sin m_{n+1}\pi x_{n+1} dx_{n+1}$$
$$= \int_0^2 \sum_{A\subset E_1} \varepsilon_1 a^A_{|m_1|\ldots|m_n|}(g_{x_{n+1}}) \cos m_{n+1}\pi x_{n+1} dx_{n+1} -$$
$$- i \int_0^2 \sum_{A\subset E_1} \varepsilon_A a^A_{|m_1|\ldots|m_n|}(g_{x_{n+1}}) \sin m_{n+1}\pi x_{n+1} dx_{n+1},$$

where $E_1 = \{1, 2, \ldots, n\}$.

Let us take $\varepsilon'_A = \varepsilon_A$ if $|m_{n+1}| \in A$, $\varepsilon'_A = \mp i\varepsilon_A$ if $|m_{n+1}| \notin A$ (depending on the sign of m_{n+1}). Then

$$a_{m_1,\ldots,m_n,m_{n+1}}(f) = a_{m_{n+1}}(h)$$

$$= \sum_{A\subset E_1} \varepsilon_A \int_0^2 \Big\{ \int_0^2 \ldots \int_0^2 f(x_1, \ldots, x_n, x_{n+1}) \prod_{i\in A} \cos m_i \pi x_i \times$$

$$\times \prod_{j\notin A} \sin m_j \pi x_j \, dx_1 \ldots dx_n \Big\} \cos m_{n+1} \pi x_{n+1} \, dx_{n+1} -$$

$$-i \sum_{A\subset E_1} \varepsilon_A \int_0^2 \Big\{ \int_0^2 \ldots \int_0^2 f(x_1, \ldots, x_n, x_{n+1}) \prod_{i\in A} \cos m_i \pi x_i \times$$

$$\times \prod_{j\notin A} \sin m_j \pi x_j \, dx_1 \ldots dx_n \Big\} \sin m_{n+1} \pi x_{n+1} \, dx_{n+1}$$

$$= \sum_{A\subset E} \varepsilon'_A \underbrace{\int_0^2 \ldots \int_0^2 \int_0^2}_{(n+1)\text{-times}} f(x_1, \ldots, x_n, x_{n+1}) \times$$

$$\times \prod_{i\in A} \cos m_i \pi x_i \cdot \prod_{j\notin A} \sin m_j \pi x_j \, dx_1 \ldots dx_n \, dx_{n+1}$$

$$= \sum_{A\subset E} \varepsilon'_A a^A_{m_1,\ldots,m_n,m_{n+1}}(f) = \sum_{A\subset E} \varepsilon_A a^A_{|m_1|\ldots|m_n|,|m_{n+1}|}(f).$$

COROLLARY. *From Lemma 2 follows immediately that*

$$|a_{m_1\ldots m_n}(f)| \leqslant \sum_{A\subset E} |a^A_{|m_1|\ldots|m_n|}(f)| = \varrho_{|m_1|\ldots|m_n|}(f). \tag{6}$$

LEMMA 3. *If* $\gamma \geqslant 1$, *then*

$$|a_{m_1\ldots m_n}(f)|^\gamma \leqslant 2^{\gamma N} \varrho^{(\gamma)}_{|m_1|\ldots|m_n|}(f),$$

where

$$\varrho^{(\gamma)}_{m_1\ldots m_n}(f) = \sum_{A\subset E} |a^A_{m_1\ldots m_n}(f)|^\gamma$$

and N denotes the number of subsets A of the set E.

Proof. By Lemma 2 and inequality (6), we have for $\gamma \geqslant 1$

$$|a_{m_1\ldots m_n}(f)|^\gamma \leqslant [\varrho_{|m_1|\ldots|m_n|}(f)]^\gamma = \Big(\sum_{A\subset E} |a^A_{|m_1|\ldots|m_n|}(f)|\Big)^\gamma$$

$$\leqslant 2^{\gamma N} \sum_{A\subset E} |a_{|m_1|\ldots|m_n|}(f)|^\gamma = 2^{\gamma N} \varrho^{(\gamma)}_{|m_1|\ldots|m_n|}(f).$$

THEOREM 3. *If* $f \in L^p$, $p = (p_1, p_2, \ldots, p_n)$, $\beta_1, \beta_2, \ldots, \beta_n \geqslant 0$ *and* $\gamma \geqslant 1$, *then*

$$\sum_{\substack{m_1,\ldots,m_n=0 \\ (m_1,\ldots,m_n)\neq 0}}^{\infty} (|m_1|+1)^{\beta_1}(|m_2|+1)^{\beta_2}\ldots(|m_n|+1)^{\beta_n}|a_{m_1\ldots m_n}(f)|^{\gamma}$$

$$\leqslant 2^n \sum_{0\neq H\subset E}\sum_{r_{k_s}=1}^{\infty}\ldots\sum_{r_{k_2}=1}^{\infty}\sum_{r_{k_1}=1}^{\infty} 2^{c(r)}[\omega_p^H(f;\ 2^{-r})]^{\gamma},$$

where

$$H = \{k_1, k_2, \ldots, k_s\}, \quad 2^{-r} = (2^{-r_1}, \ldots, 2^{-r_n}),$$

$$c(r) = c_1(r) - \tfrac{1}{2}\gamma\mu(H)$$

$$= \sum_{k=1}^{n}\beta_k r_k + \sum_{s=1}^{n-1}(r_s-1)\frac{q_s-1}{q_s} + (r_n-1)\frac{q_n-\gamma}{q_n} - \frac{\gamma}{2}\mu(H) + 2^n - 1.$$

Proof. Theorem 3 follows evidently from Lemma 3 and Theorem 1.

THEOREM 4 *If* $f\in L^p$, $p = (p_1, \ldots, p_n)$, $\beta_1, \ldots, \beta_n \geqslant 0$, $\gamma \geqslant 1$, $f \in \mathrm{Lip}_p^H a$, $a = (a_1, \ldots, a_n)$, $a_i > 0$ *for* $i \in H$, $H = \{k_1, k_2, \ldots, k_s\}$, *then series* (5) *is convergent for*

$$\gamma a_i > \beta_i + 1/p_i, \quad i \in H,\ i < n,$$

$$\gamma a_n > \beta_n + 1 - \gamma(1 - 1/p_n).$$

Proof. Similarly as Theorem 3, also Theorem 4 follows evidently from Lemma 3 and Theorem 2.

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