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A note on exhaustive measures

Abstract. The purpose of this note is to extend the Diestel–Faires characterization of Banach spaces not containing l^∞ for the case of locally convex spaces.

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1. In this paper $\mathscr{R}$ denotes a σ-ring of sets, X a locally convex Hausdorff topological vector space (= locally convex space), $\mu\colon \mathscr{R}\to X$ a finitely additive set function (= measure). A measure μ is said to be *exhaustive* iff for each sequence $(E_n) \subset \mathscr{R}$ of pairwise disjoint sets $\mu(E_n)\to 0$ $(n\to\infty)$. A measure μ is *bounded* iff it is weakly exhaustive, i.e. $\mu\colon \mathscr{R}\to X_\sigma$ is exhaustive ($X_\sigma = \{X, \sigma(X, X')\}$). We denote by c_0 the space of all sequences tending to 0, by l^∞ the space of all bounded sequences, by l_0 its subspace consisting of all sequences which take only finitely many values; l^∞, l_0 and c_0 are equipped with the supremum norm. $e^{(n)}$, $n = 1, 2, \ldots$ denotes the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with the one in the n-th position. Let two linear topologies α and β on a vector space X be given. We say that β is *α-polar* (resp. *sequentially α-polar*) if β has a base of α-closed (resp. sequentially α-closed) neighbourhoods at 0. $\boldsymbol{N}$ is a set of natural numbers 1, 2,..., $\mathscr{P}(\boldsymbol{N})$ denotes its power set.

2. Let X be a Banach space. The well-known result of Pełczyński ([2], Theorem 5) is equivalent to the statement:

(1) *Every bounded X-valued measure on a ring of sets is exhaustive iff X does not contain any isomorphic copy of c_0.*

The situation changes completely when considering measures on σ-rings. For instance, we can take $X = c_0$ since it was first proved by Diestel [3] that:

(2) *If X is separable and $\mu\colon \mathscr{R}\to X$ is bounded, then μ is exhaustive.*

It was then natural to ask a full characterization, similar to that one of Pełczyński, in the case of measures defined on σ-rings. The solution was given in [4] by Diestel and Faires:

(3) *Every bounded X-valued measure on a σ-ring of sets is exhaustive iff X does not contain any isomorphic copy of l^∞.*

In the same time, in a different context, Bennett and Kalton ([1], Theorem 1) proved that:

(4) *Every X-valued measure on a σ-ring of sets σ-additive for some locally convex Hausdorff topology coarser than the initial one on X is σ-additive iff X does not contain any isomorphic copy of l^∞.*

And the assumption on X in their theorem is weaker. They assume X to be merely a fully complete locally convex space. On the other hand the author proved that the first theorem of Diestel (2) and some similar results hold in locally convex spaces ([10], 4.6, 4.7, see also [6]). Also Tumarkin [16] showed that Pełczyński's theorem holds in sequentially complete locally convex spaces. These facts suggest that the theorem of Diestel and Faires (and Bennett and Kalton) should hold in a more general setting.

In fact, we have:

(5) *Let X be a sequentially complete locally convex space, $\mu\colon \mathscr{R}\to X$ a bounded measure. If μ is not exhaustive, then X contains an isomorphic copy of l^∞.*

Proof. If μ is not exhaustive we find a sequence $(E_n)\subset \mathscr{R}$ of pairwise disjoint sets and a continuous seminorm p such that $p(\mu(E_n)) = p(x_n) \geqslant 1$. Thus we can find a sequence of equicontinuous linear functionals (f_n) such that $|f_n(x_n)|\geqslant 1$ for all n. Define $S\colon l_0\to X$ as "the integral" of "simple functions" $a\in l_0$ by:

If $a = (a_n) = \sum_{i=1}^{m} a^i\chi(\Delta_i)$, *where* $\Delta_i\subset \boldsymbol{N}$, $\Delta_i = \{n\colon a_n = a^i\}$, $i = 1, 2, \ldots, m$, *then*

$$S(a) = \sum_{i=1}^{\infty} a^i\mu(\Delta_i) = \sum_{i=1}^{\infty} a^i\mu\Big(\bigcup_{n\in\Delta_i} E_n\Big).$$

The measure μ being bounded, is weakly exhaustive. Consequently, for each $x'\in X'$, $x'\circ\mu$ is exhaustive ($=$ bounded) scalar valued measure. Thus, clearly, $S\colon l_0\to X$ is bounded, thus continuous. X being sequentially complete, we can extend S to $V\colon l^\infty\to X$. Define also a continuous linear mapping $U\colon X\to l^\infty$ by $U(x) = \{f_n(x)\}$. Consider $UV\colon l^\infty\to l^\infty$; it is continuous. Now we can apply without change the reasoning of Bennett and Kalton used in the proof of Theorem 1 in [1]. $\|UV(e^{(n)})\| = \|U(x_n)\| = |f_n(x_n)|\geqslant 1$ for all n. Consequently, $\sum_{n=1}^{\infty} UV(e^{(n)})$ cannot converge weakly subseries, and it follows that UV is not weakly compact. By

Corollary 1.4 of [14], UV is an isomorphism on some subspace H of l^∞ which is itself isomorphic to l^∞. Hence $V(H)$ is isomorphic to l^∞.

As corollaries we get:

(6) *Let (X, β) be a sequentially complete locally convex space, a a linear topology on X. Suppose that X does not contain any isomorphic copy of l^∞ and β is sequentially a-polar. If $\mu: \mathscr{R}\to(X, a)$ is a σ-additive measure, then $\mu: \mathscr{R}\to(X, \beta)$ is σ-additive.*

In fact, we can suppose that $\mathscr{R} = \mathscr{P}(\boldsymbol{N})$. By [9], Theorem 3, $\mu: \mathscr{P}(\boldsymbol{N})\to(X, \beta)$ is bounded, hence by (5) exhaustive. It is σ-additive by the routine argument.

(7) *Let X be a sequentially complete locally convex space. The following are equivalent:*

(a) *Every X-valued bounded measure on a σ-ring is exhaustive.*

(b) *X does not contain any isomorphic copy of l^∞.*

Proof. Let Y be a subspace of X isomorphic to l^∞. Denote by j the isomorphism from l^∞ onto Y. We define a bounded additive set function on a σ-ring which is not exhaustive. Let $\Delta \subset \boldsymbol{N}$ and put $m(\Delta) = \chi(\Delta) \in l^\infty$, and then $\mu(\Delta) = jm(\Delta)$. Clearly, $\mu: \mathscr{P}(\boldsymbol{N})\to X$ is bounded non-exhaustive.

We were interested in the weak exhaustivity (= boundedness). In that case the strong restrictions, as e.g. full completness, as we have seen are not necessary. On the other hand, Bennett and Kalton using the full completness obtained the theorem in a sense much deeper as they assume the σ-additivity in an arbitrary locally convex Hausdorff topology coarser than the initial one only (4). Changing their proof slightly (as in (5)) one obtains the following assertion on exhaustive measures:

(8) *Let X be a fully complete locally convex space, a an arbitrary locally convex Hausdorff topology coarser than the initial one on X. The following are equivalent:*

(a) *Every X-valued measure on a σ-ring of sets bounded in (X, a) is exhaustive for the initial topology.*

(b) *X contains no isomorphic copy of l^∞.*

Now the full completness is essential as the closed graph theorem (for the domain space which is merely barrelled) is used. Let us still notice that if $\mu: \mathscr{R}\to(X, a)$ is bounded, then by [6], 2.7, it is bounded in X and (8) reduces to (7). Though one does not use the closed graph theorem explicitely, the full completness is still essential (cf. [6], loc. cit).

A short examination of the proof of (5) shows that the result of Diestel and Faires is in fact the theorem on linear mappings from l^∞. More generally, we can place us in the setting of operators acting from $C(K)$ spaces, where $C(K)$ is a space of continuous real-valued functions on a compact Hausdorff space with the supremum norm (l^∞ is a $C(K)$ space since it can be identified with the space of continuous functions on the Stone–Čech compactification of N). Such mappings were extensively studied in connection with the vector-valued form of Riesz representation theorem and various analogs of the notion of exhaustivity for vector measures were formulated (see [5], [7], [8], [12], [13], [15]).

Following Kalton [8], a linear map $V\colon C(K)\to X$ is said to be *exhaustive* iff:

(+) $V(f_n)\to 0$ *for any sequence* $(f_n)\subset C(K)$ *such that* $f_n \geqslant 0$ *and* $\sup_t \sum_{n=1}^{\infty} f_n(t) < \infty$.

If X is complete topological vector space, then it was shown in [15] and [8] that the exhaustivity is a necessary and sufficient condition for V to posess a regular integral extension to the space of bounded Borel functions. In [15], Theorem 3.3, Thomas gives many, equivalent to (+), necessary and sufficient conditions for the existence of such extension. Using this theorem (precisely conditions (7) and (9)) we can easily prove another condition equivalent to (+) which is still more close to the notion of exhaustivity used for measures:

(++) $V(f_n)\to 0$ *for any sequence* $(f_n)\subset C(K)$ *such that* $f_n\geqslant 0$, $\sup_t f_n(t)\leqslant 1$ *and* f_n *"are disjoint" i.e.* $\operatorname{supp} f_n \cap \operatorname{supp} f_m = \emptyset$ *for* $n\neq m$.

An operator V satisfying (++) will be still said to be *exhaustive.*

Now we can formulate the following analog of (7) for linear mappings:

Let X be a locally convex space. The following are equivalent:

(a) *Every linear bounded map on a space l^∞ to X is exhaustive.*

(b) *X contains no isomorphic copy of l^∞.*

For (a) $\Rightarrow$ (b) we define $U\colon X\to l^\infty$ as in (5), conclude that UV cannot be weakly compact, and finish as in (5).

For (b) $\Rightarrow$ (a) we take the identity on l^∞.

The theorem extends for arbitrary $C(K)$ spaces provided K is a σ-Stonian compact Hausdorff space.

Postscriptum. After the initial preparation of this note I could read a preprint of [9]. Kalton mentions therein without proof that what was proved above holds true, but restricts himself to the general non-locally convex case. He obtains various very interesting and deep results, however, they do not cover those given here completely. Thus I think that the present note is still of certain expository interest.

Added in proof. Recently L. Drewnowski and the author obtained in [17], [18] and [19] generalizations of results presented here and of those contained in [19].

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