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On a mixed problem for the biharmonic equation

The paper presents a construction of the Green function and the solution of the boundary problem for the biharmonic equation

$$(1) \qquad \Delta^2 u(x, y, z) = 0$$

in the region $D = \{(x, y, z): x > 0, -\infty < y < \infty, z > 0\}$ with the conditions

$$(2) \qquad \lim u(x, y, z) = f_1(x_0, y_0) \quad \text{when } (x, y, z) \to (x_0, y_0, 0^+), x_0 > 0,$$

$$(3) \qquad \lim \Delta u(x, y, z) = f_2(x_0, y_0) \quad \text{when } (x, y, z) \to (x_0, y_0, 0^+), x_0 > 0,$$

$$(4) \qquad \lim D_x u(x, y, z) = f_3(y_0, z_0) \quad \text{when } (x, y, z) \to (0^+, y_0, z_0), z_0 > 0,$$

$$(5) \qquad \lim D_x \Delta u(x, y, z) = f_4(y_0, z_0) \quad \text{when } (x, y, z) \to (0^+, y_0, z_0), z_0 > 0.$$

For the functions f_i $(i = 1, 2, 3, 4)$ we assume that

(6) f_i $(i = 1, 2, 3)$ are absolutely integrable,

(7) the integral

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} |f_4(t, w)| (w^2 + t^2)^{1/2} \, dt dw$$

is convergent,

(8) f_i $(i = 1, 2, 3, 4)$ are bounded and continuous.

Let us denote the boundary problem (1), (2), (3), (4) and (5) by (B.R.N).

Let us consider the following arrangements of points: $X(x, y, z) \in D$, $X_1(-x, y, z)$, $X_2(-x, y, -z)$ and $X_3(x, y, -z)$. Let $Y(s, t, w)$ be a point of E_3. Let us write

$$r_1^2 = (s-x)^2 + (t-y)^2 + (w-z)^2, \qquad r_2^2 = (s+x)^2 + (y-t)^2 + (w-z)^2,$$
$$r_3^2 = (s+x)^2 + (t-y)^2 + (w+z)^2, \qquad r_4^2 = (s-x)^2 + (t-y)^2 + (w+z)^2.$$

Let us denote two half-planes by S_1 and S_2

$$S_1 = \{x, y, z): x > 0, |y| < \infty, z = 0\},$$
$$S_2 = \{x, y, z): x = 0, |y| < \infty, z > 0\}.$$

THEOREM 1. *The Green function for the problem (B.R.N) with the pole X is of the form*

$$G(X, Y) = r_1 + H(X, Y), \tag{9}$$

where

$$H(X, Y) = r_2 - r_4 - r_3.$$

Proof. The function $G(X, Y)$ is biharmonic as a function of Y (1). It is easy to prove that

$$\Delta_Y G(X, Y) = 2(r_1^{-1} - r_4^{-1} + r_2^{-1} - r_3^{-1}). \tag{10}$$

We shall now prove that the function $G(X, Y)$ satisfies the boundary conditions

$$G(X, Y)|_{w=0} = \Delta_Y G(X, Y)|_{w=0} = 0 \quad \text{for } Y \in S_1 \tag{a}$$

and

$$D_n G(X, Y)|_{s=0} = D_n \Delta_Y G(X, Y)|_{s=0} = 0 \quad \text{for } Y \in S_2. \tag{b}$$

For $Y \in S_1$ we have $r_1 = r_4$ and $r_2 = r_3$. Basing on (10) we obtain (a). For $Y \in S_2$ we have $r_1 = r_2$ and $r_4 = r_3$. Notice that

$$\begin{aligned} D_n G(X, Y)|_{s=0} &= D_s G(X, Y)|_{s=0} \\ &= [(s-x)r_1^{-1} - (s-x)r_4^{-1} + (s+x)r_2^{-1} - (s+x)r_3^{-1}]|_{s=0} = 0, \\ D_n \Delta_Y G(X, Y)|_{s=0} &= D_s \Delta_Y G(X, Y)|_{s=0} \\ &= -[(s-x)r_1^{-3} + (s+x)r_2^{-3} - (x+s)r_3^{-3} - (s-x)r_4^{-3}]|_{s=0} \\ &= 0. \end{aligned}$$

Thus (b) is satisfied. Moreover,

$$\lim \Delta_Y G(X, Y) = 0 \quad \text{when } OY \to \infty \tag{c}$$

for an arbitrary fixed X.

Applying the fundamental formula for the biharmonic equation (1)

$$\iiint_D (u\Delta^2 v - v\Delta^2 u)\, dV + \iint_{\partial D} (\Delta u D_n v - v D_n \Delta u + u D_n \Delta v - \Delta v D_n u)\, dS = 0$$

to the function $v = G(X, Y)$ presented by formula (9), and taking into consideration boundary conditions (a), (b) for the function $G(X, Y)$, and boundary conditions (2), (3), (4) and (5) for the function $u(X)$ we

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(1) M. Krzyżański, *Partial differential equations of second order*, vol. II, p. 234, Warszawa 1971.

obtain

$$(11)\quad u(X) = \frac{1}{8\pi}\int_0^\infty\int_{-\infty}^\infty f_1(s,t)D_w\Delta_Y G(X,Y)\big|_{w=0}\,dsdt + \frac{1}{8\pi}\int_0^\infty\int_{-\infty}^\infty f_2(s,t)D_w G(X,Y)\big|_{w=0}\,dsdt + \frac{-1}{8\pi}\int_{-\infty}^\infty\int_0^\infty f_3(t,w)\Delta_Y G(X,Y)\big|_{s=0}\,dtdw + \frac{-1}{8\pi}\int_{-\infty}^\infty\int_0^\infty f_4(t,w)G(X,Y)\big|_{s=0}\,dtdw.$$

Let us prove now that the function given by formula (11) is the solution of the problem (B.R.N).

Let

$$R_1^2 = (x-s)^2+(y-t)^2+z^2, \qquad R_2^2 = (x+s)^2+(y-t)^2+z^2,$$

$$R_3^2 = x^2+(y-t)^2+(z-w)^2, \qquad R_4^2 = x^2+(y-t)^2+(z+w)^2,$$

and

$$N_1(X,s,t) = zR_1^{-3}, \qquad N_2(X,s,t) = zR_2^{-3},$$

$$N_3(X,s,t) = zR_1^{-1}, \qquad N_4(X,s,t) = zR_2^{-1},$$

and

$$M_1(X,t,w) = R_3^{-1}, \qquad M_2(X,t,w) = R_4^{-1},$$

$$M_3(X,t,w) = R_3, \qquad M_4(X,t,w) = R_4.$$

We have then

$$(12)\quad D_w\Delta_Y G(X,Y)\big|_{w=0} = 4zR_1^{-3}+4zR_2^{-3} = 4[N_1(X,s,t)+N_2(X,s,t)],$$

$$(13)\quad D_w G(X,Y)\big|_{w=0} = -2zR_1^{-1}-2zR_2^{-1} = -2[N_3(X,s,t)+N_4(X,s,t)],$$

$$(14)\quad \Delta_Y G(X,Y)\big|_{s=0} = 4R_3^{-1}-4R_4^{-1} = 4[M_1(X,t,w)-M_2(X,t,w)],$$

$$(15)\quad G(X,Y)\big|_{s=0} = 2R_3-2R_4 = 2[M_3(X,t,w)-M_4(X,t,w)].$$

Substituting expressions (12), (13), (14) and (15) by (11), we obtain

$$(11a)\quad u(X) = \sum_{i=1}^4 I_i(X)+\sum_{i=1}^4 J_i(X),$$

where

$$I_i(X) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f_1(s,t)N_i(X,s,t)\,ds\,dt, \qquad i = 1, 2,$$

$$I_i(X) = \frac{-1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f_2(s,t)N_i(X,s,t)\,ds\,dt, \qquad i = 3, 4,$$

$$J_i(X) = (-1)^i\frac{1}{2\pi}\int_{-\infty}^\infty \int_0^\infty f_3(t,w)M_i(X,t,w)\,dt\,dw, \qquad i = 1, 2,$$

$$J_i(X) = (-1)^i\frac{1}{4\pi}\int_{-\infty}^\infty \int_0^\infty f_4(t,w)\,M_i(X,t,w)\,dt\,dw, \qquad i = 3, 4.$$

Let

$$I_{ij}^{pqr}(X) = \int_0^\infty \int_{-\infty}^\infty f_i(s,t)D_{x^py^qz^r}N_j(X,s,t)\,ds\,dt,$$

where $p, q, r = 0, 1, 2, 3, 4$; $i = 1, 2$; $j = 1, 2, 3, 4$.

Let

$$J_{3j}^{pqr}(X) = \int_{-\infty}^\infty \int_0^\infty f_3(t,w)D_{x^py^qz^r}M_j(X,t,w)\,dt\,dw,$$

where $p, r, q = 0, 1, 2, 3, 4$; $j = 1, 2$, and

$$J_{4j}^{pqr}(X) = \int_{-\infty}^\infty \int_0^\infty f_4(t,w)D_{x^py^qz^r}R_j\,dt\,dw,$$

where $p, q, r = 0, 1, 2, 4, 3$; $j = 4, 3$.

We shall consider two perpendicular parallelepipeds

$$W_1 = \langle A_1, B_1\rangle \times \langle A_2, B_2\rangle \times \langle A_3, B_3\rangle,$$
$$W_2 = \langle C_1, D_1\rangle \times \langle C_2, D_2\rangle \times \langle C_3, D_3\rangle,$$

where A_i $(i = 1, 2)$, B_i $(i = 1, 2, 3)$, D_i $(i = 1, 2, 3)$, C_i $(i = 2, 3)$ are arbitrary constants and A_3, C_1 are arbitrary positive constants.

Let

$$Z(t, w, \varrho) = \langle (t^2+w^2) > \varrho^2, w > 0\rangle,$$

$$Z_0(s, t, \varrho) = \langle (s^2+t^2) > \varrho^2, s > 0\rangle.$$

LEMMA 1. *Let the functions f_i $(i = 1, 2, 3, 4)$ satisfy assumptions* (6) *and* (7). *Then the integrals $I_{ij}^{pqr}(X)$ are uniformly convergent for $X \in W_1$ and the integrals $J_{ij}^{pqr}(X)$ are uniformly convergent for $X \in W_2$.*

Proof. We shall prove that the integrals I_{11}^{pqr} and J_{43}^{pqr} are uniformly convergent. The proof for the remaining integrals is the same.

The proof for the integral J_{43}^{pqr}. We have

$$D_{x^p y^q z^r} R_3 = P\left(\frac{x^{a_1}}{R_3^{a_1+b_1}}, \frac{(y-t)^{a_2}}{R_3^{a_2+b_2}}, \frac{(z-w)^{a_3}}{R_3^{a_3+b_3}}\right),$$

where $P(t_1, t_2, t_3)$ is the polynomial of the n-th degree ($n \leqslant 16$) and a_i, b_i are non-negative integers and $p+q+r > 0$.

In view of the triangle inequality we have

$$(*) \quad \bigvee_{R_0} \bigwedge_{(t,w)} \bigwedge_{(x,y,z)\in W_2} \left[t^2+w^2 > R_0^2,\ w > 0 \Rightarrow \frac{t^2+w^2}{4} \leqslant x^2+(t-y)^2+ +(z-w)^2 \leqslant 4(t^2+w^2)\right].$$

We have also

$$\left|\frac{x^{a_1}}{R_3^{a_1}}\right| \leqslant 1, \quad \left|\frac{(y-t)^{a_2}}{R_3^{a_2}}\right| \leqslant 1, \quad \left|\frac{(z-w)^{a_3}}{R_3^{a_3}}\right| \leqslant 1.$$

By (*) and from the last inequalities we obtain the estimate

$$\left|\iint_Z f_4(t, w) D_{x^p y^q z^r} R_3\, dt dw\right| \leqslant 2C \iint_Z |f_4(t, w)|(w^2+t^2)^{1/2} dt dw$$

(C a positive constant) which holds true for $\varrho > R_0$ (where R_0 is any positive constant) and every $X \in W_2$. Let ε be an arbitrary positive number. Assumption (7) shows that there exists a number R_0 such that

$$\iint_Z |f_4(t, w)|(w^2+t^2)^{1/2} dt dw < \varepsilon$$

holds true for every $\varrho > R_0$ and for every $X \in W_2$.

We will prove now that the integral $I_{11}^{pqr}(X)$ is uniformly convergent for $X \in W_1$. Let us notice that

$$D_{x^p y^q z^r} z R_1^{-3} = Q\left(z, \frac{(x-s)^{c_1}}{R_1^{3+c_1}}, \frac{(y-t)^{c_2}}{R_1^{3+c_2}}, \frac{z^{c_3}}{R_1^{3+c_3}}\right),$$

where $Q(t_1, t_2, t_3, t_4)$ is the polynomial of the n-th degree ($n \leqslant 16$).

We can also see that

$$|zR_1^{-3}|, \quad \left|\frac{(x-s)^{c_1}}{R_1^{c_1}}\right|, \quad \left|\frac{(y-t)^{c_2}}{R_1^{c_2}}\right|, \quad \left|\frac{z^{c_3}}{R_1^{c_3}}\right|$$

are not greater than 1.

Now, from the last inequalities and by inequality similar to (*), we get

$$\Big|\iint_{Z_0} f_1(s,t) D_{x^p y^q z^r} z R_1^{-3}\, ds\, dt\Big| \leqslant \frac{C}{2}\iint_{Z_0} |f_1(s,t)|\, ds\, dt$$

(C a positive constant) holds true for $\varrho > R_0$ (R_0 any positive constant) and every $X \epsilon W_1$.

Let $\varepsilon > 0$. From assumption (6) it follows that there is a number $R_0 > 0$ such that

$$\iint_{Z_0} |f_1(s,t)|\, ds\, dt < \varepsilon$$

for every $\varrho > R_0$ and for every $X \epsilon W_1$.

From Lemma 1 we get

LEMMA 2. *If the assumptions of Lemma* 1 *are satisfied, then the derivatives* $D_{x^p y^q z^r} I_i(X)$ *exist for* $z > 0$ *and we may interchange the order of the differentiation and the integration:*

$$D_{x^p y^q z^r} I_i(X) = a_l \int_0^\infty \int_{-\infty}^\infty f_l(s,t) D_{x^p y^q z^r} N_i(X, s, t)\, ds\, dt,$$

where $i = 1, 2, 3, 4$; $l = 1, 2$; a_l *are some constants and there exist the derivatives* $D_{x^p y^q z^r}\, J_i(X)$ *for* $x > 0$ *and*

$$D_{x^p y^q z^r}\, J_i(X) = \beta_l \int_{-\infty}^\infty \int_0^\infty f_l(t, w) D_{x^p y^q z^r} M_i(X, t, w)\, dt\, dw$$

where $i = 1, 2, 3, 4$; $l = 3, 4$; β_l *are some constants.*

LEMMA 3. *Let the functions* f_i ($i = 1, 2, 3, 4$) *satisfy assumptions* (6) *and* (7). *Then the function* $u(X)$, *given by formula* (11), *satisfies equation* (1) *in the region* D.

With the uniform convergence of suitable integrals and since $G(X, Y)$ is a symmetric and biharmonic function, we obtain

$$\begin{aligned} \Delta^2 u(X) = {} & \frac{1}{8\pi}\int_0^\infty \int_{-\infty}^\infty f_1(s,t) D_w \Delta_Y^3 G(X, Y)\big|_{w=0}\, ds\, dt + \\ & + \frac{1}{8\pi}\int_0^\infty \int_{-\infty}^\infty f_2(s,t) D_w \Delta_Y^2 G(X, Y)\big|_{w=0}\, ds\, dt + \\ & + \frac{-1}{8\pi}\int_{-\infty}^\infty \int_0^\infty f_3(t,w) \Delta_Y^3 G(X, Y)\big|_{s=0}\, dt\, dw + \\ & + \frac{-1}{8\pi}\int_{-\infty}^\infty \int_0^\infty f_4(t,w) \Delta_Y^2 G(X, Y)\big|_{s=0}\, dt\, dw = 0. \end{aligned}$$

In view of (11) and from the symmetry of $G(X, Y)$ and by uniform convergence of some integrals we obtain

$$\Delta u(X) = \frac{1}{8\pi}\int_0^\infty \int_{-\infty}^\infty f_1(s, t) D_w \Delta_Y^2 G(X, Y)\big|_{w=0} ds\,dt +$$

$$+\frac{1}{8\pi}\int_0^\infty \int_{-\infty}^\infty f_2(s, t) D_w \Delta_Y G(X, Y)\big|_{w=0} ds\,dt +$$

$$+\frac{-1}{8\pi}\int_{-\infty}^\infty \int_0^\infty f_3(t, w)\, \Delta_Y^2 G(X, Y)\big|_{s=0} dt\,dw +$$

$$+\frac{-1}{8\pi}\int_{-\infty}^\infty \int_0^\infty f_4(t, w)\, \Delta_Y G(X, Y)\big|_{s=0} dt\,dw.$$

Since the function $G(X, Y)$ is biharmonic, from (12) we get

$$\Delta u(X) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f_2(s, t) z R_1^{-3} ds\,dt + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f_2(s, t) z R_2^{-3} ds\,dt + \tag{16}$$

$$+\frac{-1}{2\pi}\int_{-\infty}^\infty \int_0^\infty f_4(t, w) R_3^{-1} dt\,dw + \frac{1}{2\pi}\int_{-\infty}^\infty \int_0^\infty f_4(t, w) R_4^{-1} dt\,dw,$$

$$D_x u(X) = \frac{3}{2\pi}\int_0^\infty \int_{-\infty}^\infty f_1(s, t) z(s-x) R_1^{-5} ds\,dt + \tag{17}$$

$$+\frac{-3}{2\pi}\int_0^\infty \int_{-\infty}^\infty f_1(s, t) z(x+s) R_2^{-5} ds\,dt +$$

$$+\frac{-1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f_2(s, t) z(s-x) R_1^{-3} ds\,dt +$$

$$+\frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f_2(s, t) z(x+s) R_2^{-3} ds\,dt +$$

$$+\frac{1}{2\pi}\int_{-\infty}^\infty \int_0^\infty f_3(t, w) x R_3^{-3} dt\,dw + \frac{-1}{2\pi}\int_{-\infty}^\infty \int_0^\infty f_3(t, w) x R_4^{-3} dt\,dw +$$

$$+\frac{-1}{4\pi}\int_{-\infty}^\infty \int_0^\infty f_4(t, w) x R_3^{-1} dt\,dw + \frac{1}{4\pi}\int_{-\infty}^\infty \int_0^\infty f_4(t, w) x R_4^{-1} dt\,dw$$

and

$$(18)\qquad D_x \Delta u(X) = \frac{3}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f_2(s,t)z(s-x)R_1^{-5}\,ds\,dt + $$
$$+\frac{-3}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f_2(s,t)z(x+s)R_2^{-5}\,ds\,dt +$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} f_4(t,w)R_3^{-3}x\,dt\,dw + \frac{-1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} f_4(t,w)R_4^{-3}x\,dt\,dw.$$

We shall now prove that the function $u(X)$ defined by (11) satisfies condition (2). Let

$$\bar{f}_1(s,t) = \begin{cases} f_1(s,t) & \text{for } s>0,\ t>0,\\ 0 & \text{for } s\leqslant 0,\ t\leqslant 0.\end{cases}$$

Let

$$L(X) = L_1(X)+L_2(X),$$

where

$$L_1(X) = \frac{1}{2\pi}\iint_{E_2-K_\delta} [\bar{f}_1(s,t)-f_1(x_0,y_0)]zR_1^{-3}\,ds\,dt,$$
$$L_2(X) = \frac{1}{2\pi}\iint_{K_\delta} [\bar{f}_1(s,t)-f_1(x_0,y_0)]zR_1^{-3}\,ds\,dt$$

and

$$K_\delta = \{(s,t):\ (s-x_0)^2+(t-y_0)^2<\delta^2\}.$$

LEMMA 4.

$$(2\pi)^{-1}\iint_{E_2} f_1(x_0,y_0)zR_1^{-3}\,ds\,dt = f_1(x_0,y_0).$$

Proof. Let

$$J(X) = (2\pi)^{-1}\iint_{E_2} zR_1^{-3}\,ds\,dt.$$

Applying transformation $s = x+zt_1$, $t = y+zt_2$, $|J| = z^2$ and then introducing the polar coordinates, we get

$$J(X) = (2\pi)^{-1}\iint_{E_2}(1+t_1^2+t_2^2)^{-3/2}\,dt_1\,dt_2 = 1.$$

Multiplying the last quality by $f_1(x_0,y_0)$ we obtain

$$J(X)f_1(x_0,y_0) = f_1(x_0,y_0).$$

LEMMA 5. *Let the function $f_1(s,t)$ satisfy assumption* (8) *and let the integral*

$$\int_{\frac{\delta}{4}}^{\infty}\int_{0}^{2\pi} |\bar{f}_1(x+r\cos a,\, y+r\sin a)|\,dr\,da$$

exist for every point $(x,y)\in E_2$ and every $\delta>0$.

Then $\lim L_1(X) = \lim L_2(X) = 0$ *when* $(x,y,z)\to(x_0,y,0^+)$.

Proof. Let

$$Z_1 = \left\{(s,t):\ (s-x)^2+(t-y)^2 > \frac{\delta^2}{16} \wedge (x-x_0)^2+(y-y_0)^2 < \frac{\delta^2}{4}\right\}.$$

For the integral $L_1(X)$ we have the estimate

$$|L_1(X)| \leqslant \frac{1}{2\pi}\iint_{Z_1} |\bar{f}_1(s,t)-f_1(x_0,y_0)|\,zR_1^{-3}\,ds\,dt.$$

Applying the polar coordinates $s-x = r\cos a$, $t-y = r\sin a$, where $\delta/4 < r < \infty$, $0\leqslant a\leqslant 2\pi$, we have

$$|L_1(X)| \leqslant \frac{1}{2\pi}\int_{\frac{\delta}{4}}^{\infty}\int_{0}^{2\pi} |\bar{f}_1(x+r\cos a,\, y+r\sin a)|\,z(r^2+z^2)^{-3/2}r\,dr\,da +$$

$$+\frac{1}{2\pi}\int_{\frac{\delta}{4}}^{\infty}\int_{0}^{2\pi} |\bar{f}_1(x_0,y_0)|\,z(r^2+z^2)^{-3/2}r\,dr\,da.$$

Under assumptions of Lemma 5 the last two integrals are convergent to zero when $(x,y,z)\to(x_0,y_0,0^+)$. With Lemma 4 and assumption (8), for the integral $L_2(X)$, we have the estimates

$$|L_2(X)| \leqslant \frac{1}{2\pi}\iint_{K_\delta} |\bar{f}_1(s,t)-f_1(x_0,y_0)|\,zR_1^{-3}\,ds\,dt \leqslant \frac{\varepsilon}{2\pi}\iint_{E_2} zR_1^{-3}\,ds\,dt = \varepsilon.$$

LEMMA 6. *If the function $f_1(s,t)$ satisfies assumption* (8), *then* $\lim I_1(X) = f_1(x_0,y_0)$ *when* $(x,y,z)\to(x_0,y_0,0^+)$.

Proof. The integral $\bar{I}_1(X)$ may be presented in this form:

$$\bar{I}_1(X) = \frac{1}{2\pi}\iint_{E_2} \bar{f}_1(s,t)\,zR_1^{-3}\,ds\,dt = \frac{1}{2\pi}\iint_{E_2} [\bar{f}_1(s,t)-f_1(x_0,y_0)]\,zR_1^{-3}\,ds\,dt +$$

$$+\frac{1}{2\pi}\iint_{E_2} f_1(x_0,y_0)\,zR_1^{-3}\,ds\,dt.$$

From Lemma 4 we have

$$\bar{I}_1(X) - f_1(x_0, y_0) = L(X).$$

Now, from Lemma 5 we get Lemma 6.

Lemma 7. *With assumption* (8)

$$\lim I_2(X) = \lim I_4(X) = 0 \quad \textit{when } (x, y, z) \to (x_0, y_0, 0^+).$$

Proof. We shall prove that $I_2(X) \to 0$ when $(x, y, z) \to (x_0, y_0, 0^+)$. Let $x_0/2 < x < 2x_0$ and $x+s = u$. For the integral $I_2(X)$ we have then the estimates

$$|I_2(X)| \leqslant \frac{1}{2\pi} \int\limits_{\frac{x_0}{2}}^{\infty} \int\limits_{-\infty}^{\infty} |f_1(u-x, t)| z [u^2 + (y-t^2) + z^2]^{-3/2} du\, dt$$

$$\leqslant \frac{1}{2\pi} \iint\limits_{D_1} |f_1(u-x, t)| z [u^2 + (y-t)^2 + z^2]^{-3/2} du\, dt$$

$$\leqslant M \iint\limits_{D_1} z [u^2 + (y-t)^2 + z^2]^{-3/2} du\, dt,$$

where $D_1 = \{(u, t) \colon u^2 + t^2 > x_0^2/4\}$ and M — a positive constant.

Applying the transformation $u = r\cos a$, $y = t + r\sin a$, $-\pi/2 < a < \pi/2$, $x_0/2 < r < \infty$ we obtain

$$M \int\limits_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int\limits_{\frac{x_0}{2}}^{\infty} z (r^2 + z^2)^{-3/2} r\, dr\, da = \pi M z \frac{1}{\sqrt{\left(\frac{x_0^2}{2}\right)^2 + z^2}} \to 0 \quad \text{when } z \to 0^+.$$

The proof that $I_4(X) \to 0$ is similar to that of the proof for $I_1(X)$.

Lemma 8. *If the functions* $f_3(t, w)$ *and* $f_4(t, w)$ *are absolutely integrable and continuous, then*

$$\lim [J_2(X) - J_1(X)] = \lim [J_4(X) - J_3(X)] = 0 \quad \textit{when } (x, y, z) \to (x_0, y_0, 0^+).$$

Proof. The assertion of Lemma 8 follows as a consequence of continuity of the integrals $J_i(X)$ $(i = 1, 2, 3, 4)$ which are uniformly convergent in the set W_2.

From Lemmas 6, 7 and 8 and by (11a) follows

Lemma 9. *If the assumptions of Lemmas* 6, 7 *and* 8 *are satisfied and if* $\lim I_3(X) = 0$ *when* $(x, y, z) \to (x_0, y_0, 0^+)$, *then the function* $u(X)$ *given by formula* (11a) *satisfies condition* (2).

We will prove now that the function $u(X)$, given by formula (11a), satisfies the boundary condition (3).

LEMMA 10. *If the function $f_2(s, t)$ satisfies the assumptions of Lemmas 5 and 6, then*

$$\lim\left[\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f_2(s, t) z R_1^{-3}\, ds\, dt\right] = f_2(x_0, y_0) \quad \textit{when } (x, y, z)\to(x_0, y_0, 0^+).$$

Proof. Applying Lemmas 5 and 6 to the function $f_2(s, t)$ we obtain the thesis of Lemma 10.

LEMMA 11. *If the assumptions of Lemmas 7 and 8 are satisfied, then*

$$\lim\left[\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f_2(s, t) z R_2^{-3}\, ds\, dt\right] = 0$$

and

$$\lim\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} f_4(t, w) R_4^{-1}\, dt\, dw - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} f_4(t, w) R_3^{-1}\, dt\, dw\right] = 0$$

when $(x, y, z)\to(x_0, y_0, 0^+)$.

Proof. The proof of the first thesis is similar to the proof of Lemma 7 and the second one is similar to the proof of Lemma 8.

As a consequence of Lemmas 10 and 11 and formula (14) we obtain the following

LEMMA 12. *If the assumptions of Lemmas* 10 *and* 11 *are satisfied, then the function $u(X)$ given by formula* (11a), *satisfies the boundary conditions* (3).

The integrals given by (17) and (18) are of the same type as the integrals given by (11a) and (16). For these integrals we can prove lemmas similar to Lemmas 4, 5, 6, 7 and 8 and consequently we obtain

LEMMA 13. *If assumptions* (6), (7) *and* (8) *are satisfied, then the function $u(X)$, given by formula* (11a), *satisfies boundary conditions* (4) *and* (5).

From Lemmas 2, 3, 9, 12 and 13 we get

THEOREM 2. *If the assumptions of Lemmas* 2, 3, 9, 12 *and* 13 *are satisfied, then the function $u(X)$, defined by formula* (11) *or* (11a), *is the solution of the problem* (*B.R.N*).