Non-coercive mixed problem

In recent years there appeared many papers concerning the non-coercive problems of the form

\[ Lu - \int f \quad \text{in} \quad \Omega, \]
\[ Bu|_{\partial \Omega} = g, \]

where \( L \) is a differential operator of order \( m \), elliptic in the bounded domain \( \overline{\Omega} \subset \mathbb{R}^n \). For such problems it is possible to estimate the norm \( \| u \|_{s+m-\delta} \) (\( \delta > 0 \)) in the Sobolev space \( H^{s+m-\delta}(\Omega) \) by \( \| f \|_s \) and suitable norms of the functions \( g \) on the boundary \( \partial \Omega \). From these estimates various facts concerning solvability of problem (1) follow. Problems for which such estimates do not hold have also been studied. Many results have been obtained by the method of the theory of pseudo-differential operators.

As far as we know, no papers dealing with mixed problems for hyperbolic equations with the boundary conditions of non-coercive type have been published.

The theorem stated below should be treated as the first step in this direction.

In this paper we shall consider the following mixed problem

\[ u_{tt} - Lu = f(x, t), \quad t \in (0, T), \quad x \in \Omega, \]
\[ u_x|_{\partial \Omega \times [0, T]} = u_1(x, t), \]
\[ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \]

with the additional condition

\[ u|_{\Gamma \times [0, T]} = u_0(x'), \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), such that

1° the boundary of \( \Omega \) is a smooth manifold,

2° \( \Omega \) is a convex set with respect to \( x_1 \), i.e. every set \( \{x = (x_1, x'):\ x \in \Omega, \ x' = \text{const}\} \) is convex,
3° vector field $\partial/\partial x_1$ is tangent to $\partial \Omega$ along

$$\Gamma = \{ x : x \in \partial \Omega, \ x_1 = 0 \}, \ \dim \Gamma = n - 2. $$

We assume

$$Lu = \sum_{i,j} a_{ij}(x, t) u_{x_i x_j} + \sum_{i} a_i(x, t) u_{x_i} + b(x, t) u,$$

$$L : \mathcal{C}^\infty(R^{n+1}) \to \mathcal{C}^\infty(R^{n+1}),$$

$$\sum_{i,j} a_{ij} \xi_i \xi_j \geq c \sum_{i} \xi_i^2, \ (c > 0)$$

for every $(x, t) \in \Omega \times [0, T]$ and $\xi \in R^n \setminus \{0\}$, coefficients of the operator $L$ does not depend on $x_1$. The method of investigation of problem (2)-(3) is based on the well-known facts from the theory of well posed problems for hyperbolic equations (cf. [1]).

Let $\mathcal{H}_s^1(\Omega \times [0, T])$ be a subspace of the space $H_s(\Omega \times [0, T])$ ([2]) with the norm

$$\|u\|_{\mathcal{H}_s^1([0, T])} = \|u\|_{H_s([0, T])} + \|u\|_{\mathcal{H}_s^1([0, T])},$$

where $N = \{ x : x \in \Omega, \ x_1 = 0 \}$, $s > 1$, $\| \cdot \|_{H_s([0, T])}$ the norm in $H_s(\Omega \times [0, T])$, and let $\mathcal{H}_s^2(\Omega \times [0, T])$ be a subspace with the norm

$$\|u\|_{\mathcal{H}_s^2([0, T])} = \|u\|_{H_s([0, T])} + \|u\|_{\mathcal{H}_s^1([0, T])}.$$

**Theorem.** 1° If

$$f \in \mathcal{H}_s^{2-1}(\Omega \times [0, T]), \ \varphi \in \mathcal{H}_s^{2-1}(\Omega), \ u_0 \in H_s(\Gamma \times [0, T]), \ u_1 \in H_s^{1+1/2}(\partial \Omega \times [0, T]), \ s > 1$$

and conditions (2)-(3) are fulfilled, then $u \in \mathcal{H}_s^1(\Omega \times [0, T])$ and

$$\|u\|_{\mathcal{H}_s^1([0, T])} \leq C \left( \|f\|_{\mathcal{H}_s^{2-1}} \right. + \|\varphi\|_{\mathcal{H}_s^{2-1}} + \|\psi\|_{\mathcal{H}_s^{2-1}} + \|u_1\|_{H_s^{1+1/2}} + \|u_0\|_{H_s([0, T])},$$

where $C$ does not depend on $u$.

In particular, for the energy integral the following inequality holds

$$\|u\|_{1_{\mathcal{H}_s^2([0, T])}} \leq C \left( \|f\|_{\mathcal{H}_s^{2-1}} \right. + \|\varphi\|_{\mathcal{H}_s^{2-1}} + \|\psi\|_{\mathcal{H}_s^{2-1}} + \|u_1\|_{H_s^{1+1/2}} + \|u_0\|_{H_s([0, T])},$$

2° For every

$$[f, \varphi, \psi, u_0, u_1] \in \mathcal{H}_s^{2-1}(\Omega \times [0, T]) \otimes \mathcal{H}_s^{2-1}(\Omega) \otimes \mathcal{H}_s^{2-1}(\Omega) \otimes H_s(\Gamma \times [0, T]) \otimes H_s^{1+1/2}(\partial \Omega \times [0, T]),$$

$s > 1$, exists only one $u \in \mathcal{H}_s^1(\Omega \times [0, T])$ such that conditions (2)-(3) are fulfilled.
Proof. By differentiation of both sides of equation (2) with respect to \( x_1 \), we obtain

\[
(u_{x_1})_t - Lu_{x_1} = f_{x_1}, \quad u_{x_1} \big|_{\partial \Omega \times [0, T]} = u, \\
0 \leq u_{x_1} \big|_{t=0} = \varphi_{x_1}, \quad (u_{x_1})_{t=0} = \psi_{x_1}.
\]

Hence, the function \( u_{x_1} = \mathcal{D}(f_{x_1}, u_1, \varphi_{x_1}, \psi_{x_1}) \) is defined uniquely, \( \mathcal{D}(f_{x_1}, u_1, \varphi_{x_1}, \psi_{x_1}) \in H_s(\Omega' \times [0, T]) \)

and

\[
\|u_{x_1}\|_{s \times [0, T]} \leq C_1(\|f_{x_1}\|_{s-1 \times [0, T]} + \|\varphi_{x_1}\|_s + \|\psi_{x_1}\|_{s-1} + \|u\|_{s \times [0, T]}).
\]

Hence

\[
u = \int_0^{x_1} \mathcal{D}(f_{x_1}, u_1, \varphi_{x_1}, \psi_{x_1}) \, dx_1 + w(x', t),
\]

where \( w(x', t) \) can be an arbitrary function of class \( H_s(\Omega' \times [0, T]) \), also belongs to \( H_s(\Omega' \times [0, T]) \).

We shall show that condition (3) defines the function \( w(x', t) \) uniquely.

Since the function \( u \) defined by (6) should be a solution of problem (2)-(3), it should be

\[
(w_{tt} - L_1 w = f + \int_0^{x_1} L_1 \mathcal{D} \, dx_1 + \frac{\partial}{\partial x_1} \left( \int_0^{x_1} l_1 \mathcal{D} \, dx_1 \right) + a \frac{\partial^2}{\partial x_1^2} \int_0^{x_1} \mathcal{D} \, dx_1 - \int_0^{x_1} \mathcal{D} \, dx_1 = F,
\]

where

\[
\mathcal{D} = \mathcal{D}(f_{x_1}, u_1, \varphi_{x_1}, \psi_{x_1}).
\]

Formula \( L = L_1 + \frac{\partial}{\partial x_1} l_1 + a \frac{\partial^2}{\partial x_1^2} \) was used here, where \( l_1 \) — first order differential operator with respect to \( x' \), \( L_1 \) — the elliptic operator with respect to \( x' \), coefficients of \( L_1 \) and \( l_1 \) do not depend on \( x_1 \), \( a = a(x') \).

It is easy to notice that \( \partial F/\partial x_1 = 0 \).

The function \( w(x', t) \) has to be a solution of the following mixed problem

\[
(w_{tt} - L_1 w = f + l_1 \mathcal{D} + a \mathcal{D} x_1 |_{x_1=0}, \quad w|_{t=0} = \varphi|_{x_1=0}, \quad \quad w|_{t \times [0, T]} = u_0.
\]

Hence, the second part of the theorem follows.

It is easy to see that

\[
\|w\|_{s \times [0, T]} \leq C_2(\|f\|_{s \times [0, T]} + \|l_1 \mathcal{D} \|_{s \times [0, T]} + \|a \mathcal{D} x_1 \|_{s \times [0, T]} + \|\varphi\|_s + \|\psi\|_{s-1} + \|u_0\|_s \times [0, T])
\]

\[
\leq C_3(\|f\|_{s \times [0, T]} + \]|\mathcal{D}|_{s \times [0, T]} + \|\varphi\|_s + \|\psi\|_{s-1} + \|u_0\|_s \times [0, T])
\]

\[
\leq C_4(\|f\|_{s \times [0, T]} + \|l_1 \mathcal{D} \|_{s \times [0, T]} + \|\varphi\|_s + \|\psi\|_{s-1} + \|u_0\|_s \times [0, T])
\]

\[
+ \|\varphi_x\|_{s-1} + \|\psi_x\|_{s-1} + \|u_0\|_s \times [0, T])
\]
The estimate (4) follows from inequalities (5), (8) and from the inequality (see [1])

\[ \|u\|_G^{0 \times [0,T]} \leq \|u - w\|_G^{0 \times [0,T]} + \|w\|_G^{0 \times [0,T]} \]

\[ \leq C \left( \int_0^{x_1} \|u_x\|_G^{0 \times [0,T]} + \|w\|_G^{N \times [0,T]} \right) \leq C'_1 (\|u_x\|_G^{0 \times [0,T]} + \|w\|_G^{N \times [0,T]}). \]

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References