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Non-coercive mixed problem

In recent years there appeared many papers concerning the non-coercive problems of the form

$$(1)\qquad \begin{aligned} Lu &= f \quad \text{in } \Omega, \\ Bu|_{\partial\Omega} &= g, \end{aligned}$$

where L is a differential operator of order m, elliptic in the bounded domain $\overline{\Omega} \subset R^n$. For such problems it is possible to estimate the norm $\|u\|_{s+m-\delta}$ $(\delta > 0)$ in the Sobolev space $H_{s+m-\delta}(\Omega)$ by $\|f\|_s$ and suitable norms of the functions g on the boundary $\partial\Omega$. From these estimates various facts concerning solvability of problem (1) follow. Problems for which such estimates do not hold have also been studied. Many results have been obtained by the method of the theory of pseudo-differential operators.

As far as we know, no papers dealing with mixed problems for hyperbolic equations with the boundary conditions of non-coercive type have been published.

The theorem stated below should be treated as the first step in this direction.

In this paper we shall consider the following mixed problem

$$(2)\qquad \begin{aligned} u_{tt} - Lu &= f(x, t), \qquad t \in (0, T),\ x \in \Omega, \\ u_{x_1}|_{\partial\Omega\times[0,T]} &= u_1(x, t), \\ u|_{t=0} = \varphi(x), &\qquad u_t|_{t=0} = \psi(x), \end{aligned}$$

with the additional condition

$$(3)\qquad u|_{\Gamma\times[0,T]} = u_0(x'),$$

where Ω is a bounded domain in R^n, such that

1° the boundary of Ω is a smooth manifold,

2° Ω is a convex set with respect to x_1, i.e. every set $\{x = (x_1, x'): x \in \Omega,\ x' = \text{const}\}$ is convex,

3° vector field $\partial/\partial x_1$ is tangent to $\partial\Omega$ along

$$\Gamma = \{x\colon x \in \partial\Omega,\ x_1 = 0\}, \qquad \dim\Gamma = n-2.$$

We assume

$$Lu = \sum_{i,j} a_{ij}(x,t)u_{x_ix_j} + \sum_i a_i(x,t)u_{x_i} + b(x,t)u,$$

$$L\colon C^\infty(R^{n+1}) \to C^\infty(R^{n+1}),$$

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geqslant c\sum_i \xi_i^2 \qquad (c>0)$$

for every $(x,t) \in \Omega\times[0,T]$ and $\xi \in R^n\setminus\{0\}$, coefficients of the operator L does not depend on x_1. The method of investigation of problem (2)–(3) is based on the well-known facts from the theory of well posed problems for hyperbolic equations (cf. [1]).

Let $\mathscr{H}_s^1(\Omega\times[0,T])$ be a subspace of the space $H_s(\Omega\times[0,T])$ ([2]) with the norm

$$\|u\|_{\mathscr{H}_s^1}^{\Omega\times[0,T]} = \|u\|_s^{\Omega\times[0,T]} + \left\|\frac{\partial u}{\partial x_1}\right\|_s^{\Omega\times[0,T]} + \|u\|_s^{N\times[0,T]},$$

where $N = \{x\colon x\in\Omega,\ x_1 = 0\}$, $s>1$, $\|\cdot\|_s^{\Omega\times[0,T]}$ the norm in $H_s(\Omega\times[0,T])$, and let $\mathscr{H}_s^2(\Omega\times[0,T])$ be a subspace with the norm

$$\|u\|_{\mathscr{H}_s^2}^{\Omega\times[0,T]} = \|u\|_s^{\Omega\times[0,T]} + \left\|\frac{\partial u}{\partial x_1}\right\|_{s+1/2}^{\Omega\times[0,T]} + \|u\|_s^{N\times[0,T]}.$$

THEOREM. 1° *If*

$$f\in\mathscr{H}_{s-1}^2(\Omega\times[0,T]), \qquad \varphi\in\mathscr{H}_s^2(\Omega),$$

$$\psi\in\mathscr{H}_{s-1}^2(\Omega), \qquad u_0\in H_s(\Gamma\times[0,T]), \qquad u_1\in H_{s+1/2}(\partial\Omega\times[0,T]), \qquad s>1$$

and conditions (2)–(3) *are fulfilled, then* $u\in\mathscr{H}_s^1(\Omega\times[0,T])$ *and*

$$\text{(4)}\quad \|u\|_{\mathscr{H}_s^1}^{\Omega\times[0,T]} \leqslant C(\|f\|_{\mathscr{H}_{s-1}^2}^{\Omega\times[0,T]} + \|\varphi\|_{\mathscr{H}_s^2}^{\Omega} + \|\psi\|_{\mathscr{H}_{s-1}^2}^{\Omega} + \|u_1\|_{s+1/2}^{\partial\Omega\times[0,T]} + \|u_0\|_s^{\Gamma\times[0,T]}),$$

where C does not depend on u.

In particular, for the energy integral the following inequality holds

$$\|u\|_1^{\Omega\times[0,T]} = C(\|f\|_{\mathscr{H}_0^2}^{\Omega\times[0,T]} + \|\varphi\|_{\mathscr{H}_1^2}^{\Omega} + \|\psi\|_{\mathscr{H}_0^2}^{\Omega} + \|u_1\|_{3/2}^{\partial\Omega\times[0,T]} + \|u_0\|_1^{\Gamma\times[0,T]}).$$

2° *For every*

$$[f,\varphi,\psi,u_0,u_1]\in\mathscr{H}_{s-1}^2(\Omega\times[0,T])\otimes\mathscr{H}_s^2(\Omega)\otimes\mathscr{H}_{s-1}^2(\Omega)\otimes H_s(\Gamma\times[0,T])\otimes$$
$$\otimes H_{s+1/2}(\partial\Omega\times[0,T]),$$

$s>1$, *exists only one* $u\in\mathscr{H}_s^1(\Omega\times[0,T])$ *such that conditions* (2)–(3) *are fulfilled.*

Proof. By differentiation of the both sides of equation (2) with respect to x_1 we obtain

$$(u_{x_1})_{tt} - Lu_{x_1} = f_{x_1}, \quad u_{x_1}|_{\partial\Omega\times[0,T]} = u_1,$$
$$u_{x_1}|_{t=0} = \varphi_{x_1}, \quad (u_{x_1})_t|_{t=0} = \psi_{x_1}.$$

Hence, the function $u_{x_1} = \mathscr{D}(f_{x_1}, u_1, \varphi_{x_1}, \psi_{x_1})$ is defined uniquely, $\mathscr{D}(f_{x_1}, u_1, \varphi_{x_1}, \psi_{x_1}) \in H_s(\Omega\times[0, T])$ and

$$\|u_{x_1}\|_s^{\Omega\times[0,T]} \leqslant C_1(\|f_{x_1}\|_{s-1}^{\Omega\times[0,T]} + \|\varphi_{x_1}\|_s^{\Omega} + \|\psi_{x_1}\|_{s-1}^{\Omega} + \|u_1\|_s^{\partial\Omega\times[0,T]}). \tag{5}$$

Hence

$$u = \int_0^{x_1} \mathscr{D}(f_{x_1}, u_1, \varphi_{x_1}, \psi_{x_1})dx_1 + w(x', t), \tag{6}$$

where $w(x', t)$ can be an arbitrary function of class $H_s(N\times[0, T])$, also belongs to $H_s(\Omega\times[0, T])$.

We shall show that condition (3) defines the function $w(x', t)$ uniquely. Since the function u defined by (6) should be a solution of problem (2)–(3), it should be

$$w_{tt} - L_1 w = f + \int_0^{x_1} L_1\mathscr{D}dx_1 + \frac{\partial}{\partial x_1}\Big(\int_0^{x_1} l_1\mathscr{D}dx_1\Big) + \\ + a\frac{\partial^2}{\partial x_1^2}\int_0^{x_1}\mathscr{D}dx_1 - \int_0^{x_1}\mathscr{D}_{tt}dx_1 = F, \tag{7}$$

where

$$\mathscr{D} = \mathscr{D}(f_{x_1}, u_1, \varphi_{x_1}, \psi_{x_1}).$$

Formula $L = L_1 + \frac{\partial}{\partial x_1}l_1 + a\frac{\partial^2}{\partial x_1^2}$ was used here, where l_1 — first order differential operator with respect to x', L_1 — the elliptic operator with respect to x', coefficients of L_1 and l_1 do not depend on x_1, $a = a(x')$.

It is easy to notice that $\partial F/\partial x_1 = 0$.

The function $w(x', t)$ has to be a solution of the following mixed problem

$$w_{tt} - L_1 w = (f + l_1\mathscr{D} + a\mathscr{D}_{x_1})|_{x_1=0}, \quad w|_{t=0} = \varphi|_{x_1=0},$$
$$w_t|_{t=0} = \psi|_{x_1=0}, \quad w|_{\Gamma\times[0,T]} = u_0.$$

Hence, the second part of the theorem follows.

It is easy to see that

$$\begin{aligned}\|w\|_s^{N\times[0,T]} &\leqslant C_2(\|f\|_{s-1}^{N\times[0,T]} + \|l_1\mathscr{D}\|_{s-1}^{N\times[0,T]} + \|aD_{x_1}\|_{s-1}^{N\times[0,T]} + \\ &\quad + \|\varphi\|_s^N + \|\psi\|_{s-1}^N + \|u_0\|_s^{\Gamma\times[0,T]}) \\ &\leqslant C_3(\|f\|_{s-1}^{N\times[0,T]} + \|\mathscr{D}\|_{s+1/2}^{\Omega\times[0,T]} + \|\varphi\|_s^N + \|\psi\|_{s-1}^N + \|u_0\|_s^{\Gamma\times[0,T]}) \\ &\leqslant C_4(\|f_{x_1}\|_{s-1/2}^{\Omega\times[0,T]} + \|f\|_{s-1}^{N\times[0,T]} + \|\varphi_{x_1}\|_{s+1/2}^{\Omega} + \|\varphi\|_s^N + \\ &\quad + \|\psi_{x_1}\|_{s-1/2}^{\Omega} + \|\psi\|_{s-1}^N + \|u_0\|_s^{\Gamma\times[0,T]} + \|u_1\|_{s+1/2}^{\partial\Omega\times[0,T]}).\end{aligned} \tag{8}$$

The estimate (4) follows from inequalities (5), (8) and from the inequality (see [1])

$$\|u\|_s^{\Omega\times[0,T]} \leqslant \|u-w\|_s^{\Omega\times[0,T]} + \|w\|_s^{\Omega\times[0,T]}$$
$$\leqslant C\Big(\Big\|\int_0^{x_1} u_{x_1}\Big\|_s^{\Omega\times[0,T]} + \|w\|_s^{N\times[0,T]}\Big) \leqslant C_1'(\|u_{x_1}\|_s^{\Omega\times[0,T]} + \|w\|_s^{N\times[0,T]}).$$

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References

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[2] Л. Хёрмандер, *Линейные дифференциальные операторы с частными производными*, Москва 1965.