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The determination of the two-dimensional distribution by means of conditional distributions

1. The setting of the problem. In investigations of the two-dimensional random variables it is sometimes convenient to represent the two-dimensional distribution by means of those of one dimension. It is known, however, that for instance the boundary distributions determine uniquely the two-dimensional distributions only in case of independence of random variables. In case when the random variables are dependent the two-dimensional distribution can be represented by means of boundary distribution and a corresponding conditional distributions.

H. Rumsey Jr. and E. C. Posner dealt in [2] with the problem of how to determine the density of two-dimensional distribution of the random variable with respect to the boundary distributions and some additional conditions. These additional conditions they imposed were certain equations of the form

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)r_j(x,y)\,dx\,dy = \varrho_j, \quad j = 1, 2, \ldots, k,$$

holding for given functions $r_j(x, y)$ and for given constants ϱ_j; to assure the uniqueness of $f(x, y)$ it is assumed, moreover, that this function attains its maximal entropy

$$H(f) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \ln f^{-1}\,dx\,dy.$$

The alternative approach to this problem can also be considered: since the boundary distributions do not determine the two-dimensional distributions whether then the knowledge of both conditional distributions suffices? In the sequel we shall show that the answer to the above question is in general negative. We shall show that for a large class of distributions this is, however, sufficient.

We shall now investigate these questions separately for continuous and jump-functions random variables.

2. The representation of density of the two-dimensional distribution by means of densities of conditional distributions. Let (X, Y) be the two-dimensional continuous variable with density $f(x, y)$. Let $\varphi(x)$, $\psi(x)$ $\varphi_1(x/y)$, $\psi_1(y/x)$ denote respectively densities of boundary distributions of X, Y and conditional of X/Y and Y/X. To represent the density $f(x, y)$ with respect to $\varphi_1(x/y)$ and $\psi_1(y/x)$ we shall first prove the following lemma.

LEMMA. *Let D be a measurable subset in the plane and let density $f(x, y)$ of the two-dimensional random variable (X, Y) be given. If $f(x, y) > 0$ for $(x, y) \in D$, $f(x, y) = 0$ for $(x, y) \notin D$, then*

$$\frac{\psi_1(y/x)}{\varphi_1(x/y)} = \frac{\psi(y)}{\varphi(x)} \tag{1}$$

holds almost everwhere in D.

Proof. Denote by D_x and D_y the projections of D^*, where D^* denotes a set of density points of D, onto the coordinate axes Ox and Oy. Clearly then $D^* \subset D_x \times D_y$. By the Fubini theorem there exist, for almost every x a function $\varphi(x)$ defined as $\varphi(x) = \int\limits_{D_y} f(x, y)\,dy$ and, for almost every y, a function $\psi(y) = \int\limits_{D_x} f(x, y)\,dx$.

Those functions have values greater than 0 in D_x and D_y with exception of a measure zero set in which their values can be zero. Hence the conditional densities φ_1 and ψ_1 are well defined, have positive values and satisfy the equations

$$\varphi_1(x/y) = \frac{f(x, y)}{\psi(y)}, \qquad \psi_1(y/x) = \frac{f(x, y)}{\varphi(x)}$$

almost everywhere in D. Dividing these relations by sides we accomplish the proof.

With the help of this lemma we shall prove the following theorem.

THEOREM 1. *Let D be a measurable plane set, $D_x^{y_0}$ — a projection of its intersection by the straight line $y = y_0$ onto Oy axis, D_x — a projection of D onto Ox axis. If a density $f(x, y)$ is greater than zero for $(x, y) \in D$ and if $D_x^{y_0} = D_x$, then*

$$f(x,y) = \frac{\psi_1(y/x)\varphi_1(x/y_0)}{\psi_1(y_0/x)}\left(\iint\limits_{D} \frac{\psi_1(y/x)\varphi_1(x/y_0)}{\psi_1(y_0/x)}\,dxdy\right)^{-1}$$

holds almost everwhere in D.

Proof. We have for almost all $(x, y) \in D$

$$f(x, y) = \varphi(x)\psi_1(y/x).$$

Applying lemma, since $(x, y_0) \in D$, we get

$$\varphi(x) = \frac{\psi(y_0)\varphi_1(x/y_0)}{\psi_1(y_0/x)}$$

what substituted to the previous relation yields

$$f(x, y) = \frac{\psi_1(y/x)\varphi_1(x/y_0)}{\psi_1(y_0/x)}\,\varphi(y_0) \qquad \text{for } (x, y) \in D. \tag{3}$$

Let us, moreover, use the condition that $\iint\limits_D f(x, y)\,dxdy = 1$ to express $\psi(y_0)$ and substitute to (3) and we shall get the thesis of Theorem 1.

Remark. If $D_y^{x_0} = D_y$, where D_y and $D_y^{x_0}$ denote respectively the projections of D and the projection of its intersection with $x = x_0$ onto the axis Oy, then the thesis of the theorem can be written in the form

$$f(x, y) = \frac{\varphi_1(x/y)\psi_1(y/x_0)}{\varphi_1(x_0/y)}\left(\iint\limits_D \frac{\varphi_1(x/y)\psi_1(y/x_0)}{\varphi_1(x_0/y)}\,dxdy\right)^{-1}.$$

From the above proven theorem it follows that the density of the two-dimensional random variable (X, Y) can be represented by all conditional distributions of the random variable Y/X and one conditional distribution of $X/Y = y_0$.

This problem can also be viewed at in a certain geometrical fashion. We know that the two-dimensional density represents in general some surface, whereas the conditional densities its normed intersections by the planes parallel to the Oxz and Oyz planes. The theorem says therefore that the surface $f(x, y)$ may be represented with respect to all normed intersections by the planes parallel to Oxz and one of the normed intersections by Oyz.

To fully illustrate the theorem let us explain yet the following question. Of what a nature must be a set D and what intersection of this set by a straight line $y = y_0$ should be taken?

We know that the projection of intersection by the straight line $y = y_0$ must be identical to D_x. Of such a nature is, for example a set which is a product of its projections onto the axes Ox and Oy : $D = D_x \times \times D_y$. This set can also be of a different form but then its intersection by $y = y_0$ should not be taken arbitrarily but so as to $D_x^{y_0} = D_x$ hold.

An example of such a set is shown on Fig. 1.

It means that this intersection is such that it connects all intersections by straight lines parallel to Oy axis. In the case when such an intersection does not exist (Fig. 3) the representation of the two-dimensional density by all distributions of Y/X variable and one of the distributions $X/Y = y_0$ is not feasible.

In the sequel we shall deal with a problem which can be regarded as a converse to the above, namely:

1. Does the knowledge of the functions $\psi_1(x, y)$ and $\varphi_1(x)$ treated as densities of conditional distributions Y/X and $X/Y = y_0$ suffice to anticipate the existence of some two-dimensional distribution, of which densities of respective conditional distributions are given functions?

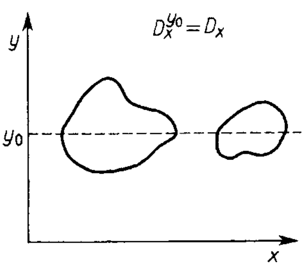

Fig. 1

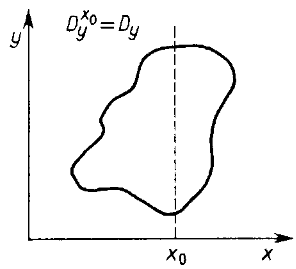

Fig. 2

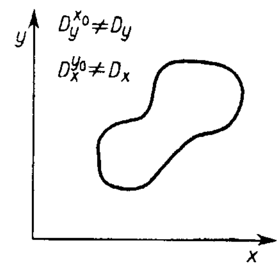

Fig. 3

2. Does the knowledge of the functions $\psi_1(x, y)$ and $\varphi_1(x)$ suffice to express uniquely the density of two-dimensional distribution $f(x, y)$ and if not, what are the additional conditions to this purpose. In what follows we shall study these problems.

3. Conditions for existence of two-dimensional distribution at given conditional distributions. Let us, firstly, introduce the notations. D denotes an arbitrary measurable plane set, D_x and D_y as well as $D_x^{y_0}$ and $D_y^{x_0}$ have the same meaning as before.

THEOREM 2 (EXISTENCE). *Given the functions* $\varphi_1(x, y)$ *and* $\varphi_1(x)$ *satisfying conditions*

1° $\psi_1(x, y) > 0$ *for* $(x, y) \in D$ *and equal* 0 *for* $(x, y) \notin D$, $\varphi_1(x) > 0$ *for* $x \in D_x^{y_0}$ *and equal* 0 *for* $x \notin D_x^{y_0}$;

2° $\int\limits_{D_y^x} \psi_1(x, y)\,dy = 1, \quad \int\limits_{D_x^{y_0}} \varphi_1(x)\,dx = 1;$

3° $\int\limits_{D_x^{y_0}} \frac{\varphi_1(x)}{\psi_1(x, y_0)}\,dx = k < +\infty.$

Then the function

$$f(x, y) = \begin{cases} \dfrac{c\varphi_1(x)\psi_1(x, y)}{\psi_1(x, y_0)} & \text{for } (x, y) \in D \cap D_y \times D_x^{y_0}, \\ \varepsilon(x)\psi_1(x, y) & \text{for } (x, y) \in D \setminus D_y \times D_x^{y_0}, \\ 0 & \text{for } (x, y) \notin D, \end{cases} \tag{4}$$

where $\varepsilon(x)$ *is any continuous positive function satisfying condition* $\int\limits_{D_x \setminus D_x^{y_0}} \varepsilon(x)dx = l < 1$, *and* c *is a positive constant such that* $ck+l = 1$, *is a density of some two-dimensional distribution* (X, Y) *for which the densities of conditional distributions are* Y/X *and* $X/Y = y_0$ *are* $\psi_1(x, y)$ *and* $\varphi_1(x)$.

Proof. (Fig. 4). Since $\psi_1(x, y_0)$ differs from zero on $D_x^{y_0}$ hence the term $\dfrac{\psi_1(x, y)\varphi_1(x)}{\psi_1(x, y_0)}$ is well-defined in $D \cap D_y \times D_x^{y_0}$. Also the integral

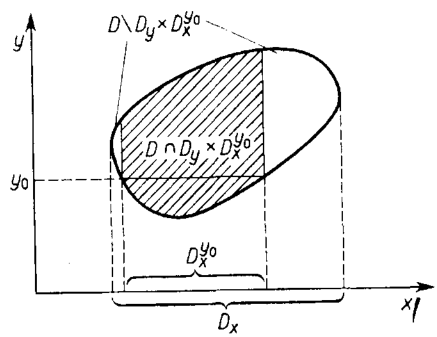

Fig. 4

$\iint\limits_{D} f(x, y)\,dx\,dy$ exists since $f(x, y)$ is non-negative nad there exists iterated integrals namely

$$\iint\limits_{D} f(x, y)dxdy = \iint\limits_{D_1(D_y \times D_x^{y_0})} f(x, y)dxdy + \iint\limits_{D \setminus (D_y \times D_x^{y_0})} f(x, y)dxdy$$

$$= \int\limits_{D_x^{y_0}} \left[\int\limits_{D_y^x} \frac{c\varphi_1(x)\psi_1(x, y)}{\psi_1(x, y)}\, dy \right] dx + \int\limits_{D_x \setminus D_x^{y_0}} \left[\int\limits_{D_y^x} \varepsilon(x)\psi_1(x, y)\, dy \right] dx$$

$$= \int\limits_{D_x^{y_0}} \frac{c\varphi_1(x)\,dx}{\psi_1(x, y_0)} + \int\limits_{D_x \setminus D_x^{y_0}} \varepsilon(x)dx = ck + l = 1.$$

Hence the function $f(x, y)$ of the theorem as non-negative in D and satisfying condition $\iint\limits_{D} f(x, y)dxdy = 1$ is density.

It remains to show that $\psi_1(x, y)$ and $\varphi_1(x)$ are the densities of distributions.

Denote by $\varphi(x)$ and $\psi(y)$ boundary densities of variables X and Y respectively and by $\psi_1(y/x)$ and $\varphi_1(x/y)$ densities of conditional distributions Y/X and X/Y. Then

$$\psi_1(y/x) = \frac{f(x, y)}{\varphi(x)}$$

and $\varphi(x)$ equals

$$\varphi(x) = \begin{cases} \int\limits_{D_y^x} \frac{c\varphi_1(x)\psi_1(x, y)}{\psi_1(x, y_0)} dy = \frac{c\varphi_1(x)}{\psi_1(x, y_0)} & \text{for } x \in D_x^{y_0}, \\ \int\limits_{D_y^x} \varepsilon(x)\psi_1(x, y)\, dy = \varepsilon(x) & \text{for } x \in D_x \setminus D_x^{y_0}, \\ 0 & \text{for } x \notin D_x. \end{cases}$$

Thus

$$\psi_1(y/x) = \begin{cases} \frac{c\varphi_1(x)\psi_1(x, y)}{\psi_1(x, y_0)} : \frac{c\varphi_1(x)}{\psi_1(x, y_0)} = \psi_1(x, y) & \text{for } (x, y) \in D \cap D_y \times D_x^{y_0}, \\ \frac{\varepsilon(x)\psi_1(x, y)}{\varepsilon(x)} = \psi_1(x, y) & \text{for } (x, y) \in D \setminus D_y \times D_x^{y_0}, \end{cases}$$

and finally $\psi_1(y/x) = \psi_1(x, y)$ for $(x, y) \in D$.

To express $\varphi_1(x/y_0)$ we shall use the relation

$$\varphi_1(x/y_0) = \frac{f(x, y)}{\psi(y_0)}.$$

We know that

$$\psi(y) = \int\limits_{D_x^y} f(x, y)\, dx,$$

hence

$$\psi(y_0) = \int\limits_{D_x^{y_0}} f(x, y_0)\, dx = \int\limits_{D_x^{y_0}} c\varphi_1(x)\, dx = c$$

and

$$\varphi_1(x/y_0) = \frac{f(x, y_0)}{\psi(y_0)} = \varphi_1(x). \qquad \text{Q.E.D.}$$

We deduce from this theorem that knowing the functions regarded as densities of conditional distributions we can find such a two-dimensional distribution, whose densities of conditional distributions are these functions.

Because density of this two-dimensional distribution depends upon choice of both the constant c and function $\varepsilon(x)$, it is not uniquely determined, i. e. taking various c and $\varepsilon(x)$ satisfying condition $ck+l = 1$ (see (4)) we shall get different two-dimensional densities $f(x, y)$ which will satisfy the required conditions.

Next chapter will be devoted to studying the following problems:

1. Under what assumptions the functions $\psi_1(x, y)$ and $\varphi_1(x)$ determine uniquely the density $f(x, y)$?

2. If uniqueness is lacking what additional conditions should be assumed in order to get the uniqueness?

4. Sufficient conditions for unique determination of the two-dimensional distribution by conditional densities.

THEOREM 3. *Given some plane set D with a point y_0 such that $D_x^{y_0} = D_x$. If there exist two functions $\varphi_1(x)$ and $\psi_1(x, y)$ satisfying the hypotheses of Theorem 2, then the function*

$$f(x, y) = \begin{cases} \dfrac{\psi_1(x, y)\varphi_1(x)}{\psi_1(x, y_0)} \left(\iint\limits_D \dfrac{\psi_1(x, y)\varphi_1(x)}{\psi_1(x, y_0)} \, dxdy \right)^{-1} & \text{for } (x, y) \in D, \\ 0 & \text{for } (x, y) \notin D, \end{cases}$$

is the only function which is a density of a two-dimensional distribution with the densities of its conditional distributions Y/X and $X/Y = y_0$ equal to $\varphi_1(x)$ and $\psi_1(x, y)$.

Proof. From the existence theorem (Theorem 2) we know that there exists a density $f(x, y)$ of which densities of conditional distributions $X/Y = y_0$ and Y/X are given functions. Therefore as the condition $D_x^{y_0} = D_x$ holds, we can apply Theorem 1 thus accomplishing the proof.

The formula for density given by this theorem is identical to that of Theorem 2 because if $D_x^{y_0} = D_x$, then $D \setminus (D_y \times D_x^{y_0})$ is a void set and hence

$$f(x, y) = \begin{cases} \dfrac{c\varphi_1(x)\psi_1(x, y)}{\psi_1(x, y_0)} & \text{for } (x, y) \in D, \\ 0 & \text{for } (x, y) \notin D, \end{cases}$$

and the constant c equals

$$c = \left(\iint\limits_D \frac{\varphi_1(x)\psi_1(x, y)}{\psi_1(x, y_0)} \, dxdy \right)^{-1}.$$

To illustrate the way of forming of $f(x, y)$ as well as to show the necessity of Assumption 3 in Theorem 2 let us consider an example.

EXAMPLE. Take the functions

$$\varphi_1(x) = \frac{1}{\pi(1+x^2)}, \qquad \psi_1(x, y) = \frac{\sqrt{1+x^2}}{\pi(1+x^2+y^2)}.$$

It is clear that

$$\int_{-\infty}^{+\infty} \varphi_1(x)\, dx = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} \psi_1(x, y)\, dy = 1$$

and so we may regard them as densities of conditional distributions of some two-dimensional distribution. Assume $y_0 = 0$. Then

$$f(x, y) = \frac{\dfrac{\sqrt{1+x^2}}{\pi(1+x^2+y^2)} \cdot \dfrac{1}{\pi(1+x^2)}}{\dfrac{1}{\pi} \cdot \dfrac{\sqrt{1+x^2}}{1+x^2}} = \frac{1}{\pi} \cdot \frac{1}{1+x^2+y^2}.$$

However, this function as non-integrable in the plane $-\infty < x$, $y < +\infty$ cannot be a density. It happens so because Assumption 3 is not satisfied, i. e. $\int\limits_{-\infty}^{+\infty} \frac{\varphi_1(x)}{\psi_1(x, y_0)}$ is not integrable.

In what follows we shall consider the following question. Since to determine uniquely the density it does not suffice the knowledge of $\varphi(x)$ and $\psi(x, y)$ when $D_x \setminus D_x^{y_0}$ is of a positive measure, what else conditions must be additionaly assumed? In order to answer this question we shall prove the following:

THEOREM 4. *Let D be a measurable plane set and let $D_x^{y_1}$ and $D_x^{y_2}$ be projections onto x axis of its intersections by the straight lines $y = y_1$ and $y = y_2$ $(y_1 \neq y_2)$. Let $D_x = D_x^{y_1} \cup D_x^{y_2}$ and let $D_x^{y_1} \cap D_x^{y_2}$ be of positive measure. If given are functions $\psi_1(x, y)$, $\varphi_1(x)$ and $\varphi_2(x)$ satisfying conditions:*

1. $\psi_1(x, y) > 0$ *for* (x, y)
   $\varphi_1(x) > 0$ *for* $x \in D_x^{y_1}$ *and equals zero outside this set;*
   $\varphi_2(x) > 0$ *for* $x \in D_x^{y_2}$
2. $\int\limits_{D_y^x} \psi_1(x, y)\,dy = 1,$

   $\int\limits_{D_x^{y_1}} \varphi_1(x)\,dx = \int\limits_{D_x^{y_2}} \varphi_2(x)\,dx = 1;$
3. $\frac{\varphi_1(x)}{\psi_1(x, y_1)} \cdot \frac{\psi_1(x_0, y_1)}{\varphi_1(x_0)} = \frac{\varphi_2(x)}{\psi_1(x, y_2)} \cdot \frac{\psi_1(x_0, y_2)}{\varphi_2(x_0)}$ *for* $x_0 \in D_x^{y_1} \cap D_x^{y_2}$;
4. *the integrals*
$$\int\limits_{D_x^{y_1}} \frac{\varphi_1(x)}{\psi_1(x, y_1)}\,dx, \qquad \int\limits_{D_x^{y_2}} \frac{\varphi_2(x)}{\psi_2(x, y_2)}\,dx$$

are finite. Then the function

$$f(x, y) = \begin{cases} \dfrac{s\psi_1(x_0, y_2)}{\varphi_1(x_0)} \cdot \dfrac{\varphi_1(x)\psi_1(x, y)}{\psi_1(x, y_1)} & \text{for } (x, y) \in D \cap D_y \times D_x^{y_1}, \\ \dfrac{s\psi_1(x_0, y_1)}{\varphi_2(x_0)} \cdot \dfrac{\varphi_2(x)\psi_1(x, y)}{\psi_1(x, y_2)} & \text{for } (x, y) \in D \cap D_y \times D_x^{y_2}, \\ 0 & \text{for } (x, y) \notin D, \end{cases} \tag{5}$$

where

$$s = \left(\frac{\psi_1(x_0, y_1)}{\varphi_1(x_0)} \int\limits_{D_x^{y}} \frac{\varphi_2(x)}{\psi_1(x, y_1)}\,dx + \frac{\psi_1(x_0, y_2)}{\varphi_2(x_0)} \int\limits_{D_x \setminus D_x^{y_1}} \frac{\varphi_2(x)}{\psi_1(x, y_2)}\,dx \right)^{-1}$$

is the only function which is a density of the two-dimensional distribution and whose densities of conditional distributions Y/X, $X/Y = y_1$, $X/Y = y_2$ are respectively the given functions $\psi_1(x, y)$, $\varphi_1(x)$, $\varphi_2(x)$.

Proof. Knowing $\psi_1(x, y)$ and $\varphi_1(x)$ we can by the existence theorem construct the two-dimensional density which on $D \cap D_y^x \times D_x^{y_1}$ takes the form

$$f(x, y) = \frac{c_1 \varphi_1(x) \psi_1(x, y)}{\psi_1(x, y_1)}.$$

Analogously, for $\psi_1(x, y)$ and $\varphi_2(x)$ on $D \cap D_y \times D_x^{y_2}$ this density equals

$$f(x, y) = \frac{c_2 \varphi_2(x) \psi_1(x, y)}{\psi_1(x, y_2)}.$$

Write

$$s_1 = \frac{c_1 \varphi_1(x_0)}{\psi_1(x_0, y_1)}, \qquad s_2 = \frac{c_2 \varphi_2(x_0)}{\psi_2(x_0, y_2)},$$

where $x_0 \in D_x^{y_1} \cap D_y^{y_2}$. Substitution of s_1 and s_2 from these relations to former equalities yields

$$f(x, y) = \begin{cases} \dfrac{s_1 \psi_1(x_0, y_1)}{\varphi_1(x_0)} \, \dfrac{\varphi_1(x) \psi_1(x, y)}{\psi_1(x, y_1)} & \text{for } (x, y) \in D \cap D_y \times D_x^{y_1}, \\[2ex] \dfrac{s_2 \psi_1(x_0, y_2)}{\varphi_2(x_0)} \, \dfrac{\varphi_2(x) \psi_1(x, y)}{\psi_1(x, y_2)} & \text{for } (x, y) \in D \cap D_y \times D_x^{y_2}. \end{cases}$$

Taking into account our third hypothesis we get that $s_1 = s_2 = s$. It remains only to take s such that $f(x, y)$ becomes a density. To this end we shall make use of the condition $f(x, y)\,dxdy = 1$.

Hence

$$\iint_D f(x, y)\,dxdy = \iint_{D \cap D_y \times D_x^{y_1}} \frac{s \psi_1(x_0, y_1) \varphi_1(x) \psi_1(x, y)}{\varphi_1(x_0) \psi_1(x, y_1)}\,dxdy +$$

$$+ \iint_{D \setminus D_y \times D_x^{y_1}} \frac{s \psi_1(x_0, y_2)}{\varphi_2(x_0)} \, \frac{\varphi_2(x) \psi_1(x, y)}{\psi_1(x, y_2)}\,dxdy$$

$$= \int_{D_x^{y_1}} \frac{s \psi_1(x_0, y_1) \varphi_1(x)}{\psi_1(x_0) \psi_1(x, y_1)}\,dx + \int_{D_x \setminus D_x^{y_1}} \frac{s \psi_1(x_0, y_2) \cdot \varphi_2(x)}{\varphi_2(x_0) \cdot \psi_1(x, y_2)}\,dx = 1$$

and

$$s = \left(\frac{\psi_1(x_0, y_1)}{\varphi_1(x_0)} \int_{D_x^{y_1}} \frac{\varphi_1(x)}{\psi_1(x, y_1)}\,dx + \frac{\psi_1(x_0, y_2)}{\varphi_2(x_0)} \int_{D_x \setminus D_x^{y_1}} \frac{\varphi_2(x)}{\psi_1(x, y_2)}\,dx \right)^{-1}.$$

To accomplish the proof we need only to show that the densities of conditional distributions $X/Y = y_1$, $X/Y = y_2$, Y/X are equal respectively to $\varphi_1(x)\varphi_2(x)$, $\psi_1(x, y)$. This can easily be verified similarly as in Theorem 1.

The above proven theorem allows us to determine the density when the projections onto Ox axis of two intersections of D by lines $y = y_1$ and $y = y_2$ cover D_x and their common part is of positive measure. This theorem could be generalized to the case when a finite number of intersections cover D_x and each of this intersections has a common part of positive measure with at most one of the remaining intersections.

Finally we conclude that the density $f(x, y)$ can be determined if all conditional distributions of Y/X are known and if given is a finite number of distributions $X/Y = y_i$ such that projections of $D_x^{y_i}$ cover

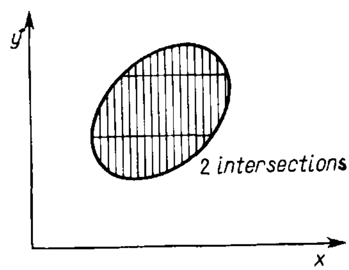

Fig. 5

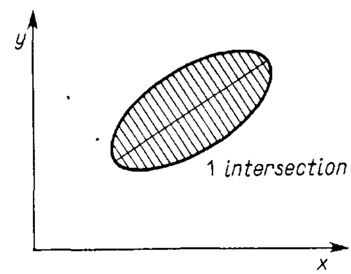

Fig. 6

D_x. The number of these intersections could be reduced if instead of taking intersections by planes parallel to the coordinate planes intersections by other planes were taken as illustrated on Figs. 5, 6.

Fig. 5 shows the case of intersections by straight lines parallel to the axes. It is seen that then to connect all intersections by lines parallel to Oy axis at least 2 intersections by lines parallel to Ox axis are needed.

Fig. 6 shows, however, that taking intersections by lines not necessarily parallel to the axes but parallel to the eclipse main axis — one intersection suffices. The number of intersections can also be reduced whenever a type of a required distribution is known. Knownig, for instance, that the two-dimensional distribution (X, Y) is normal to establish its density it suffices to give the densities of distributions of $X/Y = y_1$, $X/Y = y_2$ and $Y/X = x_1$ $(y_1 \neq y_2)$.

Therefore in this case only three normed intersections of the surface $f(x, y)$ are sufficient.

We shall now give a simple example illustrating applications of Theorem 3 to determining densities of the two-dimensional distributions at given densities of conditional distributions.

EXAMPLE. Determine the density of the two-dimensional distribution of the variables (X, Y) assuming that the densities of distributions of X/Y and $Y/X = 1$ are the functions

$$\varphi_1(x/y) = \begin{cases} ye^{-xy} & \text{for } x, y > 0, \\ 0 & \text{for others,} \end{cases}$$

$$\psi_1(y/1) = \begin{cases} \dfrac{\beta^{\alpha+1}}{\Gamma(\alpha)} y^{\alpha} e^{-\beta y} & \text{for } y > 0, \\ 0 & \text{for others.} \end{cases}$$

Since all hypotheses of Theorem 3 are fulfilled for $x, y > 0$ we have

$$f(x, y) = \frac{\varphi_1(x/y)\psi_1(y/1)}{\varphi_1(1/y)}\left(\int_1^{\infty}\int_1^{\infty}\frac{\varphi_1(x/y)\psi_1(y/1)}{\varphi_1(1/y)}\,dxdy\right)^{-1}$$
$$= \frac{(\beta-1)^{\alpha}y^{\alpha}}{\Gamma(\alpha)}e^{-xy}e^{-y(\beta-1)}.$$

5. Representation of the probability function of the two-dimensional jump distribution by means of conditional distributions. Similarly as for continuous distributions we can deal with a two-dimensional jump variable the only difference upon approach being the fact that instead of integrals in corresponding formulae the sequences and series will appear.

Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ denote jump points in boundary distributions respectively of X and Y.

Let further $p(x_j, y_k) = P(X_j = x_j, y = y_k)$ designates a joint probability distribution of variables (X, Y) whereas $\varphi(x_j)$, $\varphi(y_k)$, $\varphi_1(x_j/y_k)$ $\psi_1(y_k/x_j)$ denote probability function of boundary distribution of X, Y and conditional random variables X/Y, Y/X respectively.

THEOREM 5. *If D is a set of points (x_j, y_k) for which $p(x_j, y_k) > 0$, then for all points of D there exist $\psi_1(y_k/x_j)$ and $\psi_2(x_j/y_k)$ and the following equality holds:*

$$\frac{\psi_1(y_k/x_j)}{\varphi_1(x_j/y_k)} = \frac{\psi(y_k)}{\varphi(x_j)}.$$

Since $\sum\limits_j p_{jk}$ and $\sum\limits_k p_{jk}$ both differ from zero, the functions

$$\psi_1(y_k/x_j) = \frac{p(x_j, y_k)}{\sum\limits_k p_{jk}}, \qquad \varphi_1(x_j/y_k) = \frac{p(x_j, y_k)}{\sum\limits_j p_{jk}}$$

exist in D. Dividing these last equalities by sides we accomplish the proof.

THEOREM 6. *If D is a set of points (x_j, y_k) for which $p(x_j, y_k) > 0$, $D_x^{y_{l_0}}$ the projection of its intersection by $y = y_{l_0}$ onto Ox and D_x is the projection of D onto Ox axis and if $D_x^{y_{l_0}} = D_x$ holds in every point of D, then the relation*

$$(8) \qquad p(x_j, y_k) = \frac{\psi(y_k/x_j)\varphi_1(x_j/y_{l_0})}{\psi_1(y_{l_0}/x_j)} \left(\sum_{j,k} \frac{\psi_1(y_k/x_j)\varphi_1(x_j/y_{l_0})}{\psi_1(y_{l_0}/x_j)}\right)^{-1}$$

is satisfied.

Proof of this theorem is completely identical to this for a continuous case with a replacement of an integral by a sum.

Similarly as for continuous random variables the conditions and methods of unique determinations of the two-dimensional jump distribution by means of the conditional distributions will be given.

THEOREM 7. *Let D denote a set of points (x_j, y_k) of the plane $(j, k = 1, 2, \ldots)$. Given are functions $\varphi_1(x_j)$ and $\psi_1(x_j/y_k)$ satisfying the conditions*

1° *$\psi_1(x_j, y_k) > 0$ for $(x_j, y_k) \in D$ and $= 0$ for others, $\varphi_1(x_j) > 0$ for $x_j \in D_{x_l}^{y}$ and $= 0$ for $x_j \notin D_x^{y_l}$.*

2° $\sum\limits_k \psi_1(x_j, y_k) = 1, \ \sum\limits_j \varphi_1(x_j) = 1.$

3° $\sum\limits_j \frac{\varphi_1(x_j)}{\psi_1(x_j, y_l)} = k < \infty$ *and holds the condition $D_x^{y_l} = D_x$.*

Then the function

$$p(x_j, y_k) = \begin{cases} \dfrac{\varphi_1(x_j, y_k)\varphi_1(x_j)}{\psi_1(x_j, y_l)} \left(\sum\limits_{j,k} \dfrac{\psi_1(x_j, y_k)\varphi_1(x_j)}{\psi_1(x_j, y_l)}\right)^{-1} & \text{for } (x_j, y_k) \in D, \\ 0 & \text{for } (x_j, y_k) \notin D \end{cases}$$

is the only function representing some two-dimensional jump probability distribution and whose conditional distributions Y/X and $X/Y = y_l$ are given functions.

The proof of this theorem is analogous to that for continuous case.

COROLLARY. *If a set of points (x_j, y_k) is finite, then for the theorem to hold Condition 3 can be dropped.*

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