On convergence of bilinear integrals

0. Introduction. Let $I = (0, \infty)$, let $\mathcal{M}^-, \mathcal{M}^0, \mathcal{M}^+$ denote the sets of all monotone and non-negative functions $\mu = \mu(t) : I \to I$ for which $\mu(t) > 0$ if $t > 0$ and such that $\lim_{t \to \infty} \mu(t) = 0$, $\mu(t) = \text{const}$, $\lim_{t \to \infty} \mu(t) = \infty$, respectively. Also let $\mathcal{M}^{0+} = \mathcal{M}^0 \cup \mathcal{M}^+$ and $\mathcal{M} = \mathcal{M}^- \cup \mathcal{M}^0$. We have $\mu(0) > 0$ if $\mu \in \mathcal{M}^- \cup \mathcal{M}^0$ and $\mu(0) \geq 0$ if $\mu \in \mathcal{M}^+$; whenever $\mu \in \mathcal{M}^+$, it is essential to distinguish the case whether $\mu(0+)$ is equal to zero or is positive.

Given three Banach spaces $X, Y, Z$, we consider a continuous bilinear operator $U : X \times Y \to Z$ and functions $x = x(t) : I \to X$, $y = y(t) : I \to Y$ satisfying the following conditions:

(1) $x = x(t)$ is strongly differentiable a.e. and $x'(t)$ is locally Bochner integrable in $(0, \infty)$,

(2) $x = x(t)$ is the indefinite integral of their derivative, i.e. $x(t) = x(0) + \int_0^t x'(\tau) d\tau$ (Bochner integral),

(3) $y = y(t)$ is strongly measurable, locally Bochner integrable in $(0, \infty)$ and to each positive number $b$ there exists the improper integral $\int_0^b y(t) dt = \lim_{\varepsilon \to 0} \int_\varepsilon^b y(t) dt$.

Sometimes instead of (3) we shall take the assumption

(4) $y = y(t)$ satisfies (3), and there exists

$$\int_0^\infty y(t) dt = \lim_{b \to \infty} \left( \lim_{\varepsilon \to 0} \int_\varepsilon^b y(t) dt \right).$$

We also consider the functions $x = x(t) : (0, \infty) \to X$ satisfying (1) and such that

$$x(t) = -\int_t^\infty x'(\tau) d\tau \quad \text{(Bochner integral)}.$$
Conditions, under which \( x = x(t) \) is representable by the indefinite integral of their derivative are investigated in [3] (cf. also [4], Theorem 3.8.6). If \( X \) is the space \( R \) of reals and \( x = x(t) \) is absolutely continuous, then (2) is satisfied. If \( X \) is non-reflexive, then — as there was observed by I. Gelfand — there are strongly absolutely continuous functions nowhere differentiable: thus conditions (1)–(2) do not hold. The existence of singular functions establishes that implication (1) \( \Rightarrow \) (2) is false, already if \( X = R \).

The purpose of this note is to present some elementary investigations related with the convergence of the improper bilinear integrals of Bochner type and present some applications to Fourier integral.

There is an evident parallele between the results established here and that given in [5].

1. The space \( \mathcal{L}_\mu (X) \). Given a Banach space \( X \) and \( \mu \) in \( \mathfrak{M}^- \), we denote by \( \mathcal{L}_\mu (X) \) the set of all functions \( x = x(t) : I \to X \) satisfying (1), (2) and such that \( \mathcal{P}(x) = \int_0^\infty \mu(t)\|x'(t)\|dt < \infty \). If \( \mu \in \mathfrak{M}^{0+} \) and \( \mu(0+) > 0 \), we define the class \( \mathcal{L}_\mu (X) \) similarly, requiring supplementary that \( \lim_{t \to \infty} x(t) = 0 \). If \( \mu \in \mathfrak{M}^{0+} \) and \( \mu(0+) = 0 \), then \( \mathcal{L}_\mu (X) \) will denote the set of all functions \( x = x(t) : (0,\infty) \to X \) such that (1), (2A) and \( \mathcal{P}(x) < \infty \) are satisfied; in this case the condition \( \lim_{t \to \infty} x(t) = 0 \) holds automatically.

The space \( \mathcal{L}_\mu (X) \) is linear. If \( \|x\| = \mu(0)\|x(0)\| + \mathcal{P}(x) \) for \( \mu \) in \( \mathfrak{M}^- \) and \( \|x\| = \mathcal{P}(x) \) for \( \mu \) in \( \mathfrak{M}^{0+} \), then \( \mathcal{L}_\mu (X) \) is a Banach space.

To prove this, let us observe that the space \( L_{1,\mu} (X) \) of all functions \( u = u(t) : I \to X \), strongly measurable and Bochner integrable on \( I \) with respect to the measure \( m(A) = \int A \mu(t)dt \) is a complete space under the norm \( \|u\| = \int \|u(t)\|dm \), i. e. \( \|u\| = \int_0^\infty \mu(t)\|u(t)\|dt \).

Let \( \mu \in \mathfrak{M}^{0+} \) and let \( \{x_p\} \), where \( x_p = x_p(t) \in \mathcal{L}_\mu (X) \), be a Cauchy sequence, i. e. \( \mathcal{P}(x_p - x_q) < \varepsilon \) if \( p, q \geq N \). \( L_{1,\mu} (X) \) is complete, hence there is a function \( u = u(t) \) in \( L_{1,\mu} (X) \) such that

\[
\int_0^\infty \mu(t)\|u(t) - x'_q(t)\|dt \leq \varepsilon \quad \text{if} \quad q \geq N.
\]

Since \( \mu \in \mathfrak{M}^{0+} \) hence \( \int_0^\infty \|u(t)\|dt < \infty \) if \( \mu(0+) > 0 \) and, if \( b \) is any positive number, then \( \int_b^\infty \|u(t)\|dt < \infty \) in case if \( \mu(0+) = 0 \). Let

\[
x(t) = -\int_t^\infty u(\tau)d\tau, \quad \text{i. e.} \quad x(t) = -\int_t^\infty x'(\tau)d\tau.
\]
Thus \( x \in \mathcal{L}_\mu(X) \), \( \|x - x_q\| \leq \varepsilon \) if \( q \geq N \) and \( \mu(0+) = 0 \). If \( \mu(0+) > 0 \), taking \( x(0) = -\int_0^\infty x'(t)dt \), we get \( x = x(t) \in \mathcal{L}_\mu(X) \) and \( \|x - x_q\| \leq \varepsilon \) if \( q \geq N \).

Let \( \mu \in \mathfrak{B}^- \) and let \( \{x_p\} \) be a Cauchy sequence in \( \mathcal{L}_\mu(X) \). Then there is a \( u = u(t) \) in \( L_{1,\mu}(X) \) such that \( x'_p \to u \) as \( p \to \infty \). Taking

\[
x = x(t), \quad \text{where} \quad x(t) = \lim_{p \to \infty} x_p(0) + \int_0^t u(\tau)d\tau,
\]

we get (1), (2) and \( \varphi(x) < \infty \); moreover, \( \|x - x_p\| \leq \varepsilon \) if \( q \geq N \).

Thus, \( \mathcal{L}_\mu(X) \) is complete.

Let \( \mu \in \mathfrak{B} \), let \( Y \) be a Banach space and let \( \mathcal{G}_\mu(Y) \) denote the set of all functions \( y = y(t) : I \to Y \) such that:

- if \( \mu \in \mathfrak{B}^+ \), then condition (3) is satisfied and
  \[
  \|y\| = \sup_t \frac{1}{\mu(t)} \|\int_0^t y(\tau)d\tau\| < \infty,
  \]

- if \( \mu \in \mathfrak{B}^- \), then condition (4) is satisfied and
  \[
  \|y\| = \sup_t \frac{1}{\mu(t)} \|\int_t^\infty y(\tau)d\tau\| < \infty.
  \]

\( \mathcal{G}_\mu(Y) \) is a normed space but not a Banach space. Indeed, if we consider e.g. the space \( \mathcal{G}_1 \), i.e. \( \mathcal{G}_\mu(Y) \), where \( Y = \mathbb{R} \), \( \mu(t) = 1 \), it is easy to observe that \( \mathcal{G} \) is non-complete: the uniform limit of a sequence of absolutely continuous functions does not necessarily be absolutely continuous.

Let \( 1 \leq p \leq \infty \), and let \( \langle X \times Y; \|\cdot\|_p \rangle \) denote the Cartesian product \( X \times Y \) with the norm \( \|(x, y)\|_p = (\|x\|^p + \|y\|^p)^{1/p} \).

**Lemma 1.** Let \( p \geq 1 \) and let \( p^{-1} + q^{-1} = 1 \). The spaces \( \langle X \times Y; \|\cdot\|_p \rangle^* \) and \( \langle X^* \times Y^*; \|\cdot\|_q \rangle \) are isomorphic if \( p > 1 \) and equivalent if \( p = 1 \).

**Proof.** We consider the correspondence

\[
\langle X \times Y; \|\cdot\|_p \rangle^* \ni f \xrightarrow{V} (\varphi_X, \varphi_Y) \in \langle X^* \times Y^*; \|\cdot\|_q \rangle,
\]

where \( \varphi_X(x) = f(x, 0) \), \( \varphi_Y(y) = f(0, y) \). \( V \) is linear and continuous, since \( \|(\varphi_X, \varphi_Y)\|_q \leq 2^{1/q}\|f\| \); it is also a map onto. Indeed, denoting for \( (\varphi, \psi) \) in \( \langle X^* \times Y^*; \|\cdot\|_q \rangle \), \( f = \varphi + \psi \), we get \( f \in \langle X \times Y; \|\cdot\|_p \rangle^* \), since

\[
|f(x, y)| \leq |\varphi(x)| + |\psi(y)| \leq \|(\varphi, \psi)\|_q \|(x, y)\|_p;
\]

we also have \( \varphi = \varphi_X, \psi = \varphi_Y \). Thus our statement follows by the inequality

\[
2^{-1/q}\|(\varphi_X, \varphi_Y)\|_q \leq \|f\| \leq \|(\varphi_X, \varphi_Y)\|_q.
\]
From the above proof we get a fact useful in the proof of Theorem 6: every \( F \) in \( \langle X \times Y : \| \cdot \|_1 \rangle^* \) is of the form \( F = \varphi + \psi \), where \( \varphi \in X^* \), \( \psi \in Y^* \) and \( \| F \| = \max \{ \| \varphi \|, \| \psi \| \} \).

Let \( L_1(X) = L_{1,\mu}(X) \) if \( \mu(t) = 1 \); if \( X = R \), instead of \( L_1(R) \) we shall write \( L_1 \) — the space of all real functions, integrable over \( I \).

**Lemma 2.** If \( \mu \in W^0^+ \), then \( L_\mu(X) \) and \( L_1(X) \) are equivalent. If \( \mu \in W^- \), then \( L_\mu(X) \) is equivalent to \( \langle X \times L_1(X) ; \| \cdot \|_1 \rangle \).

**Proof.** Let \( \mu \in W^0^+ \), \( x = x(t) \in L_\mu(X) \) and let \( y = y(t) \in L_1(X) \). The correspondence \( x \rightarrow y \), where \( y(t) = \mu(t)x'(t) \) is linear and \( \| x \| = \| y \| \).

To each \( y \) in \( L_1(X) \) there exists an uniquely determined \( x \) in \( L_\mu(X) \) such that \( y = \mu(t)x'(t) \): it suffices to take \( x(t) = - \int_0^\infty \frac{y(\tau)}{\mu(\tau)} d\tau \).

Let \( \mu \in W^- \) and let \( (u, y) \in \langle X \times L_1(X) ; \| \cdot \|_1 \rangle \). The correspondence \( (u, y) \rightarrow x \), where

\[
x(t) = \frac{u}{\mu(0)} + \int_0^t \frac{y(\tau)}{\mu(\tau)} d\tau,
\]

is the desired linear isometry of \( \langle X \times L_1(X) ; \| \cdot \|_1 \rangle \) onto \( L_\mu(X) \).

**2. Tests for convergence.**

**Lemma 3.** Let \( \mu \in W \), \( x \in L_\mu(X) \) and \( y \in \mathcal{G}_\mu(Y) \). Then for each interval \( \langle a, b \rangle \subset I \),

\[
\int_a^b U(x(t), y(t)) dt = U(x(b), \int_a^b y(\tau) d\tau) - \int_a^b U(x'(t), \int_a^t y(\tau) d\tau) dt.
\]

**Proof.** It suffices to consider the case when \( \mu \) is in \( W^0^+ \). Let \( \mu \in W^0^+ \) and \( \mu(0+) > 0 \). Evidently, \( \lim_{h \rightarrow 0} \int_a^b y(\tau) d\tau = \int_a^t y(\tau) d\tau \) for \( a \leq t \leq b \), and

\[
\lim_{h \rightarrow 0} \frac{1}{h} \int_a^t y(\tau) d\tau = y(t), \quad \lim_{h \rightarrow 0} \frac{1}{h} (x(t + h) - x(t)) = x'(t)
\]
a.e. in \( \langle a, b \rangle \). We put

\[
\varphi(t) = U\left(x(t), \int_a^t y(\tau) d\tau\right), \quad a \leq t \leq b.
\]

Then

\[
\varphi'(t) = U\left(x'(t), \int_a^t y(\tau) d\tau\right) + U\left(x(t), y(t)\right), \quad \text{a.e. in } \langle a, b \rangle.
\]
The function \( q'(t) \) is Bochner integrable on \( \langle a, b \rangle \): the strong measurability is evident; moreover,

\[
\|q'(t)\| \leq \|U\| \left( \left\| x'(t) \right\| + \left\| \int_a^t y(\tau) \, d\tau \right\| + \left\| x(t) \right\| \left\| y(t) \right\| \right)
\]
a.e. in \( \langle a, b \rangle \), and the functions in parenthesis are integrable on \( \langle a, b \rangle \).

Let

\[
K = \sup_{a \leq \beta \leq b} \left( \left\| x(\beta) \right\|; \left\| \int_a^\beta y(\tau) \, d\tau \right\| \right) \|U\|.
\]

The functions \( x(t), \int_a^t y(\tau) \, d\tau \) are strongly absolutely continuous, hence so is \( q(t) \), because

\[
\|q(\beta) - q(a)\| \leq K \left( \|x(\beta) - x(a)\| + \left\| \int_a^\beta y(\tau) \, d\tau \right\| \right).
\]

Let \( f \in Z^* \) and \( a \leq t \leq b \); \( f\{q(t)\} \) is absolutely continuous, hence

\[
f\left( \int_a^t q'(\tau) \, d\tau \right) = \int_a^t f(q'(\tau)) \, d\tau = \left[ f(q(\tau)) \right]' \, d\tau = f(q(t)).
\]

Taking \( t = b \), by the arbitrariness of \( f \) in \( Z^* \) we get

\[
\int_a^b q'(\tau) \, d\tau = q(b),
\]

which proves our statement in the case \( \mu(0+) > 0 \). Thus the thesis of lemma is also true in case \( \mu(0+) = 0 \), but only for intervals \( \langle a, b \rangle \subset (0, \infty) \). The proof will be concluded, if we show — using the Lebesgue bounded convergence theorem for Bochner integrals — that

\[
\lim_{a \to 0} \int_a^b U(x'(t), \int_a^t y(\tau) \, d\tau) \, dt = \int_0^b U(x'(t), \int_0^t y(\tau) \, d\tau) \, dt.
\]

Let \( \chi_{\langle a, b \rangle} \) denote the indicator of \( \langle a, b \rangle \subset (0, \infty) \) and let

\[
g_a(t) = U\left(x'(t), \chi_{\langle a, b \rangle}(t) \int_a^t y(\tau) \, d\tau \right), \quad G_a(t) = g_a(t) - U\left(x'(t), \int_0^t y(\tau) \, d\tau \right).
\]

We easily observe that \( G_a(t) \) is convergent in measure to 0 as \( a \to 0 \). The function \( g_a \) can be majorized by an integrable function. Indeed, if \( 0 \leq t \leq a \), then \( g_a(t) = 0 \) and

\[
\|g_a(t)\| \leq \|U\| \|\mu(t)\| \|x'(t)\| \sup_{a \leq t \leq b} \frac{1}{\mu(t)} \left\| \int_a^t y(\tau) \, d\tau \right\| \leq 2 \|U\| \|\mu(t)\| \|x'(t)\| \|y\|.
\]
a.e. in \( \langle a, b \rangle \).
The space $C_{\infty}$ of all real continuous and bounded functions $f$ on $I$ for which there exists $\lim_{t \to \infty} f(t)$ is a Banach space if $\|f\| = \sup |f(t)|$. The linear span of $\{f_n\}$, where $f_n = e^{-nt}$, $n = 0, 1, 2, \ldots$, is dense in $C_{\infty}$ (cf. [7], p. 27).

**Lemma 4.** Let $\mu \in \mathcal{M}^+$ and let $\mu$ be absolutely continuous. If $f \in C_{\infty}$, then

$$\lim_{s \to \infty} \frac{1}{\mu(s)} \int_0^s \mu'(t)f(t)dt = \lim_{t \to \infty} f(t)$$

([2], p. 352 as an Exercise).

**Proof.** We have $\mu'(t) \geq 0$ a.e. in $I$. Let us consider the following functionals $\Phi_s$ ($s > 0$) and $\Psi$ on $C_{\infty}$:

$$\Phi_s(f) = \frac{1}{\mu(s)} \int_0^s f(t)\mu'(t)dt, \quad \Psi(f) = \lim_{t \to \infty} f(t).$$

Then $\|\Phi_s\| = O(1)$ if $s \to \infty$, since

$$\|\Phi_s\| \leq \frac{1}{\mu(s)} \int_0^s \mu'(t)dt = \frac{1}{\mu(s)} (\mu(s) + \text{const}).$$

We prove that $\lim_{s \to \infty} \Phi_s(f_n) = \Psi(f_n)$ if $n = 0, 1, 2, \ldots$ This is evident for $n = 0$. If $n = 1, 2, \ldots$ if suffices to prove that $\lim_{s \to \infty} \Phi_s(f_n) = 0$. Since

$$\Phi_s(f_n) = e^{-ns} - \frac{\mu(0)}{\mu(s)} + \frac{n}{\mu(s)} \int_0^s e^{-nt} \mu(t)dt,$$

it is sufficient to prove that

$$\lim_{s \to \infty} \frac{1}{\mu(s)} \int_0^s \mu(t)e^{-nt}dt = \lim_{s \to \infty} \int_0^s \frac{\mu(t)}{\mu(s)} \chi_{(0,s)}(t)e^{-nt}dt = 0.$$

We denote

$$g_s(t) = \chi_{(0,s)}(t) \frac{\mu(t)}{\mu(s)} e^{-nt}, \quad 0 < s, t < \infty.$$  

Then $g_s(t) \leq g(t)$, where $g(t) = e^{-nt} \mu$ if $t \leq s$ and $g(t) = 0$ if $t > s$; moreover, if $t \geq 0$, then $\lim_{s \to \infty} g_s(t) = 0$. Now we apply the well-known Lebesgue theorem.

By the Banach–Steinhaus theorem, there exists $\lim_{s \to \infty} \Phi_s(f)$ and is equal to $\Psi(f)$; moreover, $f$ is in $C_{\infty}$. 
Lemma 5. If \( \mu \) in \( M^+ \) is absolutely continuous and if for a real, measurable and locally integrable function \( u(t) \) on \( I \) the improper integral \( \int_0^\infty u(t) dt \) exists, then
\[
\int_0^t \mu(\tau) u(\tau) d\tau = o(\mu(t)) \quad \text{as } t \to \infty.
\]

Proof. Dividing the identity
\[
\int_0^t \mu(\tau) u(\tau) d\tau = \mu(t) \int_0^t u(\tau) d\tau - \int_0^t \mu'(\tau) \left( \int_0^\tau u(s) ds \right) d\tau
\]
by \( \mu(t) \), passing to the limit with \( t \to \infty \) and applying Lemma 4 (for \( f(t) = \int_0^\tau u(s) ds \)) we get the statement.

Lemma 6. If \( x \in L_\mu^+(X) \) and \( \mu \) in \( M^- \) is absolutely continuous, then \( \mu(t) x(t) = o(1) \) if \( t \to \infty \).

Proof. Since \( \int_0^\infty \mu(t) \| x'(t) \| dt < \infty \), \( 1/\mu(t) \) is absolutely continuous and is an element of \( M^+ \), by the preceding lemma (for \( u(t) = \mu(t) \| x'(t) \| \)) we get \( \mu(t) \int_0^t \| x'(\tau) \| d\tau \to 0 \) as \( t \to \infty \). Moreover,
\[
\| x(t) - x(0) \| = \left\| \int_0^t x'(\tau) d\tau \right\| \leq \int_0^t \| x'(\tau) \| d\tau.
\]
Hence it suffices to note the relation \( \lim_{t \to \infty} \mu(t) \| x(t) - x(0) \| = 0 \).

Lemma 7. If \( \mu \in M^0+ \) and \( x \in L_\mu(X) \), then \( \lim_{t \to \infty} \mu(t) x(t) = 0 \).

Proof. We have
\[
\lim_{t \to \infty} \int_0^t \mu(\tau) \| x'(\tau) \| d\tau = 0,
\]
and therefore
\[
\mu(t_0) \| x(t_0) \| \leq \int_{t_0}^t \mu(\tau) \| x'(\tau) \| d\tau.
\]

Theorem 1. Let \( \mu \in M^0+ \), let \( x \in L_\mu(X) \), and let a set \( \{ y_b \}_{b \geq 0} \) of elements in \( L_\mu(Y) \) be given such that \( \| y_{b_1} - y_{b_2} \| \to 0 \) as \( b_1, b_2 \to \infty \). Then there exists the limit
\[
\lim_{b \to \infty} \int_0^t U(x(t), y_b(t)) dt
\]
and
\[
\lim_{b \to \infty} \left\| \int_0^b U(x(t), y_b(t)) \, dt \right\| \leq \|U\| \|x\| \lim_{b \to \infty} \|y_b\|.
\]

**Proof.** Since \(\|y_b\| = O(1)\), \(\mu(b)\|x(b)\| = o(1)\) as \(b \to \infty\) (Lemma 7), we have \(U(x(b), \int_0^b y_b(\tau) \, d\tau) \to 0\) as \(b \to \infty\). Let \(0 \leq a < b < \infty\) and let
\[
I_b = \int_0^b U\left(x'(t), \int_0^t y_b(\tau) \, d\tau\right) \, dt.
\]

Then
\[
(5) \quad I_b - I_a = \int_a^b U\left(x'(t), \int_0^t (y_b(\tau) - y_a(\tau)) \, d\tau\right) \, dt + \int_a^b U\left(x'(t), \int_0^t y_a(\tau) \, d\tau\right) \, dt.
\]

Thus \(I_b - I_a \to 0\) if \(a, b \to \infty\) and \(\lim_{b \to \infty} I_b\) exist. The existence of the desired limit follows by Lemma 3. Moreover,
\[
\lim_{b \to \infty} \int_0^b U(x(t), y_b(t)) \, dt = -\lim_{b \to \infty} \int_0^b U\left(x'(t), \int_0^t y_b(\tau) \, d\tau\right) \, dt.
\]

**Remark 1.** If we consider another system of assumptions for \(\{y_b\}\), namely if
\[
y_b \in \mathcal{G}_\mu(Y), \quad \sup_b \|y_b\| < \infty, \quad y_b(t) = 0 \quad \text{for} \ t > b,
\]
\[
y_{b_2}(t) = y_{b_1}(t) \quad \text{for} \ 0 \leq t \leq b_1 \text{ and } b_1 < b_2,
\]
then the thesis of Theorem 1 remains also true; in the inequality written there we must replace \(\lim_{b \to \infty} \|y_b\|\) by \(\sup_b \|y_b\|\). To make the proof similar to that as in Theorem 1, it suffices to observe that
\[
I_b - I_a = \int_a^b U\left(x'(t), \int_0^t y_b(\tau) \, d\tau\right) \, dt.
\]

This follows, by (5) and by the fact that \(y_b(t) - y_a(t) = 0\) outside \(\langle a, b \rangle\).

**Remark 2.** A “sequence” \(\{y_b\}\) of type (6) is a Cauchy sequence if and only if
\[
\lim_{a, b \to \infty} \sup_{a < t < b} \frac{1}{\mu(t)} \left\| \int_a^t y_b(\tau) \, d\tau \right\| = 0.
\]
Indeed, if \( 0 \leq a < b < \infty \), then

\[
\left\| \int_0^t y_b(\tau) d\tau - \int_0^t y_a(\tau) d\tau \right\| = \begin{cases} 
0 & \text{for } 0 \leq t \leq a, \\
\left\| \int_a^t y_b(\tau) d\tau \right\| & \text{for } a \leq t \leq b, \\
\left\| \int_a^b y_b(\tau) d\tau \right\| & \text{for } t \geq b
\end{cases}
\]

and therefore

\[
\|y_b - y_a\| = \sup_{a \leq t \leq b} \frac{1}{\mu(t)} \left\| \int_a^t y_b(\tau) d\tau \right\|.
\]

Consider e.g. \( \mu(t) = 1 \), \( y_b(t) = \sin t \) for \( 0 \leq t \leq b \) and \( y_b(t) = 0 \) for \( t > b \). Then (6) is satisfied, but the Cauchy condition does not hold (the integral \( \int_0^\infty \sin t dt \) is divergent). Thus, in Theorem 1 as well as in Remark 1 there are given quite different (sufficient) tests for convergence.

An easy consequence of Theorem 1 is

**Corollary 1.** Let \( \mu \in \mathbb{M}^0 \), let \( x \in \mathcal{L}_\mu(X) \) and let \( y \in \mathcal{F}_\mu(Y) \). Then the improper integral

\[
\int_0^\infty U(x(t), y(t)) dt
\]

is convergent in \( Z \) and

\[
\int_0^\infty U(x(t), y(t)) dt = -\int_0^\infty U(x'(t), \int_0^t y(\tau) d\tau) dt
\]

(on the right-hand side is Bochner integral). Moreover,

\[
\left\| \int_0^\infty U(x(t), y(t)) dt \right\| \leq \|U\| \|x\| \|y\|.
\]

**Theorem 2.** Let \( \mu \) in \( \mathbb{M}^- \) be absolutely continuous, let \( x \in \mathcal{L}_\mu(X) \), \( y_b \in \mathcal{F}_\mu(Y) \), where \( b \geq 0 \), and let \( \|y_b - y_\beta\| \to 0 \) as \( b, \beta \to \infty \). Then there exists \( \lim_{b \to \infty} \int_a^b U(x(t), y_b(t)) dt \), where \( a \geq 0 \) and

\[
\lim_{b \to \infty} \left\| \int_a^b U(x(t), y_b(t)) dt \right\| \leq \|U\| \left( \|\mu(a)\| \|x(a)\| + \int_a^\infty \|\mu(t)\| \|x'(t)\| dt \right) \lim_{b \to \infty} \|y_b\|.
\]
Proof. The improper integral \( \int_0^\infty y_b(t) \, dt \) exist for each \( b \geq 0 \). By Lemma 3, we get

\[
\int_a^b U(x(t), y_b(t)) \, dt = -U(x(a), \int_a^\infty y_b(t) \, dt) + U(x(b), \int_0^b y_b(t) \, dt) + \int_a^b U(x'(t), \int_0^\infty y_b(\tau) \, d\tau) \, dt.
\]

Let \( 0 \leq a < \beta < b < \infty \) and let

\[
I_b = \int_a^b U(x'(t), \int_0^\infty y_b(\tau) \, d\tau) \, dt.
\]

Then

\[
I_b - I_\beta = \int_a^b U(x(t), \int_0^\infty (y_b(\tau) - y_\beta(\tau)) \, d\tau) \, dt + \int_\beta^b U(x(t), \int_0^\infty y_\beta(\tau) \, d\tau) \, dt,
\]

\[
\|I_b - I_\beta\| \leq \|U\| \left( \int_a^b \mu(t) \|x'(t)\| \|y_b - y_\beta\| + \sup_{\beta} \|y_\beta\| \int_\beta^b \mu(t) \|x'(t)\| \, dt \right)
\]

and therefore \( \|I_b - I_\beta\| \to 0 \) as \( b, \beta \to \infty \). To conclude the proof it suffices to apply Lemma 6 and to observe that there exists \( \lim_{b \to \infty} \int_0^b y_b(t) \, dt \).

**Corollary 2.** If \( \mu \in L^- \) is absolutely continuous, if \( x \in L_\mu(X) \), \( y \in G_\mu(Y) \), then the improper integral (7) is norm convergent in \( Z \) and if \( a \geq 0 \), then

\[
\left\| \int_a^\infty U(x(t), y(t)) \, dt \right\| \leq \|U\| \|\mu(a)\| \|y(a)\| + \int_a^\infty \mu(t) \|x'(t)\| \, dt.
\]

**Remark 3.** (i) Without the assumption of absolute continuity of \( \mu \), the statements of Theorem 2 and Corollary 2 are also valid, but under supplementary assumption that \( \mu(t) \|x(t)\| = o(1) \) as \( t \to \infty \).

(ii) If \( \mu \in M \), then the map \( \tilde{U}: L_\mu(X) \times G_\mu(Y) \to Z \), where \( \tilde{U}(x, y) = \int_0^\infty U(x(t), y(t)) \, dt \), is bilinear and continuous.

We denote by \( L_\infty \) the space of all real measurable and essentially bounded functions \( g = g(t) \) on \( I \). Let \( \|g\| = \text{esssup} |g(t)| \). Similarly as in the theory of unconditionally convergent series (e. g. [1], p. 159) we prove the following

**Lemma 8.** If a function \( u = u(t) : I \to X \) satisfies (3), then the following conditions are equivalent:

(i) if \( f \in X^* \), then \( \int_0^\infty |f(u(t))| \, dt < \infty \),
(ii) there is a positive constant $K$ such that if $g \in L_\infty$ and $b > 0$, then
\[ \left\| \int_a^b g(t) u(t) \, dt \right\| \leq K \| g \|, \]

(iii) if $\gamma \in L_\infty$ and $\| \gamma \chi_{(a, \beta)} \| \to 0$ as $a, \beta \to \infty$, then there exists the improper integral $\int_0^\infty \gamma(t) u(t) \, dt$ and
\[ \left\| \int_0^\infty \gamma(t) u(t) \, dt \right\| \leq K \| \gamma \|. \]

Proof. Let $k, n = 1, 2, \ldots$ and
\[ A_{k,n} = \left\{ f \in X^* : \int_0^b |f(u(t))| \, dt \leq k \right\}, \quad A_k = \cap_{n=1}^\infty A_{k,n}. \]

The sets $A_k$ are closed and (i) implies $\bigcup_{k=1}^\infty A_k = X^*$, hence according to Baire theorem there is an integer $k_0$ such that $A_{k_0}$ contains some ball $K(f_0, r)$. The definition of $A_{k_0}$ enables us to take $f_0 = 0$. Thus, there exists a positive constant $K$ such that if $f \in X^*$ and $\| f \| \leq 1$, then
\[ \int_0^\infty |f(u(t))| \, dt \leq K, \quad \text{i.e. } \sup_{\| f \| \leq 1} \int_0^\infty |f(u(t))| \, dt \leq K. \]

If $g \in L_\infty$, then
\[ \left\| \int_a^b g(t) u(t) \, dt \right\| = \sup_{\| f \| \leq 1} \left| f \left( \int_0^\infty u(t) \chi_{(a, b)}(t) g(t) \, dt \right) \right| \leq \sup_{\| f \| \leq 1} \int_0^\infty \chi_{(a, b)}(t) |g(t)| f(u(t)) \, dt \leq \sup_{\| f \| \leq 1} \int_0^\infty |f(u(t))| \, dt \| \chi_{(a, b)} g \|, \]

hence (ii). If (ii), then (iii), since for $\gamma$ in $L_\infty$, $\| \gamma \chi_{(a, \beta)} \| \to 0$ with $a, \beta \to \infty$ implies
\[ \left\| \int_a^\beta \gamma(t) u(t) \, dt \right\| = \left\| \int_a^\beta \chi_{(a, \beta)}(t) \gamma(t) u(t) \, dt \right\| \leq K \| \chi_{(a, \beta)} \gamma \| \to 0 \]
as $a, \beta \to \infty$. If (iii), we put $\gamma(t) = \chi_{(a, \beta)}(t) \text{sign } f(u(t))$. Then
\[ \int_0^b |f(u(t))| \, dt = \int_0^\infty \gamma(t) f(u(t)) \, dt = f \left( \int_0^\infty \gamma(t) u(t) \, dt \right) \leq K \| f \| \| \gamma \| = K \| f \| \]
and we get (i).

**Theorem 3.** Let $\mu \in M^+$. Given a function $x = x(t) : I \to X$ we suppose the following conditions: (1), (2), $x(t) \overset{\omega}{\to} 0$ (i.e. $x(t)$ converges weakly to 0)
as $t \to \infty$ and $\int_0^\infty \mu(t) |f(x'(t))| \, dt < \infty$ for every $f$ in $X^*$. Moreover, let $y = y(t)$ be a real function satisfying (3) and $\int_0^t y(\tau) \, d\tau = o(\mu(t))$ if $t \to \infty$. Then the improper integral $\int_0^\infty x(t)y(t) \, dt$ is norm convergent in $X$ and

$$\left\| \int_0^\infty x(t)y(t) \, dt \right\| \leq \sup_{t \in \mathbb{R}} \frac{1}{\mu(t)} \left| \int_0^t y(\tau) \, d\tau \right| \sup_{\|f\| \leq 1} \int_0^\infty \mu(t) |f(x'(t))| \, dt.$$  

**Proof.** If $y \to \infty$, then the convergence of the integral

$$\int_0^b x'(t) \left( \int_0^t y(\tau) \, d\tau \right) \, dt = \int_0^b \mu(t)x'(t) \left( \frac{1}{\mu(t)} \int_0^t y(\tau) \, d\tau \right) \, dt$$

is a consequence of Lemma 8. If $t_0$ is a positive number and $f$ is in $X^*$, then

$$\mu(t_0) |f(x(t_0))| \leq \int_{t_0}^\infty \mu(t) |f(x'(t))| \, dt,$$

hence $\mu(t) \|x(t)\| = O(1)$ as $t \to \infty$. The convergence of the desired integral follows now, by Lemma 3. We omit the easy proof of the inequality given in the statement.

3. Linear functionals on $L^\mu$. We denote by $L^\mu$ the space $L^\mu(X)$, where $X = R$. Similar notations shall be used for the space $\mathcal{S}^\mu(Y)$. As above, we denote by $L^\infty$ the space of all real measurable and essentially bounded functions $g = g(t)$ defined on $I$ with the norm

$$\|g\| = \text{ess sup} |g(t)|.$$  

Let us notice that if two Banach spaces are isomorphic (equivalent), then so are their conjugate. Thus, Lemmas 2 and 1 yield:

$$L^\mu \cong L^\infty \quad \text{if } \mu \in \mathcal{M}^+ \quad \text{and} \quad L^\mu \cong (R \times L^\infty; \| \cdot \|_\infty) \quad \text{if } \mu \in \mathcal{M}^-.$$  

More precisely, we have the following

**Theorem 4.** Each continuous linear functional on $L^\mu$, where $\mu \in \mathcal{M}^+$ is of the form

$$f(x) = \int_0^\infty x'(t) \mu(t) g(t) \, dt,$$

where $g \in L^\infty$ and $\|f\| = \|g\|$.  

**Proof.** We known (see the proof of Lemma 2) that the correspondence $y \to x$, where $x = Ay = -\int_0^\infty \frac{y(\tau)}{\mu(\tau)} \, d\tau$, is a linear isometry of $L_1$ onto
Convergence of bilinear integrals

Applying the representation theorem for continuous linear functionals on $L_\mu$ we get

$$f(x) = f(\lambda y) = \psi(y) = \int_0^\infty y(t)g(t)dt,$$

where $g \in L_\infty$ and $\|g\| = \|\psi\|$. Since $\lambda$ is a linear isometry of $L_1$ onto $L^*_\mu$, the adjoint operator $\lambda^*$ is a linear isometry of $L^*_\mu$ onto $L^*_1 \simeq L_\infty$, hence $\|f\| = \|\psi\|.$

**Theorem 5.** If $\mu \in \mathcal{M}^+$, then each continuous linear functional on $L_\mu$ is of the form

$$f(x) = c\mu(0)x(0) + \int_0^\infty x'(t)\mu(t)g(t)dt,$$

where $c = \text{const}$, $g \in L_\infty$ and $\|f\| = \max\{|c|, \|g\|\}$.

**Proof.** After the end of proof of Lemma 1 there was observed that each continuous and linear functional $F$ on $\langle X \times Y ; \| \cdot \|_1 \rangle$ is of the form $F = \varphi + \psi$, where $\varphi \in X^*$, $\psi \in Y^*$, and $\|F\| = \max\{\|\varphi\|, \|\psi\|\}$. Since $\mu$ is in $\mathcal{M}^+$ and the linear isometry $B$ of $\mathcal{L}^*$ onto $\langle B \times L_1 ; \| \cdot \|_1 \rangle$ is given by $Bx = (u, y)$, where $u = \mu(0) x(0)$ and $y = \mu(t)x'(t)$, we can write

$$f(x) = f(B^{-1}(u, y)) = F(u, y) = \varphi(u) + \psi(y)$$

$$= c\mu(0)x(0) + \int_0^\infty y(t)g(t)dt = c\mu(0)x(0) + \int_0^\infty x'(t)\mu(t)g(t)dt,$$

where $c = \text{const}$, $g \in L_\infty$, $\|f\| = \|F\|$ and $\|F\| = \max\{|c|, \|g\|\}$.

In the following we shall suppose that $\mu$ is in $\mathcal{M}^{0+}$. $\mathcal{G}_\mu$ is non-complete, hence it is non-equivalent to $\mathcal{L}^*_\mu$. We shall prove that the same situation takes place with the space $\mathcal{G}^\sim_\mu = \text{Compl} \mathcal{G}_\mu$ — the Hausdorff completion of $\mathcal{G}_\mu$. This space consists of the set of all classes of equivalent Cauchy sequences. If $\eta \in \mathcal{G}^\sim_\mu$, and $\{y_n\} \in \eta$, then

$$\|\eta\| = \limsup_{n \to \infty} \frac{1}{\mu(t)} \left| \int_0^t y_n(\tau) d\tau \right|.$$

To any $\eta$ in $\mathcal{G}^\sim_\mu$ there corresponds a continuous linear functional $\Phi$ on $\mathcal{L}_\mu$,

$$\Phi(x) = \lim_{n \to \infty} \int_0^\infty x(t)y_n(t)dt$$

with $\|\Phi\| = \|\eta\|$, but this is not the general representation of elements of $\mathcal{L}^*_\mu$. 
Indeed, $\eta \rightarrow \{y_n\}$, hence by Theorem 1, $\Phi$ is continuous and

$$
\Phi(x) = \lim_{n \to \infty} \int_0^\infty x'(t) \mu(t) \left( -x_{<0,n>}(t) \frac{1}{\mu(t)} \int_0^t y_n(\tau) d\tau \right) dt.
$$

According to the representation Theorem 4 there is only one function $g$ in $L_\infty$ such that

$$
g(t) = -\lim_{n \to \infty} \frac{1}{\mu(t)} \int_0^t y_n(\tau) d\tau \quad \text{a.e. in } I.
$$

Thus, $\|\Phi\| = \|g\|$, i.e. $\|\Phi\| = \sup_{t \to \infty} \frac{1}{\mu(t)} \left| \int_0^t y_n(\tau) d\tau \right|$ and by a simple calculation we get also $\|\Phi\| = \|\eta\|$. Should such functionals $\Phi$ exhausted the whole space $\mathcal{L}_\mu^*$, then the correspondence $\mathcal{G}_\mu \ni \eta \rightarrow g \in L_\infty$ defined by (8) would be a linear isometry of $\mathcal{G}_\mu$ onto $L_\infty$. But this is impossible, because the expression on the right-hand side of (8) is continuous except of a denumerable set in $I$.

The question concerning representation of bounded linear functionals on $\mathcal{L}_\mu$ independently from the derivative $x'$ remains open.

We yet prove

THEOREM 6. Let $\mu \in \mathfrak{M}$, let $X = R$, $Z = Y$ and let (3) be satisfied.

If the improper integral

$$
\int_0^\infty x(t)y(t) dt
$$

is convergent whenever $x = x(t) \in \mathcal{L}_\mu$, then $y = y(t) \in \mathcal{G}_\mu(Y)$.

Proof. Let $\mu \in \mathfrak{M}^{+}$ and $\mu(0^+) > 0$. We denote

$$
\Phi_b(x) = \int_0^b x(t)q(y(t)) dt,
$$

where $q \in Y^*$ and $b$ is a positive number. Then $\Phi_b \in \mathcal{L}_\mu^*$ and

$$
\|\Phi_b\| = \sup_{0 \leq t \leq b} \frac{1}{\mu(t)} \left| q \left( \int_0^t y(\tau) d\tau \right) \right|.
$$

There exists $\lim_{b \to \infty} \Phi_b(x)$ for $x$ in $\mathcal{L}_\mu$, hence $\sup_b \|\Phi_b\| \leq K_q < \infty$. The arbitrariness of $q$ in $Y^*$ yields $y = y(t) \in \mathcal{G}_\mu(Y)$.

If $\mu \in \mathfrak{M}^{+}$ and $\mu(0^+) = 0$, let

$$
f_\mu(x) = \int_a^\infty x(t)q(y(t)) dt, \quad 0 < a < \infty.
$$
Then $f_a \in \mathcal{L}_{\mu}^*$ and

$$
\|f_a\| = \sup_{t \geq a} \frac{1}{\mu(t)} \left| \varphi \left( \int_{a}^{\infty} y(\tau) d\tau \right) \right|.
$$

Banach–Steinhaus theorem yields $\|f_a\| = O(1)$ as $a \to 0$. Thus $y \in \mathcal{G}_{\mu}(Y)$.

If $\mu \in \mathcal{M}^-$, we use once more the functional $\Phi_b$. Since

$$
\Phi_b(x) = x(0) \int_{0}^{b} \varphi(y(\tau)) d\tau + \int_{0}^{b} \mu(t) x'(t) \left( \frac{1}{\mu(t)} \int_{t}^{b} \varphi(y(\tau)) d\tau \right) dt,
$$

we get $\Phi_b \in \mathcal{L}_{\mu}^*$, and by Theorem 5, we get

$$
\|\Phi_b\| = \sup_{0 \leq t \leq b} \frac{1}{\mu(t)} \left| \varphi \left( \int_{t}^{\infty} y(\tau) d\tau \right) \right|.
$$

The improper integral $\int_{0}^{\infty} y(t) dt$ exists, $\|\Phi_b\| = O(1)$ as $b \to \infty$. Consequently,

$$
\sup_{t} \frac{1}{\mu(t)} \left| \varphi \left( \int_{t}^{\infty} y(\tau) d\tau \right) \right| \leq K_{\varphi} < \infty.
$$

Now we take into account the arbitrariness of $\varphi$ in $Y^*$.

4. Some application to Fourier integral.

I. Let $\mu(t) = t^{1/p}$, where $t \geq 0$, $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and let $L_q = L_q(-\infty, \infty)$. If $x = x(t) \in \mathcal{L}_{\mu}$, then we put $x(t) = 0$ for $t < 0$. The Fourier transform $x^*$ of $x$:

$$
x^*(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt
$$

is $L_q$-convergent and

$$
\|x^*\|_{L_q} \leq \frac{C_q}{\sqrt{2\pi}} \|x\|_{\mathcal{L}_{\mu}}, \tag{10}
$$

where

$$
C_q = 2^{2/q} \left( \int_{0}^{\infty} \left| \frac{\sin u}{u} \right|^q du \right)^{1/q}.
$$

Proof. Let $X = \mathbb{R}$, $Z = Y = L_q$ and let $U: X \times Y \to Z$ be of the form $U(x, y) = xy$. Given a fixed positive number $T$, we consider a vector function $y_T = y_T(t)$, where

$$
y_T(t) = \begin{cases} 
\exp(-i\omega t) & \text{for } (t, \omega) \in A, \\
0 & \text{for } (t, \omega) \notin A,
\end{cases}
A = \{(t, \omega) \in \mathbb{R}^2: |t| \leq T \& |\omega| \leq T\}.
$$
and the following bilinear integral

\[ \int_0^T x(t) y_T(t) dt = \int_0^T x(t) e^{-i\omega t} dt. \]

Applying a variant of Theorem 1 given in Remark 1, we get the \( L_q \)-convergence of this integral with \( T \) passing to infinity, if we verify that the vector-valued function \( y_T \) satisfies all the assumptions required there.

It is obvious that \( y_T(t) \in L_q \) for a fixed \( t \) in \( R \). By the representation theorem for \( L_q^* \) we observe that if \( g \) is in \( L_p \), then the complex-valued function

\[ \varphi_q(t) = \int_{-\infty}^{\infty} g(\omega) y_T(t) d\omega = \begin{cases} \int_{-t}^{t} e^{-i\omega t} g(\omega) d\omega & \text{for } |t| \leq T, \\ 0 & \text{for } |t| > T \end{cases} \]

is measurable; thus, \( y_T = y_T(t) \) being weakly measurable, it is also strongly measurable, since \( L_q \) is separable. To prove that the function \( y_T = y_T(t) \) satisfies (3) it suffices to observe that the real function \( ||y_T(t)||_{L_q} \) is integrable over any interval \( (0, T) \).

Since

\[ \left| \int_0^t y_T(\tau) d\tau \right| = \begin{cases} \frac{2}{|\omega|} \left| \sin \frac{\omega t}{2} \right| & \text{for } (t, \omega) \in A, \\ 0 & \text{for } (t, \omega) \notin A, \end{cases} \]

\[ \left\| \int_0^t y_T(\tau) d\tau \right\|_{L_q} = 2^{2/q} \left( \int_0^1 \left| \sin \frac{u}{u} \right|^q du \right)^{1/q} |t|^{1/p}, \]

where

\[ a_t = \begin{cases} \frac{1}{2} |t| T & \text{for } |t| \leq T, \\ \frac{1}{2} T^2 & \text{for } |t| > T, \end{cases} \]

hence (3) is satisfied and

\[ \sup_{T \geq 0} \sup_{t \geq 0} t^{-1/p} \left\| \int_0^t y_T(\tau) d\tau \right\|_{L_q} = C_q < \infty \]

i.e. \( y_T = y_T(t) \in G_\mu(L_q) \). Consequently there exists the desired limit:

\[ L_q - \lim_{T \to \infty} \int_0^T x(t) y_T(t) dt = L_q - \lim_{T \to \infty} \int_0^T x(t) e^{-i\omega t} dt = V2\pi x^*(\omega). \]

Adopting the inequality described in Remark 1 to our case we get (10):

\[ \left( \int_{-\infty}^{\infty} |x^*(\omega)|^q d\omega \right)^{1/q} \leq 2^{2/q} \left( \int_0^1 \left| \frac{\sin u}{u} \right|^q du \right)^{1/q} \int_0^\infty t^{1/p} |x'(t)| dt. \]
Similar, the cosine and sine transform $x_c^*(\omega)$, $x_s^*(\omega)$ of $x \in L_\mu (\mu(t) = t^{1/p})$:

$$x_c^*(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty x(t) \cos \omega t \, dt, \quad x_s^*(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty x(t) \sin \omega t \, dt$$

are $L_q$-convergent and

$$\left( \int_0^\infty |x_c^*(\omega)|^q \, d\omega \right)^{1/p} \leq \sqrt{\frac{2}{\pi}} \left( \int_0^\infty \left( \frac{\sin u}{u} \right)^q \, du \right)^{1/q} \int_0^\infty t^{1/p} |x'(t)| \, dt,$$

$$\left( \int_0^\infty |x_s^*(\omega)|^q \, d\omega \right)^{1/p} \leq 2^{1/q} \sqrt{\frac{2}{\pi}} \left( \int_0^\infty \left( \frac{\sin^2 u}{u} \right)^q \, du \right)^{1/q} \int_0^\infty t^{1/p} |x'(t)| \, dt.$$
As an application of Corollary 2 and Theorem 6 we get some results similar to that given in [6] for Fourier series:

III. Let $0 < \alpha < 1$, $\mu(t) = (1 + t)^{-\alpha}$ for $t \geq 0$ and let $f \in L_2(0, \infty)$. Then $f \in \text{Lip}(\alpha, 2)$ iff the improper integral

$$\int_0^\infty x(\omega)f^{*}(\omega)\cos \omega t d\omega \quad (\text{resp. } \int_0^\infty x(\omega)f^{*}(\omega)\sin \omega t d\omega)$$

is $L_2$-convergent for each $x$ in $L_\mu$. Moreover, if $x \in L_\mu$, then

$$\|x(f^*)\|_{L_2} \leq \sqrt{\frac{2}{\pi}} \sup_{T \geq T_0} (1 + T)^{\alpha}\|x(T)\|_{L_2} \left(\frac{|x(T_0)|}{(1 + T_0)^{\alpha}} + \int_0^\infty \frac{|x'(t)|}{(1 + t)^{\alpha}} dt\right),$$

where $f^* = f^*_c$ or $f^*_s$, $(f^*)_T(\omega) = f^{*}(\omega)\chi_{(T, \infty)}(\omega)$ and $T_0 \geq 0$.

We also have

IV. Let $f \in \text{Lip}(\alpha, 2)$, let $0 < \eta < 2$, $\gamma_0 = \frac{1}{2}(1 + 2\alpha)\eta - 1$ and let $\delta \geq 1 + \frac{1}{2} \gamma$, where $0 < \alpha < 1$. We denote by $v = v(t)$ a real absolutely continuous function defined and non-negative for $t \geq 0$ such that

$$\int_0^\infty |v'(t)| dt < \infty \quad \text{and} \quad \int_0^\infty \frac{v(t)}{1 + t} dt < \infty.$$

Then

$$\int_0^\infty \omega^{\gamma_0} |v(\omega)|^{\delta} |f^{*}(\omega)|^{\eta} d\omega < \infty \quad (f^* = f^*_c \text{ or } f^*_s).$$

The proof is similar to that given in Theorem 2 of [6].

References