Particular spectral theory in finite-dimensional spaces

A formula is derived for the matrix-polynomial of lowest degree equivalent to a matrix function, for the class $\mathcal{F}(\mathcal{C})$ of functions holomorphic in an open set containing the spectrum of a given square matrix $\mathcal{C}$. Some applications of the formula are deduced. By a matrix function $f(\mathcal{C})$ we mean a matrix assigned to matrix $\mathcal{C}$ by means of a function $f$ from class $\mathcal{F}(\mathcal{C})$, considered as an operator. In the literature, within the framework of general spectral theory in finite-dimensional unitary spaces, a similar formula occurs; however, the polynomial appearing there is not of lowest degree for an arbitrary matrix $f(\mathcal{C})$.

Essential to the present work was a characteristic expansion of an arbitrary polynomial of the complex variable $z$ in a neighbourhood of an arbitrary finite number of points. It will be our aim to find this expansion and to apply it appropriately.

**Lemma 1.** We now proceed to a generalization of the usual Vandermonde determinant.

**Definition.** We define a generalized Vandermonde determinant of degree $p$ as a determinant of the form:

$$\begin{vmatrix} K_{p_1}(z_1) & K_{p_2}(z_2) & \cdots & K_{p_s}(z_s) \\ K_{p_1}(z_1) & K_{p_2}(z_2) & \cdots & K_{p_s}(z_s) \\ \vdots & \vdots & \ddots & \vdots \\ K_{p_1}(z_1) & K_{p_2}(z_2) & \cdots & K_{p_s}(z_s) \end{vmatrix},$$

where $K_{p_i}(z_i)$ denotes the group of $p_i$ columns of the matrix of the form:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ z_i \\ \vdots \\ z_i^{p_i-1} \\ \vdots \\ z_i^{p_i-1} \\ \vdots \\ z_i^{p_i-1} \end{bmatrix},$$

where $p = \sum_{i=1}^{s} p_i$ and $p_i \in \mathcal{N}$.
For a determinant thus defined, the following formula holds:

\[ |K_{p_1}(z_1)|K_{p_2}(z_2)| \ldots |K_{p_s}(z_s)| = \prod_{j,i=1 \atop j \neq i}^{s} (z_j - z_i)^{p_j p_i}. \]

Proof. The proof proceeds by complete induction with respect to the degree of the determinant.

For \( p = 1 \) the formula is obvious.

Let us assume it to be correct for \( p - 1 \), i.e.

\[ |K_{p_1}(z_1)|K_{p_2}(z_2)| \ldots |K_{p_{s-1}}(z_{s-1})| = \prod_{j,i=1 \atop j > i}^{s-1} (z_j - z_i)^{p_j p_i} \prod_{i=1}^{s-1} (z_s - z_i)^{(p_s - 1)p_i}. \]

We now consider the generalized Vandermonde determinant of degree \( p \) and transform it in the following manner: starting with \( l = p \) and ending with \( l = 1 \), and applying the formulas

\[
\begin{aligned}
\left( \frac{l}{k} \right) z_i^{l-k} - \left( \frac{l-1}{k} \right) z_s^{l-k} - z_s &= \left( \frac{l-1}{k-1} \right) z_i^{l-k} \\
\left( \frac{l}{k} \right) z_i^{l-k} - \left( \frac{l-1}{k} \right) z_s^{l-k} - z_s &= \left( \frac{l-1}{k} \right) (z_i - z_s) z_i^{l-k} + \left( \frac{l-1}{k-1} \right) z_i^{l-k} 
\end{aligned}
\]

for \( i \neq s \), we multiply its \((l-1)\)-st row by \( -z_s \) and we add it to the \( l \)-th row. Then in the \( s \)-th group of columns, in the first column with the exception of the first row in which 1 appears, we now have only zeros.

We apply the Laplace expansion with respect to the \( \left( \sum_{i=1}^{r-1} p_i + 1 \right) \)-th column. Thus, \( K_{p_s}(z_s) \) goes over into \( K_{p_{s-1}}(z_s) \). Similarly, each of the \( s - 1 \) groups of columns of the new determinant is equal to \( K_{p_i}(z_i) \); after transforming it in the following manner: starting from the first column and ending by the last but one, we take out the common factor \((z_i - z_s)\), and we add the column as a whole, with opposite sign, to next column. In turn, we take out the common factor \((z_i - z_s)\) from the last column. Thus, the factor \((z_i - z_s)^{p_i} \) is taken out from the group as a whole.

Hence, with regard to the equality:

\[
(-1)^{\sum_{i=1}^{r-1} p_i + 2} \prod_{i=1}^{s-1} (z_i - z_s)^{p_i} = \prod_{i=1}^{s-1} (z_s - z_i)^{p_i}
\]

and the induction hypothesis, we obtain:

\[
|K_{p_1}(z_1)|K_{p_2}(z_2)| \ldots |K_{p_s}(z_s)| = \prod_{j,i=1 \atop j > i}^{s} (z_j - z_i)^{p_j p_i} \prod_{i=1}^{s-1} (z_s - z_i)^{(p_s - 1)p_i} = \prod_{j,i=1 \atop j > i}^{s} (z_j - z_i)^{p_j p_i}.
\]

Q.E.D.
In the sequel, \( \mathcal{Z} \) will mean the open complex plane. Let \( z_i \in \mathcal{Z} \) for \( i = 1, 2, \ldots, s \) and \( z_i \neq z_j \) for \( i \neq j \). We define polynomials of the variable \( z \in \mathcal{Z} \) by means of the formulas:

\[
\mu(z) = \prod_{j=1}^{s} (z - z_j)^{p_j}, \quad \nu_{i,k}(z) = \prod_{j=1 \atop j \neq i}^{s} (z - z_j)^{p_j}(z - z_i)^k,
\]

where \( i = 1, 2, \ldots, s, \ k = 0, 1, \ldots, n_i - 1 \).

We denote by \( A_v(z) \) the determinant whose \( \left( \sum_{j=1}^{i-1} p_j + k + 1 \right) \)-st column is given by the vector

\[
[v_{i,k}(z), (v_{i,k}(z))^{(1)}, \ldots, (v_{i,k}(z))^{(p_i - 1)}]^{tr}.
\]

Then there holds the following

**Lemma 2.** For each \( z \in \mathcal{Z} \) one has \( A_v(z) \neq 0 \).

**Proof.** Assume \( z \neq z_i \) for \( i = 1, 2, \ldots, s \). Then, taking out the common factor \( \mu(z) \) from each row of the determinant \( A_v(z) \) and applying the Leibniz formula, its \( \left( \sum_{j=1}^{i-1} p_j + k + 1 \right) \)-st column becomes:

\[
\left[ (z - z_i)^{k-p_i}, (z - z_i)^{k-p_i} + \frac{\mu^{(k)}(z)}{\mu(z)} (z - z_i)^{k-p_i} + \ldots + \sum_{l=1}^{p_i - 1} \left( \frac{(p_i - 1)\mu^{(l)}(z)}{\mu(z)} \right) (z - z_i)^{k-p_i} \right]^{tr}.
\]

Omitting in each row the linear combinations of preceding rows, it takes the form

\[
[(z - z_i)^{k-p_i}, (z - z_i)^{k-p_i} + (z - z_i)^{k-p_i} + \ldots + (z - z_i)^{k-p_i}]^{tr}.
\]

Taking out the common factor \( (-1)^{l-1}(l-1) \) from the \( l \)-th row, the column under consideration becomes

\[
\left[ \binom{p_i - k - 1}{0} (z - z_i)^{k-p_i}, \binom{p_i - k}{1} (z - z_i)^{k-p_i - 1}, \ldots, \binom{p_i - k + 2}{p_i - 2} (z - z_i)^{k-p_i} \right]^{tr}.
\]

Let \( B_v(z) \) denote the determinant constructed by means of groups of columns of the form

\[
\begin{pmatrix}
\binom{0}{0} (z - z_i)^{-1} & \binom{1}{0} (z - z_i)^{-2} & \ldots & \binom{p_i - 1}{0} (z - z_i)^{-p_i} \\
\binom{1}{1} (z - z_i)^{-2} & \binom{2}{1} (z - z_i)^{-3} & \ldots & \binom{p_i}{1} (z - z_i)^{-p_i - 1} \\
\vdots & \vdots & \ddots & \vdots \\
\binom{p_i - 1}{p_i - 1} (z - z_i)^{-p_i} & \binom{p_i}{p_i - 1} (z - z_i)^{-p_i - 1} & \ldots & \binom{2p_i - 2}{p_i - 1} (z - z_i)^{-2p_i + 1} \\
\binom{p_i - 1}{p_i - 1} (z - z_i)^{-p_i} & \binom{p_i - 1}{p_i - 1} (z - z_i)^{-p_i - 1} & \ldots & \binom{p_i - 1}{p_i - 1} (z - z_i)^{-p_i + 1}
\end{pmatrix}.
\]
then

$$A_\ast(z) = \mu(z)^p \cdot (-1)^{hp(p-1)}(p-1)!(-1)^{i} \sum_{i=1}^s p_i(p_i-1) B_i(z).$$

We transform the determinant $B_i(z)$ as follows: we begin by the last but one and we end on the first column of each group of columns, multiplying a given column by $-(z - z_i)^{-1}$ and adding to the next column. Proceeding similarly, stopping the procedure successively at the second, third, ... and finally the last but first column of a given group of columns, we finally obtain

$$\begin{bmatrix}
0 \\
0 \\
(1) (z - z_i)^{-2} \\
\vdots \\
p_i-1 (z - z_i)^{-p_i} \\
p_i-1 (z - z_i)^{-p_i} \\
(1) (z - z_i)^{-2} \\
\vdots \\
p_i-1 (z - z_i)^{-p_i} \\
p_i-1 (z - z_i)^{-p_i} \\
\end{bmatrix}$$

Taking out the common factor $(z - z_i)^{-p_i^2}$, we obtain the group of columns $K_{p_i}((z - z_i)^{-1})$. With regard to Lemma 1, we obtain

$$B_\ast(z) = \prod_{i=1}^s (z - z_i)^{-p_i} \prod_{i,j=1}^s (z - z_j)^{-p_j p_i} (z - z_i)^{-p_j (z_i - z_j)^{p_j p_i}}.$$  

Thus, after insertion into $A_\ast(z)$ we have

$$A_\ast(z) = (p-1)! \prod_{i,j=1}^s (z_j - z_i)^{p_j p_i}.$$  

Since $A_\ast(z)$ is continuous, $A_\ast(z) = \text{const} \neq 0$.

**Lemma 3.** For any polynomial $w_n$ ($n = 0, 1, \ldots$), there always exists exactly one solution of the set of linear equations of the form

$$w_n(z) = \sum_{h=0}^l \mu_h(z) \sum_{i=1}^l \sum_{k=0}^p \mu_{i,k} v_{i,k}(z),$$

where $l$ is chosen in such a manner that $n < p(l+1)$, $p = \sum_{i=1}^s p_i$, where

$$X_{i,k}^{0,n} = \frac{1}{k!} \left[ \frac{w_n(z)}{v_{i,0}(z)} \right]^{(k)} (z_i).$$
Note. By \( \cdot^{(k)}(z_t) \) we understand the derivative of order \( k \) calculated at the point \( z_t \).

Proof. We begin by proving that the set of solutions of (1) is not empty.

From Euclid's algorithm, we have

\[
W_n(z) = \sum_{h=0}^{l} \mu^{(h)}(z) \partial_{h,n}(z),
\]

where \( \partial_{h,n} \) is a polynomial, and degree \( \partial_{h,n} < p \). Moreover, degree \( \partial_{l,n} = n - lp < p \), i.e. \( n < p(l+1) \).

We now compare (3) and (1). This system is satisfied if, for \( h = 0, 1, \ldots, l \)

\[
\partial_{h,n}(z) = \sum_{i=1,2,\ldots,s}^{i} X_{i,k}^{h,n} v_{i,k}(z).
\]

Keeping \( h \) fixed, this identity is fulfilled if

\[
\{\partial_{h,n}(z)\}^{(j)} = \sum_{i=1,2,\ldots,s}^{i} X_{i,k}^{h,n} \{v_{i,k}(z)\}^{(j)} \quad \text{for} \quad j = 0, 1, \ldots, p - 1.
\]

Since \( A(z) \neq 0 \) for all \( z \in \mathcal{Z} \), it is always possible to choose constants \( X_{i,k}^{h,n} \) in a manner to fulfil the latter condition.

We now proceed to show by complete induction with respect to \( k \) at fixed \( i \) that the first \( p \) unknowns are of the form (2). For \( k = 0 \), we have by (1)

\[
w_n(z_t) = X_{i,0}^{0,n} v_{i,0}(z_t), \quad \text{i.e.} \quad X_{i,0}^{0,n} = \frac{1}{0!} \left\{ \frac{w_n(z_t)}{v_{i,0}(z_t)} \right\}^{(0)}(z_t).
\]

Now we show that if the thesis true for \( q_t - 1 \), then it is also true for \( q_t \), where \( q_t < p_t \).

We note that the system (1) can be rewritten as:

\[
w_n(z) = \sum_{k=0}^{q_t} X_{i,k}^{0,n} v_{i,k}(z) + (z - z_t)^{q_t+1} \vartheta(z),
\]

where \( \vartheta(z) \) is a polynomial.

Consequently,

\[
X_{i,q_t}^{0,n} \{v_{i,q_t}(z)\}^{q_t}(z_t) = \{w_n(z)\}^{(q_t)}(z_t) - \left\{ \sum_{k=0}^{q_t-1} X_{i,k}^{0,n} v_{i,k}(z) \right\}^{(q_t)}(z_t).
\]

With regard to the relation

\[
v_{i,k}(z) = v_{i,0}(z) \cdot (z - z_t)^{k},
\]
we have, on the one hand,
\[ \{v_{i,k}(z)\}_{q} = q! v_{i,0}(z) \neq 0 \]
and, on the other (by the Leibniz formula, dropping zero terms)
\[ \left\{ \sum_{k=0}^{q-1} X_{i,k}^{q} v_{i,k}(z) \right\}_{q} = \sum_{l=0}^{q-1} \left( \begin{array}{c} q \\ l \end{array} \right) \{v_{i,0}(z)\}_{q-l} (z) \cdot X^{q}_{i,l} \cdot l! . \]

Applying now the induction hypothesis and then the Leibniz formula, we obtain
\[ q! v_{i,0}(z) X_{i,q}^{n} = \left\{ w_{n}(z) \right\}_{q} (z) - \sum_{l=1}^{q-1} \left( \begin{array}{c} q \\ l \end{array} \right) \{v_{i,0}(z)\}_{q-l} (z) \cdot \left[ \frac{w_{n}(z)}{v_{i,0}(z)} \right]^{(l)} (z) \]
\[ = v_{i,0}(z) \left[ \frac{w_{n}(z)}{v_{i,0}(z)} \right]^{(q)} (z) . \]

This proves the lemma.

At the same time, this proves the uniqueness of the existence of the first \( p \) solutions. That of the remaining ones can be proved similarly.

Q.E.D.

It should be noted that, with regard to the preceding lemma, the set of polynomials \( v_{i,k}(z) \) is linearly independent.

Now let \( \mu(z) \) stand for the minimal zeroing polynomial of a square matrix \( \mathcal{C} \) of degree \( n \), and \( \sigma(\mathcal{C}) \) for its spectrum. By the Cayley–Hamilton theorem, the inequality: degree \( \mu \leq n \) holds (cf. [2], p. 270). Hence, Lemma 3 leads to the following theorem:

**Theorem 1.** If \( f \) denotes an arbitrary polynomial, considered as an operator acting on the matrix \( \mathcal{C} \), then
\[
(5) \quad f(\mathcal{C}) = \sum_{i=1,z,...,q}^{q-1} \sum_{k=0,1,...,p-1} \frac{1}{k!} \left\{ \frac{f(z)}{v_{i,0}(z)} \right\}^{(k)} (z) v_{i,k}(\mathcal{C}) ,
\]
where \( z \in \sigma(\mathcal{C}) \) and, if there exist \( p \) matrices \( \mathcal{C}_{i,k} \) such that there holds
\[
(6) \quad f(\mathcal{C}) = \sum_{i=1,z,...,q}^{q-1} \sum_{k=0,1,...,p-1} \frac{1}{k!} \left\{ \frac{f(z)}{v_{i,0}(z)} \right\}^{(k)} (z) \mathcal{C}_{i,k}
\]
for an arbitrary polynomial \( f \), then \( \mathcal{C}_{i,k} = v_{i,k}(\mathcal{C}) \).

**Proof.** Equation (5) results from (1) on omitting terms equal to the matrix \( \mathcal{C} \). Now, with regard to (4), one has:
\[
\frac{1}{l!} \left\{ \frac{v_{i,k}(z)}{v_{i,0}(z)} \right\}^{(l)} (z) = \delta_{i,j} \delta_{k,l}
\]
whence, on putting \( f = v_{i,k} \) in (6), we have \( v_{i,k}(\mathcal{C}) = \mathcal{C}_{i,k} \). Q.E.D.
Lemma 4. If $\mathcal{C}$ is a non-singular matrix, then

$$\mathcal{C}^{-1} = \sum_{i=1,2,\ldots,s} \frac{1}{k!} \left[ \frac{z^{-1}}{v_{i,0}(z)} \right]^{(k)} (z_i) v_{i,k}(\mathcal{C}), \text{ where } z_i \in \sigma(\mathcal{C}).$$

Proof. Let $\mu^*(z) = \prod_{i=1}^{s} (z - z_i^{-1})^{n_i}$. Then, by the equality

$$\mu(\mathcal{C}) = (-1)^p \mathcal{C}^p \prod_{i=1}^{s} z_i^{n_i} \cdot \mu^* (\mathcal{C}^{-1}),$$

$\mu^*(z)$ is the minimal polynomial zeroing the matrix $\mathcal{C}^{-1}$. We introduce the following definition:

$$v_{i,k}^*(z) = \prod_{j=1}^{s} (z - z_j^{-1})^{n_j} \cdot (z - z_i^{-1})^k.$$

Note that $z^{n_i} v_{i,k}^*(z^{-1})$ is a polynomial in the variable $z$. Thus, by Theorem 1, the matrix $\mathcal{C}^p v_{i,k}^*(\mathcal{C}^{-1})$ is a linear combination of matrices $v_{i,k}(\mathcal{C})$. Also, by Theorem 1, one has the equality

$$(\mathcal{C}^{-1})^{p+1} = \sum_{i=1,2,\ldots,s} \frac{1}{k!} \left[ \frac{z^{p+1}}{v_{i,0}(z)} \right]^{(k)} (z_i^{-1}) v_{i,k}^*(\mathcal{C}^{-1}).$$

Consequently, $\mathcal{C}^{-1}$ is a linear combination of matrices $v_{i,k}(\mathcal{C})$. Hence, if $\delta(z)$ is a polynomial corresponding to this linear combination, then the function $f(z) = z(z^{-1} - \delta(z))$ is a polynomial with the property $f(\mathcal{C}) = \delta$. As a consequence, $z^1 - \delta(z) = \mu(z) \chi(z) z^{-1}$, where $\chi(z)$ is a polynomial, and $(z^1 - \delta(z))^k(z_i) = 0$ for $i = 1, 2, \ldots, s$, $k = 0, 1, \ldots, p_i - 1$, $z_i \in \sigma(\mathcal{C})$. Further steps of the proof are identical with those of (2) in Lemma 3. Q.E.D.

Lemma 5. For any $\lambda \in \sigma(\mathcal{C})'$, the following formula holds:

$$\lambda I - \mathcal{C}^{-1} = \sum_{i=1,2,\ldots,s} \frac{1}{k!} \left[ \frac{(\lambda - z)^{p_i}}{v_{i,0}(z)} \right]^{(k)} (z_i) v_{i,k}(\mathcal{C}), \text{ where } z_i \in \sigma(\mathcal{C}).$$

Proof. Note that $\mu^*(z) = \prod_{i=1}^{s} (z - z_i + \lambda)^{p_i}$ is the minimal polynomial zeroing the matrix $\mathcal{C} - \lambda I$. We use the definition

$$v_{i,k}^*(z) = \prod_{j=1}^{s} (z - z_j + \lambda)^{p_j} \cdot (z - z_i + \lambda)^k.$$

We now have, by Lemma 4,

$$(\mathcal{C}^{-1})^{p+1} = \sum_{i=1,2,\ldots,s} \frac{1}{k!} \left[ \frac{z^{p+1}}{v_{i,0}(z)} \right]^{(k)} (z_i^{-1} - \lambda) v_{i,k}^*(\mathcal{C}^{-1} \lambda).$$
Since $v_{i,k}^*(z) = v_{i,k}(z + \lambda)$ and
\[
((\cdot)_{(z)})^{(k)}(z_i - \lambda) = (\cdot_{(z - \lambda)})^{(k)}(z_i),
\]
we obtain formula (7) which is equivalent to the thesis of this lemma. Q.E.D.

Let $\mathcal{F}(\mathcal{E})$ denote the class of functions holomorphic in an open set containing $\sigma(\mathcal{E})$. If $f \in \mathcal{F}(\mathcal{E})$, we define:

(8) $$f(\mathcal{E}) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\lambda)}{\lambda - \mathcal{E}} \, d\lambda,$$

where $\Gamma$ is a contour consisting of a finite number of curves $I_j$, enclosing $\sigma(\mathcal{E})$ and satisfying the assumptions needed for the Cauchy formula. This definition is of use in the following theorem.

**Theorem 2.** If $f \in \mathcal{F}(\mathcal{E})$, then the matrix $f(\mathcal{E})$ is uniquely determined by the formula:

(9) $$f(\mathcal{E}) = \sum_{j=1,2,\ldots, s} \sum_{k=0,1,\ldots, p_j - 1} \frac{1}{k!} \left[ \frac{f(z)}{v_{j,0}(z)} \right]^{(k)}(z_j) v_{j,k}(\mathcal{E}), \quad \text{where } z_j \in \sigma(\mathcal{E}).$$

**Proof.** By Lemma 5, the right-hand term of equation (8) is an integral of a matrix polynomial. We note that, with regard to the uniform convergence of the integrals, the following equality holds:

$$\frac{1}{2\pi i} \int_{\Gamma} \left[ (\lambda - z)^{-1} f(\lambda) \right]^{(k)} \left( z_j \right) v_{j,0}(\mathcal{E}) \, d\lambda = \left[ \frac{f(z)}{v_{j,0}(z)} \right]^{(k)}(z_j),$$

its right-hand term being independent of the contour $\Gamma$. Hence, on placing the sum before the integration sign, we obtain (9). Uniqueness of this representation follows from the linear independence of the system of matrices $v_{j,k}(\mathcal{E})$, $j = 1, 2, \ldots, s$, $k = 0, 1,\ldots, p_j - 1$, immediately.

**Theorem 3.** If $f \in \mathcal{F}(\mathcal{E})$, then equation (9) holds, if and only if, there exists a polynomial $g(z)$ such that

(10) $$f(\mathcal{E}) = g(\mathcal{E}) \quad \text{and} \quad f^{(k)}(z_i) = g^{(k)}(z_i)$$

for $k = 0, 1,\ldots, p_i - 1, z_i \in \sigma(\mathcal{E})$.

If, moreover, degree of $g$ is $\leq p$, then the polynomial $g$ is determined uniquely.

**Proof.** Sufficiency of condition (10) follows from Theorem 2. In order to prove necessity of (10), it suffices to take

$$g(z) = \sum_{j=1,2,\ldots, s} \sum_{k=0,1,\ldots, p_j - 1} \frac{1}{k!} \left[ \frac{f(z)}{v_{j,0}(z)} \right]^{(k)}(z_i) v_{j,k}(\mathcal{E}).$$
Then, \( f(\mathcal{C}) = g(\mathcal{C}) \), and formula (9) holds. Since the representation is unique, we get
\[
\begin{bmatrix}
  f(z) \\
  v_{i,0}(z)
\end{bmatrix}(z_i) = \begin{bmatrix}
  g(z) \\
  v_{i,0}(z)
\end{bmatrix}(z_i)
\quad \text{for} \quad k = 0, 1, \ldots, p_t - 1, z_i \in \sigma(\mathcal{C}).
\]

From the Leibniz formula we obtain after a rearrangement
\[
\sum_{l=0}^{k} \binom{k}{l} \left(v_{i,0}(z)^{-1}\right)^{(k-l)}(z_i) \left\{f(z) - g(z)\right\}^{(l)}(z_i) = 0
\]
for \( k = 0, 1, \ldots, p_t - 1, z_i \in \sigma(\mathcal{C}) \). Since for fixed \( i \) the determinant of this system is \( v_{i,0}(z_i)^{-p_i} \neq 0 \), relations (10) are necessary. Q.E.D.

We now proceed to formulate the theorems concerning the minimal zeroing polynomial.

**Theorem 4.** Let \( \mu(z) = \prod_{i=1}^{s} (z - z_i)^{\mu_i} \) be the minimal polynomial zeroing the matrix \( \mathcal{C} \) and let \( f \in \mathcal{F}(\mathcal{C}) \). If \( f \) possesses the following properties:

1° Let \( Q_i \) be the set of those \( i - s \) corresponding to \( z_i \in \sigma(\mathcal{C}) \), for which the function \( f \) takes the same value (denoted by \( u_i \)) and \( r \) is the number of all distinct numbers \( u_i \),

2° the point \( z_i \in \sigma(\mathcal{C}) \) is an \( r_i \)-fold point of the function \( f \),

then the minimal polynomial zeroing the matrix \( f(\mathcal{C}) \) is of the form:

\[
\mu'(z) = \prod_{i=1}^{r} (z - u_i)^{q_i} \quad \text{where} \quad q_i = \min\{q \in \mathbb{N}: r_i q \geq p_t, i \in Q_i\}.
\]

**Proof.** The function \( f \) can be dealt with as a polynomial. Since, in neighbourhood of an \( r_i \)-fold point \( z_i \), the function \( f \) can be represented in the form: \( f(z) = u_i + (z - z_i)^{r_i} g_i(z) \), where \( g_i(z_i) \neq 0 \), and \( r_i q_i \geq p_t > k \), by applying equation (9) to the polynomial \( \prod_{m=1}^{r} [f(z) - u_m]^{q_m} \) it is seen that the coefficient by the matrix \( v_{i,k}(\mathcal{C}) \) is equal to
\[
\frac{1}{k!} \left\{(z - z_i)^{r_i} g_i(z)^{q_i} v_{i,0}(z)^{-1} \int_{m=1}^{r} [f(z) - u_m]^{q_m} \right\}^{(k)}(z_i) = 0.
\]

Hence, \( \prod_{m=1}^{r} [f(\mathcal{C}) - u_m I]^{q_m} = 0 \).

If \( \mu'(z) \) denotes the minimal polynomial zeroing the matrix \( f(\mathcal{C}) \), then \( \mu'(z) \mid \prod_{m=1}^{r} (z - u_m)^{q_m} \). Since the matrix \( f(\mathcal{C}) \) is not zero for the polynomial
Because for \( h = r_i(q_i - 1) \leq (p_i - 1) \) the coefficient by the matrix \( \nu_{i,h}(\mathcal{C}) \) is

\[
g_i(z_i)^{q_i-1} \nu_{i,0}(z_i)^{-1} \prod_{m=1}^{r} (u - u_m)^{q_m} \neq 0,
\]

we have \( \mu^I(z) = \prod_{m=1}^{r} (z - u_m)^{q_m} \). Q.E.D.

The last theorem leads to the following corollaries:

**Corollary 1.** If \( f \in \mathcal{F}(\mathcal{C}) \), then \( \text{degree } \mu^I = \text{degree } \mu \), if and only if,

1. the function \( f \) is one-to-one on \( \sigma(\mathcal{C}) \),
2. for each \( z_i \in \sigma(\mathcal{C}) \) for which \( p_i \geq 2 \), \( f'(z_i) \neq 0 \).

**Corollary 2.** If \( f \in \mathcal{F}(\mathcal{C}) \), then \( \text{degree } \mu^I \leq \text{degree } \mu \).

Moreover, the following theorem holds:

**Theorem 5** If \( f \in \mathcal{F}(\mathcal{C}) \), there exists a function \( g \in \mathcal{F}(f(\mathcal{C})) \) of the property \( (g \circ f)(\mathcal{C}) = \mathcal{C} \) if and only if

\[
\text{degree } \mu^I = \text{degree } \mu.
\]

**Proof.** Since, by Corollary 2, we have

\[
\text{degree } \mu = \text{degree } \mu^{(I)} \leq \text{degree } \mu^I \leq \text{degree } \mu,
\]

it results that the condition \( \text{degree } \mu^I = \text{degree } \mu \) is necessary. Inversely, from Corollary 1 and Theorem 3 follows that the function \( f \) can be considered as a polynomial with the property \( f'(z_i) \neq 0 \) for \( i = 1, 2, \ldots, s \). Then from the fact that zeros of a holomorphic are isolated, inverse function exists in some neighbourhood of \( \sigma(f(\mathcal{C})) \). Q.E.D.

Assume \( r, q_i, u_i, Q_i \) having the same meaning as in Theorem 4.

**Theorem 6.** If \( f \in \mathcal{F}(\mathcal{C}) \) and if \( |\mathcal{C} - z| = (-1)^n \prod_{i=1}^{s} (z - z_i)^{q_i} \) denotes the characteristic polynomial of matrix \( \mathcal{C} \), then the characteristic polynomial of the matrix \( f(\mathcal{C}) \) is of the form

\[
|f(\mathcal{C}) - z\mathcal{I}| = (-1)^n \prod_{i=1}^{r} (z - u_i)^{\beta_i}, \quad \text{where } \beta_i = \sum_{i=0}^{\alpha_i} u_i \text{ and } \sum_{i=1}^{r} \beta_i = n.
\]

**Proof.** The function \( f \) can be considered as a polynomial. Applying equation (9) to the function \( |f(z) - u_i|^{\alpha_i} \) and making use of the relations

\[
\sum_{u_i \in Q_i} \frac{1}{k!} \nu_{i,0}(z)^{-1} |f(z) - u_i|^{\alpha_i} \nu_{i,k}(\mathcal{C}) = \mathcal{C},
\]

where
Finite-dimensional spaces

\[ v_{i,k}(\mathcal{C}) = \mathcal{A} \cdot \prod_{j \in Q_i} (\mathcal{C} - z_j I)^{p_j} \text{ for } i \in Q, \text{ where } \mathcal{A} \text{ is a matrix, we obtain} \]

\[ [f(\mathcal{C}) - u_i I]^{q_i} = \mathcal{B} \cdot \prod_{i \in Q_i} (\mathcal{C} - z_i I)^{p_i}, \]

where \( \mathcal{B} \) is a matrix. Hence any vector which zeroes the matrix \( (\mathcal{C} - z_i I)^{p_i} \), where \( i \in Q_i \), also zeroes the matrix \( [f(\mathcal{C}) - u_i I]^{q_i} \). By a theorem of [2] (p. 273), the multiplicity of the eigenvalue \( u_i \) which is \( \beta_i \), fulfils the inequality \( \beta_i \geq \sum_{i \in Q_i} a_i \). Since, moreover, the inequality \( n = \sum_{l=1}^{r} \beta_l \geq \sum_{l=1}^{r} \sum_{i \in Q_i} a_i = \sum_{i\in Q_i} a_i = n \) holds, we obtain \( \beta_i = \sum_{i \in Q_i} a_i \), proving the theorem.

References