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Some remarks on rearrangements of series

Let P be the set of all permutations (one-to-one mappings onto) of $N = \{1, 2, \ldots\}$, endowed with the topology of pointwise convergence on N. This topology is determined for example by the metric $d(p, q) = \sum_{i=1}^{\infty} 3^{-i}\min\{1, |p(i)-q(i)|\}$. For every $p \in P$ and $n \in N$, the open ball $B(p, 3^{-n})$ in (P, d) is equal to $\{q \in P : q(i) = p(i) \text{ for } 1 \leqslant i \leqslant n\}$. The metric space (P, d) is not complete, nevertheless

LEMMA 1 [1]. *P is of the second Baire category.*

Proof. Suppose that $P = \bigcup_{n=1}^{\infty} P_n$, where P_n are closed and boundary. Then it is easy to define a sequence (p_n) in P and an increasing sequence (k_n) in N so that $B(p_n, 3^{-k_n}) \cap P_n = \emptyset$, $B(p_{n+1}, 3^{-k_{n+1}}) \subset B(p_n, 3^{-k_n})$ and $p_n(i) = n$ for some $i \leqslant k_n$; $n \in N$. Then $p = \lim p_n$ exists in P, $p(i) = p_n(i)$ for $i \leqslant k_n$ and $p \notin P_n$; $n \in N$. This is impossible.

R. P. Agnew has proved in [1] that, roughly, if a scalar series $\sum x_i$ converges, but not unconditionally, then the set of those $p \in P$ for which $\sum x_{p(i)}$ converges is of the first category. This was recently generalized by Talaljan [5] to series in a real Hilbert space. The purpose of this note is to show that analogous results are valid in arbitrary topological groups or vector spaces.

LEMMA 2. *Let F be any family of finite sequences in N. Let $P(F) = \bigcup_{n=1}^{\infty} P_n$, where $P_n = \{p \in P : (p(i), p(i+1), \ldots, p(j)) \in F \text{ for } j \geqslant i \geqslant n\}$. Then either $P(F) = P$ or $P(F)$ is of the first category.*

Proof. Suppose that there is $q \in P \setminus P(F)$. Then we can find sequences (i_k) and (j_k) in N such that $i_k \leqslant j_k < i_{k+1}$ and $(q(i_k), \ldots, q(j_k)) \notin F$, $k \in N$. We are going to show that each P_n is nowhere dense. Take an arbitrary $p_0 \in P$ and $m \in N$, $m \geqslant n$, and then choose k so large that $\min\{q(i): i_k \leqslant i \leqslant j_k\} > \max\{p_0(i): 1 \leqslant i \leqslant m\}$. Then any permutation p such that $p(i)$

$= p_0(i)$ for $1 \leqslant i \leqslant m$ and $p(i) = q(i)$ for $i_k \leqslant i \leqslant j_k$, is in $B(p_0, 3^{-m})$ but not in P_n. Thus P_n is boundary, and being evidently closed, it is nowhere dense.

Below, G denotes a topological group, X a topological vector space, and $(x_i)_{i \in N}$ is a sequence in G or X. Group operation is written additively.

THEOREM 1. *Suppose that G is sequentially complete. Then the set P_c of those $p \in P$ for which $\sum_{i=1}^{\infty} x_{p(i)}$ converges is either identical with P or of the first category.*

Proof. Given a neighbourhood U of the origin in G, let F_U denote the set of all finite sequences $(i_1, \ldots, i_m)$ such that $\sum_{k=1}^{m} x_{i_k} \in U$. We apply Lemma 2 for $F = F_U$, writting $P(U)$ instead of $P(F_U)$. If, for some U, $P(U)$ is of the first category, then so is P_c because $P_c \subset P(U)$. Otherwise $P(U) = P$ for every neighbourhood U, thus $\sum_{i=1}^{\infty} x_{p(i)}$ satisfies the Cauchy condition for every $p \in P$. It follows that $P_c = P$.

THEOREM 2. *Let $(x_i) \subset X$ and let $P_b = \{p \in P$: the sequence $(\sum_{i=1}^{n} x_{p(i)})_{n \in N}$ is bounded$\}$. Then either $P_b = P$ or P_b is of the first category.*

Proof. There are two possibilities:

1° For every neighbourhood U of 0, $P = \bigcup_{n=1}^{\infty} P(nU)$. Then by Lemma 2, $P = P(nU)$ for some n. It follows that $P_b = P$.

2° There is U such that $P \neq \bigcup_{n=1}^{\infty} P(nU)$. Then, by Lemma 2 again, the latter set is of the first category and contains P_b.

$P_b = P$ means simply that the set of all finite sums $x_{i_1} + \ldots + x_{i_k}$ is bounded. In this case the series $\sum_{i=1}^{\infty} x_i$ is said to be *perfectly bounded* ([2], [3]). A linear topological space X is said to *satisfy condition* (0) if every perfectly bounded series of its points is convergent; each such series is then subseriesly convergent, hence also unconditionally. Every weakly sequentially complete locally convex vector space, in particular every Hilbert space, satisfies condition (0) (this follows easily from the Orlicz–Pettis theorem). (See [2], [3] and [4] Chapter III, § 8 for other spaces satisfying this condition.) Therefore the results from [1] and [5] mentionned at the beginning are contained in the following corollary to Theorem 2.

COROLLARY. *Let X satisfy condition* (0). *If a series $\sum_{i=1}^{\infty} x_i$ in X is convergent but not unconditionally, then P_b is of the first category.*

Remark. One obtains similar results (essentially due to Orlicz [3]) when instead of permutations of a series $\sum x_i$ considers its subseries $\sum x_{k_i}$ $(k_1 < k_2 < \dots)$; P is then replaced by $\mathscr{P}(\boldsymbol{N})$, the power set of $\boldsymbol{N}$.

References

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