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A note on almost K-compact spaces*

Abstract. The purpose of this paper is to present some more results concerning *almost K-compact spaces*. It is shown that almost K-compactness is preserved under finite unions with some restrictions and is inverse invariant under perfect mapping if the domain space is regular. Its relationships with other well-known spaces like compact, K-compact, a_K-realcompact spaces, are also studied.

1. INTRODUCTION

This paper is a continuation of [1], where a new class of spaces, named almost K-compact spaces, was introduced as a generalization of K-compact spaces due to Herrlich [5]. A Tychonoff space is said to be *K-compact* if every z-ultrafilter with K-intersection property (i.e., intersection of less than K members of z-ultrafilter is non-empty) is fixed. This is to be distinguished from the concept of (K, f)-compactness to be defined later. Notations and terminology used in this paper follow that of [1]. Here all the topological spaces are assumed to be Hausdorff and the letter K stands for an infinite cardinal number.

A space X is called an *almost K-compact* if and only if for every open ultrafilter $\mathscr{U}$ of X with $\bigcap \overline{\mathscr{U}} = \emptyset$, there exists a subfamily $\mathscr{V}$ of $\mathscr{U}$ such that $\bigcap \overline{\mathscr{V}} = \emptyset$, where $\mathrm{card}(\mathscr{V}) < K$. Almost K-compact spaces coincide with almost realcompact spaces, introduced by Frolík [4], for $K = \aleph_1$, where a space is said to be an *almost realcompact* if and only if for every open ultrafilter $\mathscr{U}$ with $\bigcap \overline{\mathscr{U}} = \emptyset$, there exists a countable subfamily $\mathscr{V}$ of $\mathscr{U}$ such that $\bigcap \overline{\mathscr{V}} = \emptyset$.

A subset of a topological space is said to be *closed domain* (or *regular closed*) [6] if it is the closure of its own interior or equivalently, if it is the closure of some open set. A property P of a topological space is said

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to be *feebly hereditary* if every closed domain subspace of the space has the property P. A space is said to be (K,f)-*compact* if every open covering of cardinality $< K$ has a finite subcovering. A mapping $f\colon X \to Y$ is (K,f)-perfect if f is closed and $f^{-1}(y)$ is (K,f)-compact for every y in Y.

In [1] we proved that these spaces have feebly hereditary property, productivity and are preserved under arbitrary intersection and under a (K,f)-perfect mapping which is a notion weaker than that of perfect mapping. These are also characterized in terms of completeness property.

In this paper we prove that the union of finite number of closed domain, almost K-compact spaces is almost K-compact. It is inverse invariant under a perfect mapping if the domain space is regular. Its relationships with other well-known spaces like compact, K-compact, a_K-realcompact spaces are also studied.

2 UNION OF ALMOST K-COMPACT SPACES

It can be noted that the union property of K-compact spaces is yet to be examined. Here we will study some union property of almost K-compact spaces and then with the help of this result, the union property of K-compactness is studied in Section 4.2 of this paper.

We consider the case in which a space X is the union of a finite number n of closed domain almost K-compact subsets; it is, of course, enough to deal with the case $n = 2$.

THEOREM 2.1. *If X is the union of two closed domain almost K-compact spaces, then X is almost K-compact space.*

Proof. Let $X = X_1 \cup X_2$, where X_1 and X_2 are closed domain and almost K-compact spaces. We have to prove that X is almost K-compact.

Since X_1 and X_2 are closed domain, they can be written in the form $\bar{G}_1$ and $\bar{G}_2$ respectively, where G_1 and G_2 are open sets. We have $X = \bar{G}_1 \cup \bar{G}_2 = \overline{G_1 \cup G_2}$. Let $\mathcal{U} = \{U\colon U \in \mathcal{U}\}$ be an open ultrafilter on X such that $\bigcap \overline{\mathcal{U}} = \emptyset$. Since $G_1 \cup G_2$ is dense in X, it follows that $G_1 \cup G_2 \in \mathcal{U}$. But $\mathcal{U}$ is prime also, so $G_1 \in \mathcal{U}$ or $G_2 \in \mathcal{U}$. Let us assume that $G_1 \in \mathcal{U}$ and set $\mathcal{U}' = \{U' = U \cap \bar{G}_1\colon U \in \mathcal{U}\}$. Thus $\mathcal{U}'$ is an open ultrafilter on the almost K-compact space $\bar{G}_1$ such that $\bigcap \overline{\mathcal{U}'} = \emptyset$. By almost K-compactness of $\bar{G}_1$, there exists a subfamily $\{U'_a\colon a \in A \text{ and } \operatorname{card}(A) < K\}$ of $\mathcal{U}'$ such that $\bigcap_{a \in A} \bar{U}'_a = \emptyset$. Now let $\{U_a\colon U_a \in \mathcal{U} \text{ and } a \in A,\ \operatorname{card}(A) < K\}$ be a subfamily of $\mathcal{U}$ such that for each a, $U'_a = U_a \cap \bar{G}_1$. Since U_a and G_1 are open, $U_a \cap G_1 \in \mathcal{U}$ for each a and $\overline{U_a \cap \bar{G}_1} = \overline{U_a \cap G_1}$. Then $\bigcap \{\overline{U_a \cap G_1}\} = \emptyset$ and hence X is almost K-compact.

With the help of the following Lemma 2.1 proved in [1], we can prove Corollary 2.1.

LEMMA 2.1. *A closed domain subset of almost K-compact space is almost K-compact.*

COROLLARY 2.1. *If X is the union two closed, almost K-compact spaces, then X is almost K-compact.*

Proof. Let $X = F_1 \cup F_2$, where F_1 and F_2 are closed, almost K-compact spaces. Let $G_1 = X - F_1$ and $G_2 = X - \overline{X - F_1}$. It can be easily seen that $\bar{G}_1 \subseteq F_2$ and $\bar{G}_2 \subseteq F_1$. Since G_1 and G_2 are open sets, $\bar{G}_1$ and $\bar{G}_2$ are closed domain subsets of F_2 and F_1 respectively. By Lemma 2.1, $\bar{G}_1$ and $\bar{G}_2$ are almost K-compact spaces. Further, $X = \bar{G}_1 \cup \bar{G}_2$. For if $\bar{G}_1 \subseteq F_2$ and $\bar{G}_2 \subseteq F_1$, $\bar{G}_1 \cup \bar{G}_2 \subseteq F_1 \cup F_2 = X$. Again since $\bar{G}_1 \cup G_2 = \bar{G}_1 \cup \cup (X - \bar{G}_1) = X$, we have $\bar{G}_1 \cup \bar{G}_2 \supseteq X$. Thus $X = \bar{G}_1 \cup \bar{G}_2$. So by Theorem 2.1, the corollary follows.

Remark. It is doubtful that almost K-compact spaces are preserved under arbitrary union.

3. MAPPINGS AND ALMOST K-COMPACT SPACES

It was established in [1] that almost K-compactness is invariant under (K, f)-perfect mappings and that inverse image of each almost K-compact subset of a regular space under a continuous mappings is almost K-compact provided the domain space is almost K-compact. In this section, the inverse invariance of almost K-compactness under perfect mapping is investigated.

In his paper [4], Frolik proved the following lemma.

LEMMA 3.1. *Let f be a perfect mapping of a regular space X onto a space Y. If $a = \{\mathscr{U}\}$ is a complete collection of open coverings of Y, then the family $f^{-1}(a)$ of all coverings $f^{-1}(\mathscr{U}) = \{f^{-1}(U) : U \in \mathscr{U}\}$, where $\mathscr{U} \in a$, is a complete collection.*

In [1] we have characterized almost K-compactness in terms of completeness property as follows:

LEMMA 3.2. *A space X is almost K-compact if and only if the collection δ of all open coverings with cardinality less than K, of X is complete.*

Now the main theorem of this section is obvious from the above lemmas.

THEOREM 3.1. *The inverse image under a perfect mapping of an almost K-compact space is almost K-compact provided the domain space is regular.*

4. ALMOST K-COMPACTNESS AND OTHER SPACES

In this section, relationships of almost K-compact space with compact, K-compact and a_K-realcompact space are studied.

4.1. Almost K-compactness and compactness. Every compact space is almost K-compact but the converse may not be true. However, in a regular space almost K-compact space coupled with (K, f)-compact is compact.

THEOREM 4.1.1. *Let X be a regular space. If X is (K, f)-compact and almost K-compact space, then X is compact.*

Proof. Let X be the given space and $\mathfrak{F}$ be a centred family of closed sets of X. Let us set $\mathscr{U} = \{U : U \text{ is open such that } U \supset F \in \mathfrak{F}\}$. Now $\mathscr{U}$ is a centred family of open sets of X and is contained in some maximal centred family $\mathfrak{M}$ of open sets. By the (K, f)-compactness of X, we have

$$\bigcap\{\bar{M}_a\colon M_a \in \mathfrak{M},\ a \in A,\ \operatorname{card}(A) < K\} \neq \emptyset.$$

Now since X is an almost K-compact space, $\bigcap\overline{\mathfrak{M}} \neq \emptyset$, i. e., there is at least a point x in $\bigcap\{\bar{M} : M \in \mathfrak{M}\}$. Finally we have to show that $x \in \bigcap\mathfrak{F}$. Suppose otherwise. Since X is regular, there is an open set M_1 containing F and an open set M_2 containing the point x such that M_1, $M_2 \in \mathfrak{M}$ with $M_1 \cap M_2 = \emptyset$, contradicting the centredness of $\mathfrak{M}$. Hence $x \in \bigcap\mathfrak{F}$. Thus X is compact.

4.2. Almost K-compactness and K-compactness. It is easy to see that every K-compact space is almost K-compact. Now we will show that the converse also holds under some additional conditions. For this, we have to introduce the following concept.

DEFINITION 4.2.1. A space X is said to be *weak Kb-space* if given a downward directed family $\{F_d : d \in D,\ \operatorname{card}(D) < K\}$ of closed domain sets in X with $\bigcap_{d \in D} F_d = \emptyset$, there exists a family $\{Z_d : d \in D,\ \operatorname{card}(D) < K\}$ of zero-sets with $\bigcap_{d \in D} Z_d = \emptyset$, such that $Z_d \supset F_d$ for each $d \in D$.

Here by a downward directed family we mean a family which is indexed by a directed set D such that for any $\alpha, \beta \in D$, $\alpha \dashv \beta$, we have $F_\alpha \supset F_\beta$, for any pair of members $F_\beta, F_\alpha \in \mathfrak{F}$.

Weak Kb-spaces give us weak cb-spaces, when we take D to be the set of all natural numbers with the usual ordering.

THEOREM 4.2.1. *A completely regular, weak Kb- and almost K-compact space is K-compact.*

Proof. Let $\mathscr{Z}$ be a Z-ultrafilter on the given space X with $\bigcap\mathscr{Z} = \emptyset$ and let us define a family $\mathscr{U}$ of the form $\{U;\ U \subset X \text{ is open and there exists a } Z \in \mathscr{Z} \text{ such that } Z \subset U\}$. Clearly $\mathscr{U}$ is an open filter on X and is contained in an open ultrafilter $\mathfrak{M}$. Since the space X is completely regular and Hausdorff, it is regular and by the regularity of X, $\bigcap \mathfrak{M} = \emptyset$. Otherwise, if $\bigcap \mathfrak{M} \neq \emptyset$, let for any point $x \in \bigcap \mathfrak{M}$. Then there exists a $Z \in \mathscr{Z}$ such that $x \notin Z$ and so by regularity of X, there exists an open set M_1 containing the point x and an open set M_2 containing the zero-set Z (hence closed set) such that $M_1, M_2 \in \mathfrak{M}$ with $M_1 \cap M_2 = \emptyset$. This gives a contradiction of the filtredness of $\mathfrak{M}$. Hence our supposition holds ture.

Now since X is an almost K-compact space, we have $\bigcap \{\overline{M}_a : M_a \in \mathfrak{M}, a \in A, \operatorname{card}(A) < K\} = \emptyset$. Let β be the family of finite subsets of A; then β is a directed set. Let us now set $O_B = \bigcap\limits_{a \in B} M_a$ for each $B \in \beta$. We have $\operatorname{card}(\beta) < K$ and that $\{O_B : B \in \beta\}$ is a downward directed system of open sets with $\bigcap\limits_{B \in \beta} \overline{O}_B = \emptyset$.

Since the space X is weak Kb, there exist zero-set $\{Z_B : B \in \beta, \operatorname{card}(\beta) < K\}$ such that for each $B \in \beta$, $Z_B \supset \overline{O}_B$ and $\bigcap\limits_{B \in \beta} Z_B = \emptyset$. Now for each B, $\overline{O}_B$ intersects every member of $\mathscr{Z}$. Thus $Z_B \in \mathscr{Z}$. Therefore, there exists a subfamily $\mathscr{Z}'$ of $\mathscr{Z}$, with cardinality $< K$ such that $\bigcap \mathscr{Z}' = \emptyset$. Hence the theorem.

Now from Theorem 4.2.1 and Corollary 2.1, we have the following theorem concerning the union of K-compact spaces.

THEOREM 4.2.2. *Let X be a Tychonoff, weak Kb-space. If X is the union of two closed K-compact spaces, then X is K-compact.*

4.3. Almost K-compactness and a_K-realcompactness. In [2], we introduced a concept of a_K-realcompact space as a generalization of a-realcompact space due to Dykes [3]. A space X is said to be *a_K-realcompact* if $\mathfrak{F}$ is a closed ultrafilter with K-intersection property, then $\bigcap \mathfrak{F} \neq \emptyset$. The following theorem relates almost K-compact spaces with a_K-realcompact spaces as follows:

THEOREM 4.3.1. *In a regular space, almost K-compactness implies a_K-realcompactness.*

Proof. Let X be the given space and $\mathfrak{F}$ be a free closed ultrafilter of X. Let $\mathscr{U}$ be the family $\{U : U$ is open and there exists a $F \in \mathfrak{F}$ such that $F \subset U\}$. Now $\mathscr{U}$ is an open filter and is contained in an open ultrafilter $\mathfrak{M}$. By regularity of X, $\bigcap \overline{\mathfrak{M}} = \emptyset$. Since X is almost K-compact, there exists a subfamily $\mathfrak{M}'$ of $\mathfrak{M}$ with $\operatorname{card}(\mathfrak{M}') < K$ such that $\bigcap \overline{\mathfrak{M}'} = \emptyset$.

If $M \in \mathfrak{M}$, then $\overline{M}$ intersects each member of $\mathfrak{F}$ and by maximality of $\mathfrak{F}$, $\overline{M} \in \mathfrak{F}$. Thus $\overline{\mathfrak{M}'}$ is a subafmily of $\mathfrak{F}$, with $\operatorname{card}(\mathfrak{M}') < K$ and it has empty intersection. It follows, then, that X is a_K-realcompact.

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