

E. ŚLIWIŃSKI (Kraków)

The Dirichlet and Neumann problem for Laplace equation in an angular region

1. In this paper we shall construct the Green function and find the solution of the Dirichlet and Neumann problem for the angular region D described by the inequalities

$$x > 0, \quad 0 < y < \sqrt{3}x.$$

Let us consider the straight lines:

$$L_1: y = 0, \quad L_2: y = \sqrt{3}x, \quad L_3: y = -\sqrt{3}x.$$

Let $X = (x, y) \in D$, $Y = (s, t) \in D$ and let denote by $X_1 = (x_1, y_1)$, $X_2 = (x_2, y_2)$, ..., $X_5 = (x_5, y_5)$ the symmetric images of X in L_2 , X_1 in L_3 , X_2 in L_1 , X_3 in L_1 , X_4 in L_3 , respectively.

Hence

$$\begin{aligned} x_1 &= \frac{\sqrt{3}y - x}{2}, & y_1 &= \frac{\sqrt{3}x + y}{2}, \\ x_2 &= \frac{-\sqrt{3}y - x}{2}, & y_2 &= \frac{\sqrt{3}x - y}{2}, \\ x_3 &= -\frac{\sqrt{3}y + x}{2}, & y_3 &= \frac{y - \sqrt{3}x}{2}, \\ x_4 &= \frac{\sqrt{3}y + x}{2}, & y_4 &= \frac{-\sqrt{3}x - y}{2}, \\ x_5 &= x, & y_5 &= -y. \end{aligned}$$

Let

$$\begin{aligned} r &= [(s-x)^2 + (t-y)^2]^{\frac{1}{2}}, & r_1 &= [(s-x_1)^2 + (t-y_1)^2]^{\frac{1}{2}}, & r_2 &= [(s-x_2)^2 + (t-y_2)^2]^{\frac{1}{2}}, \\ r_3 &= [(s-x)^2 + (t-y)^2]^{\frac{1}{2}}, & r_4 &= [(s-x_4)^2 + (t-y_4)^2]^{\frac{1}{2}}, & r_5 &= [(s-x_5)^2 + (t-y_5)^2]^{\frac{1}{2}}. \end{aligned}$$

2. We now proceed to solve the Dirichlet problem for the Laplace equation in the region D .

THEOREM 1 ⁽¹⁾. *The Green's function for the Dirichlet problem concerning the equation $\Delta u = 0$ in the region D with pole at X is of the form*

$$(1) \quad G(X, Y) = \ln r - \ln r_5 + \ln r_2 - \ln r_3 + \ln r_4 - \ln r_1.$$

Proof. The function G being the sum of harmonic functions is itself harmonic and satisfies the boundary condition

$$\begin{aligned} G(X, Y)|_{t=0} &= 0 & \text{since } r &= r_5, r_1 = r_4, r_2 = r_3, \\ G(X, Y)|_{t=\sqrt{3}s} &= 0 & \text{since } r &= r_1, r_2 = r_5, r_3 = r_4. \end{aligned}$$

We have further

$$(2) \quad \left. \frac{\partial G}{\partial t} \right|_{t=0} = \frac{-2y}{(s-x)^2 + y^2} + \frac{y - \sqrt{3}x}{(s+2x)^2 + (\sqrt{3}s+2y)^2} + \frac{\sqrt{3}x+y}{(s+2x)^2 + (\sqrt{3}s-2y)^2},$$

$$(3) \quad \left. \frac{dG}{dn} \right|_{t=\sqrt{3}s} = \frac{1}{2} \left[\sqrt{3} \frac{\partial G}{\partial s} - \frac{\partial G}{\partial t} \right] \Big|_{t=\sqrt{3}s} = \frac{2(y-\sqrt{3}x)}{(s-x)^2 + (\sqrt{3}s-y)^2} + \frac{2(\sqrt{3}x+y)}{(s-x)^2 + (\sqrt{3}s+y)^2} - \frac{4y}{(2s+x)^2 + y^2}.$$

Let us consider the function

$$(4) \quad u(X) = \frac{-1}{\pi} \int_0^\infty f_1(s) \left. \frac{\partial G}{\partial t} \right|_{t=0} ds + \frac{1}{\pi} \int_0^\infty f_2(s) \left. \frac{dG}{dn} \right|_{t=\sqrt{3}s} ds.$$

By (2) and (3)

$$(4a) \quad u(X) = \sum_{i=1}^6 V_i(X),$$

where

$$\begin{aligned} V_1(X) &= \frac{1}{\pi} \int_0^\infty f_1(s) \frac{y ds}{(s-x)^2 + y^2}, \\ V_2(X) &= \frac{-1}{2\pi} \int_0^\infty f_1(s) \frac{(y - \sqrt{3}x) ds}{(s+2x)^2 + (\sqrt{3}s+2y)^2}, \end{aligned}$$

⁽¹⁾ B. M. Budak, A. A. Samarski and A. N. Tichonow, *Zadania i problemy fizyki matematycznej*, PWN, Warszawa 1965.

$$V_3(X) = \frac{-1}{2\pi} \int_0^\infty f_1(s) \frac{(\sqrt{3}x+y) ds}{(s+2x)^2 + (\sqrt{3}s-2y)^2},$$

$$V_4(X) = \frac{1}{\pi} \int_0^\infty f_2(s) \frac{(y-\sqrt{3}x) ds}{(s-x)^2 + (\sqrt{3}s-y)^2},$$

$$V_5(X) = \frac{1}{\pi} \int_0^\infty f_2(s) \frac{(\sqrt{3}x+y) ds}{(s-x)^2 + (\sqrt{3}s+y)^2},$$

$$V_6(X) = \frac{2}{\pi} f_2(s) \frac{y ds}{(2s+x)^2 + y^2}.$$

LEMMA 1. Let a be any real number and let $y - ax \neq 0$; then

$$I(X) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{(y - ax) ds}{(s - x)^2 + (a \cdot s - y)^2} = 1.$$

Proof. Upon changing the variable in the above integral

$$(5) \quad s - x = t(y - a \cdot x)$$

we get

$$a \cdot s - y = ax + ta(y - ax) - y = (y - ax)(t \cdot a - 1),$$

whence

$$I(X) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{(1 + a^2)(t^2 - 2 \cdot at + 1)} = 1.$$

LEMMA 2. Let f_2 be continuous and bounded function on the positive real half-line let $y - ax \neq 0$ for a fixed real number a . If the integral $V_4(X)$ converges uniformly in the set $y + \delta < \sqrt{3}x$, $\delta > 0$, then $\lim_{v \rightarrow \sqrt{3}x} V_4(X) = f_2(x)$.

Proof. Let

$$\bar{f}_1(s) = \begin{cases} f_1(s) & \text{for } 0 \leq s < \infty, \\ 0 & \text{for } s < 0; \end{cases}$$

then

$$V_4(X) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \bar{f}_1(s)(y - \sqrt{3}x)[(s-x)^2 + (\sqrt{3}s-y)^2]^{-1} ds.$$

We may write $V_4(X) = J_1(X) + J_2(X)$, where

$$J_1(X) = \frac{1}{\pi} \int_{-\infty}^{+\infty} [\bar{f}_1(s) - \bar{f}_1(x_0)](y - \sqrt{3}x)[(s-x)^2 + (\sqrt{3}s-y)^2]^{-1} ds,$$

$$J_2(X) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \bar{f}_1(x_0)(y - \sqrt{3}x)[(s-x)^2 + (\sqrt{3}s-y)^2]^{-1} ds.$$

By Lemma 1 we have $J_2(X) = \bar{f}_1(x_0)$ for $\alpha = \sqrt{3}$. Let us check that $J_1(X) \rightarrow 0$ as $X \rightarrow (x, 0)$. Given any positive number ε there exists a $\delta > 0$ such that $s \in K_\delta = \{u: |u - x_0| < \delta\}$ implies

$$|\bar{f}_1(s) - \bar{f}_1(s_0)| < \varepsilon.$$

Let $J_1(X) = H_1(X) + H_2(X)$, where

$$H_1(X) = \frac{1}{\pi} \int_{K_\delta} [\bar{f}_2(s) - \bar{f}_2(x_0)](y - \sqrt{3}x) [(s-x)^2 + (\sqrt{3}s-y)^2]^{-1} ds,$$

$$H_2(X) = \frac{1}{\pi} \int_{E_1 - K_\delta} [\bar{f}_2(s) - \bar{f}_2(x_0)](y - \sqrt{3}x) [(s-x)^2 + (\sqrt{3}s-y)^2]^{-1} ds.$$

By Lemma 1

$$|H_1(X)| \leq \frac{\varepsilon}{\pi} \int_{-\infty}^{+\infty} |y - \sqrt{3}x| [(s-x)^2 + (\sqrt{3}s-y)^2]^{-1} ds,$$

$$|H_2(X)| \leq \frac{2M}{\pi} \int_{E_1 - K_\delta} |y - \sqrt{3}x| [(s-x)^2 + (\sqrt{3}s-y)^2]^{-1} ds = K_1(X) + K_2(X),$$

where

$$K_1(X) = \frac{2M}{\pi} \int_{-\infty}^{x-\delta} |y - \sqrt{3}x| [(s-x)^2 + (\sqrt{3}s-y)^2]^{-1} ds,$$

$$K_2(X) = \frac{2M}{\pi} \int_{x+\delta}^{+\infty} |y - \sqrt{3}x| [(s-x)^2 + (\sqrt{3}s-y)^2]^{-1} ds,$$

$$M = \sup |f_1(s)|.$$

Introducing the change a variables (5) in $K_1(X)$, $K_2(X)$ we obtain

$$K_1(X) = \frac{2M}{\pi} \int_{-\infty}^{-A} [4(t^2 - 2\sqrt{3}t + 1)]^{-1} dt \rightarrow 0, \quad y \rightarrow \sqrt{3}x,$$

$$K_2(X) = \frac{2M}{\pi} \int_A^{+\infty} [4(t^2 - 2\sqrt{3}t + 1)]^{-1} dt \rightarrow 0, \quad y \rightarrow \sqrt{3}x,$$

where

$$A = \frac{\delta}{\sqrt{3}x - y}.$$

LEMMA 3. Let $f_1(s)$ be continuous and bounded on the positive real half-line; then $V_1(X) \rightarrow f_1(x)$ as $y \rightarrow 0^+$.

The proof is similar to that of Lemma 1.

LEMMA 4. Let $f_2(s)$ be continuous and bounded on the positive half-line; then $[V_5(X) + V_6(X)] \rightarrow 0$ and $[V_1(X) + V_3(X)] \rightarrow 0$ as $y \rightarrow \sqrt{3}x$.

Proof. By assumption the integrals $V_5(X)$ and $V_6(X)$ are uniformly convergent for $y > a > 0$ whence represent continuous function for $y = \sqrt{3}x, x > 0$. Thus

$$\begin{aligned} &V_1(x, \sqrt{3}x) + V_2(x, \sqrt{3}x) \\ &= \frac{2}{\pi} \int_0^\infty f_2(s) (2\sqrt{3}x - 2\sqrt{3}x) [(s-x)^2 + (\sqrt{3}s + \sqrt{3}x)^2]^{-1} ds. \end{aligned}$$

The proof of the second part of the lemma is similar.

LEMMA 5. Let $f_1(s)$ be continuous and bounded on the positive half-line; then $V_2(X) \rightarrow 0$ as $y \rightarrow \sqrt{3}x$.

Proof. By our assumption

$$|V_2(X)| \leq \frac{M}{\pi} (y - \sqrt{3}x) \int_0^\infty (a^2 + s^2)^{-1} ds$$

a being a positive constant independent of y , and $M = \sup |f_1(s)|$. Hence $V_2(X) \rightarrow 0$ as $y \rightarrow \sqrt{3}x$.

LEMMA 6. Let $f_1(s)$ be continuous and bounded on the positive half-line; then $[V_1(X) + V_2(X)] \rightarrow 0, [V_4(X) + V_5(X)] \rightarrow 0,$ and $V_6(X) \rightarrow 0$ as $y \rightarrow 0^+$.

Proof. Continuity of the involved integrals implies easily that $V_1(X) + V_2(X)$ and $V_4(X) + V_5(X)$ tend to zero as $y \rightarrow 0^+$. The proof concerning the integral $V_6(X)$ is similar to that of Lemma 5.

LEMMA 7. Let the functions $f_i(s), i = 1, 2$ be continuous and bounded on the positive half-line; then the integrals

$$V_1^1(X) = \frac{1}{\pi} \int_0^\infty f_1(s) \frac{\partial}{\partial x} [P(X)] ds,$$

$$V_1^2(X) = \frac{1}{\pi} \int_0^\infty f_1(s) \frac{\partial}{\partial y} [P(X)] ds,$$

$$V_1^3(X) = \frac{1}{\pi} \int_0^\infty f_1(s) \frac{\partial^2}{\partial x^2} [P(X)] ds,$$

$$V_1^4(X) = \frac{1}{\pi} \int_0^\infty f_1(s) \frac{\partial^2}{\partial y^2} [P(X)] ds$$

converge uniformly for $a < y < A$, $a < x < A$, where a, A are positive constants and $P(X) = y[(s-x)^2 + y^2]^{-1}$.

Proof. All the integrals admit the common majorant

$$\frac{MA}{\pi} \int_0^{\infty} \frac{ds}{(s-x)^2 + a^2} \leq \frac{MA}{\pi} \int_{-\infty}^{+\infty} \frac{ds}{(s-x)^2 + a^2} < \frac{Ma}{\pi a} \int_{-\infty}^{+\infty} \frac{dt}{t^2 + 1}.$$

The uniform convergence of the integrals $V_2^i(X)$, $V_3^i(X)$, $i = 1, 2, 3, 4$ in region $|y| < A$, $a \leq x < B$ and of the integrals $V_5^i(X)$, $V_6^i(X)$, $i = 1, 2, 3, 4$ in the region $a \leq y < A$, $|x| < A$ may be checked analogously. Let us check the uniform convergence of the integrals $V_4^i(X)$, $i = 1, 2, 3, 4$, in the set $y + a < \sqrt{3}x$, $a < x < A$.

All the integrals possess the common majorant

$$\frac{M}{\pi} \int_{-\infty}^{+\infty} \frac{|y - \sqrt{3}x| ds}{(s-x)^2 + (\sqrt{3}s-y)^2} = \frac{M}{\pi}$$

in virtue of Lemma 1.

Lemma 7 implies that the function $u(X)$ defined by formula (4) is of class O^2 in the set $a < x < A$, $a < y < \sqrt{3}x - a$. From uniform convergence of the integrals V_k^i , $i = 0, 1, 2, 3, 4$, $k = 1, 2, 3, 4, 5, 6$ and from the fact that Green's function is harmonic we infer easily that $u(X)$ is harmonic too, i. e.

$$\begin{aligned} \Delta u(X) &= \frac{-1}{\pi} \int_0^{\infty} f_1(s) \frac{\partial}{\partial t} [\Delta_X G(X, Y)] \Big|_{t=0} ds + \\ &+ \frac{1}{\pi} \int_0^{+\infty} f_2(s) \frac{d}{dn} [\Delta_X G(X, Y)] \Big|_{t=\sqrt{3}s} ds = 0. \end{aligned}$$

From Lemmas 2-7 follows

THEOREM 2. *Let the functions $f_i(s)$, $i = 1, 2$, be continuous and bounded on the positive half-line; then the function $u(X)$ defined by (4) solves the Dirichlet problem for the region D , the boundary conditions being $u(x, 0) = f_1(x)$, $u(x, \sqrt{3}x) = f_2(x)$.*

3. We now proceed to solve the Neumann problem for the Laplace equation in the region D .

Let $\vec{n}(\cos \alpha, \cos \beta)$ denote the normal at the boundary of D directed to the interior of D . Then

$$\frac{dr}{dn} = \cos \varphi, \quad \frac{dr_1}{dn} = -\cos \varphi.$$

Hence for $Y \in L_2, \vec{r} = \vec{r}_1$ and

$$(6) \quad \frac{dr}{dn} + \frac{dr_1}{dn} = 0.$$

So we have proved

LEMMA 8. Let $X_1(x_1, y_1)$ be the symmetric image of the point X the straight line L_2 ; then (6) holds for $Y \in L_2$.

LEMMA 9. Let the function $v(r)$ be of class C^1 and let $X_1(x_1, y_1)$ be as in Lemma 8; then

$$\frac{dv(r)}{dn} + \frac{dv(r_1)}{dn} = 0 \quad \text{for } Y \in L_2.$$

Proof. We have

$$\frac{dv(r)}{dn} = v'(r) \frac{dr}{dn}, \quad \frac{dv(r_1)}{dn} = v'(r_1) \frac{dr_1}{dn}.$$

Now $r = r_1, v'(r) = v'(r_1)$ for $Y \in L_2$ whence by Lemma 8

$$\frac{dr}{dn} = -\frac{dr_1}{dn}$$

which implies our assertion.

THEOREM 3. The Green's function concerning the Neumann problem for the Laplace equation in the region D of form

$$(7) \quad G(X, Y) = \ln r + \sum_{i=1}^5 \ln r_i.$$

Proof. By (7) and Lemma 7

$$\left. \frac{dG}{dn} \right|_{t=0} = \frac{d}{dn} [\ln r_0 + \ln r_5] + \frac{d}{dn} [\ln r_1 + \ln r_4] + \frac{d}{dn} [\ln r_2 + \ln r_3] = 0$$

and

$$\left. \frac{dG}{dn} \right|_{t=\sqrt{3}s} = \frac{d}{dn} [\ln r_0 + \ln r_1] + \frac{d}{dn} [\ln r_2 + \ln r_5] + \frac{d}{dn} [\ln r_3 + \ln r_4] = 0.$$

By (7)

$$G(X, Y)|_{t=\sqrt{3}s} = 2 \{ \ln [(s-x)^2 + (\sqrt{3}s-y)^2] + \ln [(s-x)^2 + (\sqrt{3}s+y)^2] + \ln [(2s+x)^2 + y^2] \}$$

and

$$G(X, Y)|_{t=0} = 2 \left\{ \ln[(s-x)^2 + y^2] + \ln\left[\left(\frac{1}{2}s+x\right)^2 + \left(\frac{\sqrt{3}}{2}s-y\right)^2\right] + \right. \\ \left. + \ln\left[\left(\frac{1}{2}s+x\right)^2 + \left(\frac{\sqrt{3}}{2}s+y\right)^2\right] \right\}.$$

Whence

$$u(X) = \frac{1}{\pi} \int_0^{\infty} f_1(s) G(X, Y)|_{t=0} ds + \frac{1}{\pi} \int_0^{\infty} f_2(s) G(X, Y)|_{t=\sqrt{3}s} ds = \sum_{i=1}^6 W_i(X),$$

where

$$W_1(X) = \frac{1}{2\pi} \int_0^{\infty} f_1(s) \ln[(s-x)^2 + y^2] ds,$$

$$W_2(X) = \frac{1}{2\pi} \int_0^{\infty} f_1(s) \ln\left[\left(\frac{s}{2}+x\right)^2 + \left(\frac{\sqrt{3}}{2}s-y\right)^2\right] ds,$$

$$W_3(X) = \frac{1}{2\pi} \int_0^{\infty} f_1(s) \ln\left[\left(\frac{s}{2}+x\right)^2 + \left(\frac{\sqrt{3}}{2}s+y\right)^2\right] ds,$$

$$W_4(X) = \frac{1}{2\pi} \int_0^{\infty} f_1(s) \ln[(s-x)^2 + (\sqrt{3}s-y)^2] ds,$$

$$W_5(X) = \frac{1}{2\pi} \int_0^{\infty} f_2(s) \ln[(s-x)^2 + (\sqrt{3}s+y)^2] ds,$$

$$W_6(X) = \frac{1}{2\pi} \int_0^{\infty} f_2(s) \ln[(2s+x)^2 + y^2] ds.$$

LEMMA 10. Let the integrals $W_i(X)$, $i = 1, 2, 3, 4, 5, 6$, be uniformly convergent in the set $a < x < A$, $a < y < A$, and let $W_4(X)$ be uniformly convergent for $a < x < A$, $0 < y < \sqrt{3}x$, and let the functions f_1, f_2 be continuous and bounded on the positive half-line, the function $u(X)$ defined by (8) is harmonic in the region $a < x < A$, $0 < y < \sqrt{3}x - a$.

Proof. Proceeding as in the proof of Lemma 7 one can prove that the integrals $W_i^k(X)$, $i = 1, 2, 3$, $k = 0, 1, 2, 3, 4$, are uniformly convergent for $a < x < A$, $0 < y < \sqrt{3}x - a$.

Therefore $u(X)$ is of class C^2 in D . By (8)

$$\Delta u(X) = \frac{1}{2\pi} \int_0^{\infty} f_1(s) \Delta_X G(X, Y)|_{t=0} ds + \frac{1}{\pi} \int_0^{\infty} f_2(s) \Delta_X G(X, Y)|_{t=\sqrt{3}s} ds = 0,$$

$$\frac{\partial u}{\partial y} = \sum_{i=1}^6 W_i^2(X),$$

where

$$\begin{aligned}
 W_1^2(X) &= \frac{1}{\pi} \int_0^\infty f_1(s) \frac{y ds}{(s-x)^2 + y^2}, \\
 W_2^2(X) &= \frac{1}{\pi} \int_0^\infty f_1(s) \frac{\left(y - \frac{\sqrt{3}}{2}s\right) ds}{\left(\frac{s}{2} + x\right)^2 + \left(\frac{\sqrt{3}}{2}s - y\right)^2}, \\
 W_3^2(X) &= \frac{1}{\pi} \int_0^\infty f_1(s) \frac{\left(\frac{\sqrt{3}}{2}s + y\right) ds}{\left(\frac{s}{2} + x\right)^2 + \left(\frac{\sqrt{3}}{2}s + y\right)^2}, \\
 W_4^2(X) &= \frac{1}{\pi} \int_0^\infty f_2(s) \frac{(y - \sqrt{3}) ds}{(s-x)^2 + (\sqrt{3}s - y)^2}, \\
 W_5^2(X) &= \frac{1}{\pi} \int_0^\infty f_2(s) \frac{(y + \sqrt{3}s) ds}{(s-x)^2 + (\sqrt{3}s + y)^2}, \\
 W_6^2(X) &= \frac{1}{\pi} \int_0^\infty f_2(s) \frac{y ds}{(2s+x)^2 + y^2}, \\
 \frac{du}{dn} &= \frac{1}{2} \left[\sqrt{3} \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right] = \sum_{i=1}^6 U_i(X),
 \end{aligned}$$

where

$$\begin{aligned}
 U_1(X) &= \frac{1}{\pi} \int_0^\infty f_1(s) \frac{\sqrt{3}x - \sqrt{3}s - y}{(s-x)^2 + y^2} ds, \\
 U_2(X) &= \frac{1}{\pi} \int_0^\infty f_1(s) \frac{\sqrt{3}s + \sqrt{3}x - y}{\left(\frac{s}{2} + x\right)^2 + \left(\frac{\sqrt{3}}{2}s - y\right)^2} ds, \\
 U_3(X) &= \frac{1}{\pi} \int_0^\infty f_1(s) \frac{\sqrt{3}x - y}{\left(\frac{s}{2} + x\right)^2 + \left(\frac{\sqrt{3}}{2}s + y\right)^2} ds, \\
 U_4(X) &= \frac{1}{\pi} \int_0^\infty f_2(s) \frac{\sqrt{3}x - y}{(s-x)^2 + (\sqrt{3}s - y)^2} ds,
 \end{aligned}$$

$$U_5(X) = \frac{1}{\pi} \int_0^\infty f_2(s) \frac{\sqrt{3}x - y - 2\sqrt{3}s}{(s-x)^2 + (\sqrt{3}s+y)^2} ds,$$

$$U_6(X) = \frac{1}{\pi} \int_0^\infty f_2(s) \frac{2\sqrt{3}s + \sqrt{3}x - y}{(2s+x)^2 + y^2} ds.$$

LEMMA 11. *Let the functions $f_i(s)$, $i = 1, 2$, satisfy the assumptions of Lemma 10; then*

$$(9) \quad \lim \frac{\partial u(X)}{\partial y} = f_1(x) \quad \text{as } y \rightarrow 0^+,$$

$$(10) \quad \lim \frac{du(X)}{dn} = f_2(x) \quad \text{as } y \rightarrow \sqrt{3}x, y < \sqrt{3}x.$$

Proof. By Lemma 3 $W_1^2(X) \rightarrow f_1(x)$ as $y \rightarrow 0^+$,

$$[W_2^2(X) + W_3^2(X)] \rightarrow W_2^2(X, 0) + W_3^2(X, 0) \rightarrow 0,$$

$$[W_4^2(X) + W_5^2(X)] \rightarrow W_4^2(X, 0) + W_5^2(X, 0) \rightarrow 0 \quad \text{and} \quad W_6^2(X) \rightarrow 0$$

as $y \rightarrow 0^+$ since the involved integrals are continuous.

This implies (9). By Lemma 2 for $y \rightarrow \sqrt{3}x$

$$U_4(X) \rightarrow f_2(x),$$

$$[U_1(X) + U_2(X)] \rightarrow U_1(x, \sqrt{3}x) + U_2(x, \sqrt{3}x) = 0, \quad U(X) \rightarrow 0 \quad \text{as } y \rightarrow \sqrt{3}x,$$

$$[U_5(X) + U_6(X)] \rightarrow U_5(x, \sqrt{3}x) + U_6(x, \sqrt{3}x) = 0.$$

Lemmas 10 and 11 imply

THEOREM 4. *Let the functions $f_i(s)$, $i = 1, 2$, satisfy the assumptions of Lemma 10; then the function $u(X)$ defined by (8) solves the Neumann problem in D with the boundary conditions (9) and (10).*