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The mixed boundary conditions of Dirichlet-Neumann type for the Laplace equation in the positive quadrant of the plane

In this paper we solve the Laplace equation in the region $D = \{(x_1, x_2)\colon x_1 > 0, x_2 > 0\}$ with Dirichlet condition an one half-line at the boundary of D and with the Neumann condition on the another one.

In the paper [2] we solved only the Neumann problem for this region.

The Green function by means of which we solve the problem, is constructed by aid of the symmetric images.

1. Let $X = X_{11}(x_1, x_2)$, $Y = Y(y_1, y_2)$ denote two different points in D. Let $X_{-11} = (-x_1, x_2)$, $X_{1-1} = (x_1, -x_2)$ be symmetric images of X in the axis Oy_2 and Oy_1, respectively. The point $X_{-1-1} = (-x_1, -x_2)$ is the symmetric image of X_{-11} in the axis Oy_1 or of X_{1-1} in the axis Oy_2.

Let

$$\begin{aligned} r_{11}^{-1} &= \overline{YX_{11}} = [(y_1-x_1)^2+(y_2-x_2)^2]^{-\frac{1}{2}}, \\ r_{-11}^{-1} &= \overline{YX_{-11}} = [(y_1+x_1)^2+(y_2-x_2)^2]^{-\frac{1}{2}}, \\ r_{1-1}^{-1} &= \overline{YX_{1-1}} = [(y_1-x_1)^2+(y_2+x_2)^2]^{-\frac{1}{2}}, \\ r_{-1-1}^{-1} &= \overline{YX_{-1-1}} = [(y_1+x_1)^2+(y_2+x_2)^2]^{-\frac{1}{2}} \end{aligned}$$

and let denote l_1 the half-line $x_2 = 0, x_1 > 0$ and l_2 the half-line $x_1 = 0$, $x_2 > 0$.

THEOREM 1. *The Green function harmonic in D and satisfying the boundary conditions*

$$G(X, Y)|_{y_2=0} = 0 \quad \textit{for } Y \in l_1, \tag{1}$$

$$\left.\frac{\partial G(X, Y)}{\partial y_1}\right|_{y_1=0} = 0 \quad \textit{for } Y \in l_2 \tag{2}$$

is of form

$$G(X, Y) = -\log r_{11} - \log r_{-11} + \log r_{1-1} + \log r_{-1-1}. \tag{3}$$

Proof. The function $G(X, Y)$ of form

$$-\log r_{11} + H(X, Y),$$

where

$$H(X, Y) = -\log r_{-11} + \log r_{1-1} + \log r_{-1-1}$$

is harmonic with respect to Y. The function $G(X, Y)$ satisfies condition (1). For $Y \in l_1$ we get

$$\log r_{11} = \log r_{1-1} \quad \text{and} \quad \log r_{-11} = \log r_{-1-1}.$$

For $Y \in l_2$ we have by (3)

$$\frac{\partial G(X, Y)}{\partial y_1}\Big|_{y_1=0}$$
$$= \left[\frac{2x_1}{x_1^2+(y_2-x_2)^2} - \frac{2x_1}{x_1^2+(y_2-x_2)^2} + \frac{2x_1}{x_1^2+(y_2+x_2)^2} - \frac{2x_1}{x_1^2+(y_2+x_2)^2}\right] = 0.$$

2. Using some lemmas dealing with convergence of certain integrals we shall prove that the function

$$(4) \quad u(x_1, x_2) = \frac{1}{2\pi}\int_0^\infty f_1(y_1)\frac{\partial G}{\partial y_2}\Big|_{y_2=0} dy_1 - \frac{1}{2\pi}\int_0^\infty f_2(y_2)G(X, Y)\Big|_{y_1=0} dy_2,$$

where $G(X, Y)$ is defined by (3) solves in D the Laplace equation and satisfies the mixed boundary conditions

$$(5) \qquad \lim u(x_1, x_2) = f_1(x_1^0) \quad \text{as } (x_1, x_2) \to (x_1^0, 0),\ x_1^0 > 0,$$

$$(6) \qquad \lim \frac{\partial u}{\partial x_1} = f_2(x_2^0) \quad \text{as } (x_1, x_2) \to (0, x_2^0),\ x_2^0 > 0.$$

Indeed

$$\frac{\partial G}{\partial y_2}\Big|_{y_2=0} = 2x_2\{[(y_1-x_1)^2+x_2^2]^{-1} + [(y_1+x_1)^2+x_2^2]^{-1}\}.$$

Hence by (3) the function $u(x_1, x_2)$ defined by (4) takes the form

$$(4a) \qquad u(x_1, x_2) = \sum_{i=1}^{4} I_i(X),$$

where

$$I_1(X) = \frac{x_2}{\pi}\int_0^\infty f_1(y_1)[(y_1-x_1)^2+x_2^2]^{-1}dy_1,$$

$$I_2(X) = \frac{x_2}{\pi}\int_0^\infty f_1(y_1)[(y_1+x_1)^2+x_2^2]^{-1}dy_1,$$

$$I_3(X) = \frac{1}{2\pi}\int_0^\infty f_2(y_2)\log[x_1^2+(y_2-x_2)^2]dy_2,$$

$$I_4(X) = \frac{-1}{2\pi}\int_0^\infty f_2(y_2)\log[x_1^2+(y_2+x_2)^2]dy_2.$$

Let $a > 0$ and let

$$W_1(a, A, B) = \langle -B, B\rangle \times \langle a, A\rangle,$$
$$W_2(a, A, B) = \langle a, A\rangle \times \langle -B, B\rangle.$$

LEMMA 1. *Let the function $f_1(y_1)$ be Lebesgue integrable and let for every rectangle $W_1(a, A, B)$ exist a constant $N(W_1) > 0$ such that the integral*

$$\int_N^\infty |f_1(y_1)|\cdot y_1^{\frac{1}{2}} dy_1$$

converges; then the integrals

$$I_1^s(X) = (\pi)^{-1} x_2 \int_0^\infty f_1(y_1) \frac{\partial^k}{\partial x_i^k} [(y_1 - x_1)^2 + x_2^2]^{-1} dy_1$$

and

$$I_2^s(X) = (\pi)^{-1} x_2 \int_0^\infty f_1(y_1) \frac{\partial^k}{\partial x_i^k} [(y_1 + x_1)^2 + x_2^2]^{-1} dy_1,$$

where $k = 0, 1, 2$; $i = 1, 2$; $s = 0, 1, \ldots, 7$ exist and are almost uniformly convergent in the half-plane $-\infty < x_1 < +\infty, x_2 > 0$.

Proof. The proof will be carried only for $I_1^s(X)$, $k = 0$, $i = 1$, $s = 0$; the proof in remaing case is similar because those integrals admit an analogous majorant.

Let $d_1^2 = (y_1 - x_1)^2 + x_2^2$. By the inequality of the triangle: $\frac{1}{2} y_1 < d_1 < 2y_1$. For $y_1 > N(W_1)$ and for each $(x_1, x_2) \in W_1(a, A, B)$. Hence

$$\int_N^\infty |f_1(y_1)| x_2 [(y_1 - x_1)^2 + x_2^2]^{-1} dy_1$$
$$\leqslant \int_N^\infty |f_1(y_1)| x_2 [(y_1 - x_1)^2 + x_2^2]^{-\frac{1}{2}} [(y_1 - x_1)^2 + x_2^2]^{-\frac{1}{2}} dy_1$$
$$\leqslant \sqrt{2} \int_N^\infty |f_1(y_1)| y_1^{-\frac{1}{2}} dy_1.$$

LEMMA 2. *Let $f_2(y_2)$ be Lebesgue integrable and let for each rectangle $W_2(a, A, B)$ exist a constant $N(W_2)$ such that the integral*

$$\int_N^\infty |f_2(y_2)| \log y_2 \, dy_2$$

converges. Then the integrals

$$I_3^p(X) = \frac{1}{2\pi} \int_0^\infty f_2(y_2) \frac{\partial^k}{\partial x_i^k} \{\log [x_1^2 + (y_2 + x_2)^2]\} dy_2,$$

$$I_4^p(X) = -\frac{1}{2\pi} \int_0^\infty f_2(y_2) \frac{\partial^k}{\partial x_i^k} \{\log [x_1^2 + (y_2 - x_2)^2]\} dy_2,$$

where $k = 0, 1, 2$; $i = 1, 2$; $p = 0, 1, \ldots, 7$ *exist and are almost uniformly convergent in the half-plane* $x_1 > 0$, $-\infty < x_2 < +\infty$.

Proof. We will prove thesis for the integrals $I_3^1(X)$ and $I_4^1(X)$ for $i = 1$; $k = 0$; $p = 1$.

Let $d^2 = x_1^2 + (y_2 - x_2)^2$. By the triangle inequality $\frac{1}{2} y_2 < d < 2y_2$ for $y_2 > N(W_2)$ and for each $(x_1, x_2) \in W_2(a, A, B)$.

Hence

$$I_4^N(X) = \int_N^\infty |f_2(y_2)| \log[x_1^2 + (y_2 - x_2)^2] dy_2$$
$$\leqslant 2 \int_N^\infty |f_2(y_2)| \log y_2 dy_2 \quad \text{for } N(W_2).$$

For the integral $I_3^1(X)$ we obtain

$$I_3^N(X) = 2 \int_N^\infty |f_2(y_2)| x_1 [x_1^2 + (y_2 + x_2)^2]^{-1} dy_2$$
$$\leqslant 2 \int_N^\infty |f_2(y_2)| x_1 [x_1^2 + y_2^2]^{-1} dy_2 < 2 \int_N^\infty |f_2(y_2)| \log y_2 dy_2$$

for $N(W_2)$ sufficiently large and for each $(x_1, x_2) \in W_2(a, A, B)$. This majorant applies also to the remaining integrals.

THEOREM 2. *Under the assumptions of Lemmas* 1 *and* 2 *the function* $u(x_1, x_2)$ *defined by* (4) *or* (4a) *is harmonic in* D.

Proof. By Lemmas 1 and 2 the integrals $I_1^s(X)$, $I_2^s(X)$, $I_3^p(X)$, $I_4^p(X)$ converge almost uniformly, whence we can interchange the integration with the differentiation.

Therefore

$$\Delta u(x_1, x_2) = \frac{x_2}{\pi} \int_0^\infty f_1(y_1) f_1(y_1) \Delta [(y_1 \pm x_2)^2 + x_2^2]^{-1} dy_1 \pm$$
$$\pm \frac{x_2}{\pi} \int_0^\infty f_2(y_2) \Delta \{\log[x_1^2 + (y_2 \pm x_2)^2]\} dy_2 = 0.$$

The Laplacjan Δ under the sign of the integral vanishes, since the involved function is harmonic.

We shall prove now that the boundary conditions (5) and (6) are satisfied.

LEMMA 3. *Let the function* $f_1(y_1)$ *satisfy the assumptions of Lemma* 1 *and let it be continuous at* x_1^0, *moreover, for each* $\delta > 0$ *let*

$$\int_{|y_1 - x_1^0| > \delta} x_2 |f_1(y_1) - f_1(x_1^0)| [(y_1 - x_1)^2 + x_2^2]^{-1} dy_1 \to 0$$

as $x_1 \to x_1^0$, $x_2 \to 0$; *then*

$$I_1(X) \to f_1(x_1^0) \quad \textit{as } (x_1, x_2) \to (x_1^0, 0).$$

Proof. Let

$$\bar{f}_1(y_1) = \begin{cases} 0 & \text{for } y_1 < 0, \\ f_1(y_1) & \text{for } y_1 \geqslant 0. \end{cases}$$

Hence

$$I_1(X) = \frac{x_2}{\pi} \int_{-\infty}^{+\infty} \bar{f}_1(y_1)[(y_1 - x_1)^2 + x\ ]^{-1} dy_1.$$

It is known ([1], p. 270) that

$$\frac{x_2}{\pi} \int_{-\infty}^{+\infty} [(y_1 - x_1)^2 + x_2^2]^{-1} dy_1 = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{ds}{1+s^2} \equiv 1. \tag{8}$$

Whence

$$\frac{x_2}{\pi} \int_{-\infty}^{+\infty} f_1(x_1^0)[(y_1 - x_1)^2 + x_2^2]^{-1} dy_1 = f_1(x_1^0). \tag{9}$$

Let us write $I_1(X)$ as

$$I_1(X) = J(X) + \frac{x_2}{\pi} \int_{-\infty}^{+\infty} f_1(x_1^0)[(y_1 - x_1)^2 + x_2^2]^{-1} dy_1,$$

where

$$J(X) = \frac{x_2}{\pi} \int_{-\infty}^{+\infty} [\bar{f}_1(y_1) - f_1(x_1^0)][(y_1 - x_1)^2 + x_2^2]^{-1} dy_1.$$

By (8) and (9)

$$I_1(X) - f_1(x_1^0) = J(X).$$

Let $\varepsilon > 0$; continuity of f_1 at x_1^0 implies existence of $\delta > 0$ such that $|y_1 - x_1^0| < \delta$ implies

$$|\bar{f}_1(y_1) - f_1(x_1^0)| < \varepsilon/2.$$

Now

$$J(X) = J_1(X) + J_2(X),$$

where

$$J_1(X) = \frac{x_2}{\pi} \int_{|y_1 - x_1^0| < \delta} [\bar{f}_1(y_1) - f_1(x_1^0)][(y_1 - x_1)^2 + x_2^2]^{-1} dy_1$$

and

$$J_2(X) = \frac{x_2}{\pi} \int_{|y_1 - x_1^0| > \delta} [\bar{f}_1(y_1) - f_1(x_1^0)][(y_1 - x_1)^2 + x_2^2]^{-1} dy_1.$$

Thus applying (9) we obtain

$$J_1(X) \leqslant \frac{x_2}{\pi} \int\limits_{|y_1-x_1^0|<\delta} |\bar{f}_1(y_1)-f_1(x_1^0)|\,[(y_1-x_1)^2+x_2^2]^{-1}dy_1$$

$$\leqslant \frac{\varepsilon}{2}\frac{x_2}{\pi}\int\limits_{-\infty}^{+\infty}[(y_1-x_1)^2+x_2^2]^{-1}dy_1 = \frac{\varepsilon}{2}.$$

It follows that $J_1(X)\to 0$ as $x_1\to x_1^0$, $x_2\to 0^+$.

By our assumption x_1 being sufficiently close to x_1^0 and x_2 being sufficiently close to zero, we have

$$|J_2(X)| \leqslant \varepsilon/2,$$

whence

$$|J| < \varepsilon.$$

Lemma 4. *Let the functions $f_1(y_1)$ and $f_2(y_2)$ satisfy the assumptions of Lemmas* 1, 2, 3; *then*

(a) $$I_2(X)\to 0 \quad \textit{as } x_2\to 0^+,\ x_2>0,$$

(b) $$[I_3(X)+I_4(X)]\to 0 \quad \textit{as } x_2\to 0^+,\ x_2\to 0.$$

Proof. ad (a) The integral $I_2(X)$ satisfies the inequality

$$I_2(X)\leqslant \frac{x_2}{\pi}\int\limits_0^{\infty}|f_1(y_1)|(y_1^2+x_1^2+x_2^2)^{-1}dy_1$$

$$\leqslant \frac{x_2}{\pi}\int\limits_0^{\infty}|f_1(y_1)|(x_1^2+y_1^2)^{-1}dy_1.$$

The last integral is independent of x_2.

ad (b) By Lemma 2 the integrals $I_3(X)$ and $I_4(X)$ converge uniformly for $x_1\geqslant a>0$, $-\infty<x_2<+\infty$, a being an arbitrary positive number, whence those integrals are continuous functions and for $x_2=0$ we have

$$I_3(X)+I_4(X) = \frac{1}{2\pi}\int\limits_0^{\infty} f_2(y_2)\,[\log(x_1^2+y_2^2)-\log(x_1^2+y_2^2)] = 0.$$

Lemmas 3 and 4 imply that the boundary condition is satisfied. Let us prove now the boundary condition (6). By Lemmas 1 and 2

$$\frac{\partial u}{\partial x_1} = \sum_{i=1}^{4} S_i(X),$$

where

$$S_1(X) = \frac{\partial I_1(X)}{\partial x_1} = \frac{2x_2}{\pi} \int_0^\infty f_1(y_1)(y_1 - x_1)[(y_1 - x_1)^2 + x_2^2]^{-2} dy_1,$$

$$S_3(X) = \frac{\partial I_3(X)}{\partial x_1} = \frac{x_1}{\pi} \int_0^\infty f_2(y_2)[x_1^2 + (y_2 - x_2)^2]^{-1} dy_2,$$

$$S_2(X) = \frac{\partial I_2(X)}{\partial x_1} = \frac{-2x_2}{x} \int_0^\infty f_1(y_1)(y_1 + x_1)[(y_1 + x_1)^2 + x_2^2]^{-2} dy_1,$$

$$S_4(X) = \frac{\partial I_4(X)}{\partial x_1} = \frac{-x_1}{\pi} \int_0^\infty f_2(y_2)[x_1^2 + (y_2 + x_2)^2]^{-1} dy_2.$$

LEMMA 5. *Let the function $f_1(y_1)$ satisfy the assumption of Lemma 1, let $f_2(y_2)$ be continuous at x_0 and satisfy the assumptions of Lemma 2, and let for every $\delta > 0$*

$$\int_{|y_2 - x_2^0| > \delta} x_1 |f_2(y_2) - f_2(x_2^0)| [x_1^2 + (y_2 - x_2)^2]^{-1} dy_2 \to 0$$

as $x_2 \to x_2^0$, $x_1 \to 0$, $x_1 > 0$; then

(c) $$S_3(X) \to f_2(x_2^0) \quad \text{as } (x_1, x_2) \to (0, x_2^0);$$

(d) $$S_4(X) \to 0 \quad \text{as } (x_1, x_2) \to (0, x_2^0);$$

(e) $$[S_1(X) + S_2(X)] \to 0 \quad \text{as } (x_1, x_2) \to (0, x_2^0).$$

Proof of (c) is similar to this of Lemma 3, and this of (d) and (e) to this of Lemma 4.

Lemma 5 implies the boundary condition (6). Lemmas 1–5 and Theorem 2 imply .

THEOREM 3. *The assumptions of Lemmas 1–4 being supposed, the function defined by (4) or (4a) forms the solution of the Laplace equation in the quadrant $x_1 > 0$, $x_2 > 0$ with boundary conditions (5) and (6).*

References

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[2] J. Musiałek, *Funkcja Greena oraz rozwiązanie zagadnienia Neumanna i Dirichleta*, Zeszyty Naukowe AGH, Kraków 1970.