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On some extremal problem for certain classes of close-to-starlike functions

1. Let $\wp$ denote the class of Carathéodory functions, i. e. of functions

$$P(z) = 1 + b_1 z + \ldots \tag{1.1}$$

regular in the disc $K = \{z\colon |z| < 1\}$ and satisfying the condition $\operatorname{re} P(z) > 0$ for every $z \in K$.

Next, for any arbitrary fixed number a, $a \in [0, 1)$, denote by $\wp(a)$ its subclass of functions of order a, i. e. of functions $P(z)$ such that $\operatorname{re} P(z) > a$ for $z \in K$.

Moreover, for $a \in (0, 1]$ let $\wp\big((a)\big)$ be the class of all functions (1.1) regular in K and satisfying the condition

$$|P(z) - 1| < a|P(z) + 1| \tag{1.2}$$

for every $z \in K$.

Evidently $\wp\big((a)\big) \subseteq \wp$ and $\wp\big((1)\big) \equiv \wp$. It can easily be verified that if $P(z) \in \wp\big((a)\big)$, then the values of $P(z)$ for every $z \in K$ belong to the disc with the centre $\tilde{c}$ and the radius $\tilde{\varrho}$, where

$$\tilde{c} = \frac{1+a^2}{1-a^2}, \quad \tilde{\varrho} = \frac{2a}{1-a^2}.$$

DEFINITION 1. A function $P(z)$ regular in K and satisfying (1.1) is *of type* a, $a \in (0, 1]$, if and only if satisfy (1.2) for all z in K.

Denote by $\tilde{S}$ the family of functions

$$f(z) = z + a_2 z^2 + \ldots \tag{1.3}$$

regular in K and by $S^*(a)$, $a \in [0, 1)$, its subclass of functions starlike of order a in the disc K, i. e. of all functions of form (1.3) such that $\operatorname{re}\dfrac{zf'(z)}{f(z)} > a$ for every $z \in K$.

It is known that any function $f(z)$ of $\tilde{S}$ belongs to $S^*(a)$ if and only if $\dfrac{zf'(z)}{f(z)} \in \wp(a)$.

Next, denote by $S^*((\alpha))$ the subclass of functions of $\tilde{S}$ such that

$$\frac{zf'(z)}{f(z)} = P(z) \tag{1.4}$$

for some $P(z)$ in $\wp((\alpha))$ and all z in K.

Evidently $S^*((\alpha))$ represents a subclass of S^*, where S^* is the class of all functions (1.3) starlike in K.

Definition 2. A function $f(z)$ regular in K and satysfying (1.3) is *starlike of type* α, $\alpha \in (0, 1]$, if and only if it satisfies (1.4) for some $P(z)$ in $\wp$ of type α.

Now we consider the following four clasess of functions (1.3)

$$S(\alpha, \beta), \quad S(\alpha, (\beta)), \quad S((\alpha), \beta), \quad S((\alpha), (\beta)) \tag{1.5}$$

defined as follows:

Definition 3. Let $f(z) \in \tilde{S}$ and let

$$f(z) = F(z) \cdot P(z) \tag{1.6}$$

for some $F(z)$ in S^* and $P(z)$ in $\wp$. Then

1° $f(z) \in S(\alpha, \beta)$ if and only if $F(z)$ is of order α and $P(z)$ is of order β,

2° $f(z) \in S(\alpha, (\beta))$ if and only if $F(z)$ is of order α and $P(z)$ is of type β,

3° $f(z) \in S((\alpha), \beta)$ if and only if $F(z)$ is of type α and $P(z)$ is of order β,

4° $f(z) \in S((\alpha), (\beta))$ if and only if $F(z)$ is of type α and $P(z)$ is of type β.

Thus families (1.5) represent a subclasses of the family $\hat{S}$ of all functions (1.3) close-to-starlike in K introduced by Read [2], i.e. of functions satisfying (1.6) for some $F(z)$ in S^* and $P(z)$ in $\wp$. Evidently

$$S(0, 0) \equiv S(0, (1)) \equiv S((1), 0) \equiv S((1), (1)) \equiv \hat{S}.$$

Every function of each family (1.5) is starlike and hence univalent in a neighborhood of the origin. In this paper we determine the exact value of the radius starlikeness for every family (1.5) of close-to-starlike functions.

2. We denote at present the arbitrary family (1.5) by $S(\{\alpha\}, \{\beta\})$ and the corresponding families of starlike functions and of Carathéodory functions by $S^*(\{\alpha\})$ and $\wp(\{\beta\})$, respectively, where $\{\alpha\}$ designes α or (α) and $\{\beta\}$ designes β or (β).

Let T be an arbitrary subclass of $\tilde{S}$. If f is in T, then r. s. $\{f\}$, the radius of starlikeness of f is

$$\text{r. s. } \{f\} = \sup\left[r\colon \operatorname{re}\frac{zf'(z)}{f(z)} > 0,\ |z| < r\right]$$

and r. s. T, the radius of starlikeness of T, is

$$\text{r. s. } T = \inf_{f\in T}[\text{r. s. } \{f\}].$$

If T is compact, then problem of finding r. s. T is reduced to finding the greatest value of r, $0 < r \leqslant 1$, for which

$$\operatorname{re}\frac{zf'(z)}{f(z)} \geqslant 0$$

for every $|z| \leqslant r$ and every function $f(z) \in T$. It is easy to prove that every family $S(\{a\}, \{\beta\})$ is compact, hence r. s. $S(\{a\}, \{\beta\})$ equals the smallest root r_0, $0 < r_0 \leqslant 1$ of the equation $w(r) = 0$, where

$$w(r) = \min\left\{\operatorname{re}\frac{zf'(z)}{f(z)}\colon |z| = r < 1, f \in S(\{a\}, \{\beta\})\right\}. \tag{2.1}$$

Let $f(z) \in S(\{a\}, \{\beta\})$. Then in view of Definition 3

$$f(z) = F(z)\cdot P(z) \tag{2.2}$$

for some functions $F(z) \in S^*(\{a\})$ and $P(z) \in \wp(\{\beta\})$.

Differentiating (2.2) we obtain

$$\frac{zf'(z)}{f(z)} = \frac{zF'(z)}{F(z)} + \frac{zP'(z)}{P(z)}.$$

Thus, on $|z| = r$, $0 < r < 1$, we have

(2.3)

$$\operatorname{re}\frac{zf'(z)}{f(z)} \geqslant \min\left\{\operatorname{re}\frac{zF'(z)}{F(z)}\colon F \in S^*(\{a\})\right\} + \min\left\{\operatorname{re}\frac{zP'(z)}{P(z)}\colon P \in \wp(\{\beta\})\right\}.$$

Next, for arbitrary fixed numbers A, B, $-1 < A \leqslant 1$, $-1 \leqslant B < A$, denote by $\wp(A, B)$ the subclass of $\wp$ which contains all functions $P(z)$ such that $P(z)$ is in $\wp(A, B)$ if and only if

$$P(z) = \frac{1 + A\cdot\vartheta(z)}{1 + B\cdot\vartheta(z)}$$

for some function $\vartheta(z)$ regular in K, satisfying the conditions $\vartheta(0) = 0$, $|\vartheta(z)| < 1$ for $z \in K$.

The family $\wp(A, B)$ has been introduced by Janowski [1].

It is easy to prove that

$$\wp(a)' \equiv \wp(1-2a, -1) \tag{2.4}$$

and

$$\wp\big((a)\big) \equiv \wp(a, -a). \tag{2.5}$$

According to Theorem 3 of [1], for all $P(z)$ in $\wp(A, B)$ and $|z| = r$, $0 < r < 1$,

$$\operatorname{re}\frac{zP'(z)}{P(z)} \geqslant \begin{cases} Y_1(r;\ A, B) & \text{for } 0 < r \leqslant r^*, \\ Y_2(r;\ A, B) & \text{for } r^* < r < 1, \end{cases} \tag{2.6}$$

where

$$Y_1(r;\ A, B) = -\frac{(A-B)r}{(1-Ar)(1-Br)}, \tag{2.7}$$

$$Y_2(r;\ A, B) = 2\frac{\sqrt{\mathfrak{A}\mathfrak{B}}-(1-ABr^2)}{(A-B)(1-r^2)} + \frac{A+B}{A-B}, \tag{2.8}$$

$$\mathfrak{A} = \mathfrak{A}(r;\ A, B) = (1-B)(1+Br^2), \quad \mathfrak{B} = (1-A)(1+Ar^2) \tag{2.9}$$

and

$$r^* = r^*(A, B) \tag{2.10}$$

is the unique root of the polynomial

$$g(r;\ A, B) = -ABr^4+2ABr^3-(2A+2B-AB-1)r^2+2r-1 \tag{2.11}$$

in the interval $(0, 1]$.

The bounds (2.6) are sharp, being attained at the point $z = \varepsilon\cdot r$, $|\varepsilon| = 1$, by

$$P^*(z;\ A, B) = \frac{1-A\bar{\varepsilon}z}{1-B\bar{\varepsilon}z} \quad \text{for } 0 < r \leqslant r^* \tag{2.12}$$

and by

$$P^{**}(z;\ A, B) = \frac{1-(1-A)d\bar{\varepsilon}z-A\bar{\varepsilon}^2z^2}{1-(1-B)d\bar{\varepsilon}z-B\bar{\varepsilon}^2z^2} \quad \text{for } r^* < r < 1 \tag{2.13}$$

respectively, where

$$d = d(r;\ A, B) = \frac{1}{r}\,\frac{(1-Br^2)s-(1-Ar^2)}{(1-B)s-(1-A)}, \quad s = \sqrt{\mathfrak{B}\mathfrak{A}^{-1}}. \tag{2.14}$$

Let $S^*(A, B)$ denote the subclass of $\tilde{S}$ which contains all functions $F(z)$ such that $f(z)$ is in $S^*(A, B)$ if and only if

$$\frac{zF'(z)}{F(z)} = P(z) \tag{2.15}$$

for some $P(z)$ in $\wp(A, B)$ and all z in $K[1]$. Evidently,

$$S^*(\alpha) = S^*(1-2\alpha,\ -1) \tag{2.16}$$

and

$$S^*((\alpha)) = S^*(\alpha,\ -\alpha). \tag{2.17}$$

It is easy to verify that if $P(z) \in \wp(A, B)$, then

$$P(z) = \frac{(1+A)p(z)+1-A}{(1+B)p(z)+1-B} \tag{2.18}$$

for some $p(z)$ in $\wp$ and all z in K. Since every function $p(z)$ in $\wp$ is subordinate to

$$p_0(z) = \frac{1+z}{1-z},$$

i. e., $p(z) \prec p_0(z)$, it follows that $P(z) \prec P_0(z)$, where $P(z)$ is given by (2.18) and

$$P_0(z) = \frac{1+Az}{1+Bz}.$$

Therefore, the images of the closed disc $|z| \leqslant r$ under all the transformation $w = P(z), P(z) \in \wp(A, B)$, are contained in the closed disc with the centre c and the radius ϱ, where

$$c = \frac{1-ABr^2}{1-B^2r^2}, \quad \varrho = \frac{(A-B)r}{1-B^2r^2}.$$

With these considerations we find that for all $F(z)$ in $S^*(A, B)$ and $|z| = r, 0 < r < 1$

$$\operatorname{re}\frac{zF'(z)}{F(z)} \geqslant \frac{1-Ar}{1-Br} \tag{2.19}$$

with equality holding in $z = \varepsilon r$, $|\varepsilon| = 1$, for

$$F^*(z;\ A, B) = \begin{cases} (1-B\bar{\varepsilon}z)^{(A-B)/B} & \text{if } B \neq 0, \\ \exp(A\bar{\varepsilon}z) & \text{if } B = 0. \end{cases} \tag{2.20}$$

3. THEOREM 1. *Let*

$$D_1 = \{(\alpha, \beta) : 0 \leqslant \alpha < 1, 0 \leqslant \beta \leqslant \beta(\alpha)\},$$

$$D_2 = \{(\alpha, \beta) : 0 \leqslant \alpha < 1, \beta(\alpha) < \beta < 1\},$$

where

$$\beta(\alpha) = (L+\sqrt{L^2+4M})^{-1}, \tag{3.1}$$

$$L = L(\alpha) = 2\alpha^3-3\alpha^2+6\alpha+1, \qquad M = M(\alpha) = 1-6\alpha. \tag{3.2}$$

Then, the radius of starlikeness for the family $S(\alpha, \beta)$,

$$\text{r. s. } S(\alpha, \beta) = \begin{cases} r_1 & \textit{if } (\alpha, \beta) \in D_1, \\ r_2 & \textit{if } (\alpha, \beta) \in D_2, \end{cases}$$

where

$$r_1 = (2-\alpha-2\beta+\sqrt{\alpha^2+4\beta^2-2\alpha-6\beta+3})^{-1}, \tag{3.3}$$

r_2 is the smallest root of the polynomial

$$v(r) = [(a^2+2a-1)\beta-a^2]r^4+2(1-a)[(1+a)\beta-a]r^3+ \\ +[(a^2-4a+1)\beta-(1-a)^2]r^2-2(1-a)\beta r+\beta \tag{3.4}$$

in the interval $(r^*, 1)$ *and* $r^* = r^*(1-2\beta, -1)$ (cf. (2.10)–(2.11)).

Equality r. s. $\{f\}$ = r. s. $S(a, \beta)$ *holds for function* (2.2), *where*

$$F(z) = F^*(z;\ 1-2a,\ -1), \tag{3.5}$$

$$P(z) = \begin{cases} P^*(z;\ 1-2\beta,\ -1) & \textit{for } (a, \beta)\in D_1, \\ P^{**}(z;\ 1-2\beta,\ -1) & \textit{for } (a, \beta)\in D_2 \end{cases}$$

and $d = d(r_2;\ 1-2\beta, -1)$ (cf. (2.12)–(2.16) and (2.19)–(2.20)); re $\dfrac{zf'(z)}{f(z)} = 0$ *for* $z = \bar{\varepsilon} r_0$, *where* $r_0 = r_1$ *or* $r_0 = r_2$, *respectively.*

Proof. a. In view of (2.1)–(2.4), (2.6)–(2.9) and (2.19) we obtain

$$w(r) = \begin{cases} u(r)/u_1(r) & \text{for } 0 < r \leqslant r^*, \\ v(r)/v_1(r) & \text{for } r^* < r < 1, \end{cases} \tag{3.6}$$

where

$$u(r) = (1-2a)(1-2\beta)r^2-2(2-a-2\beta)r+1,$$

$v(r)$ is given by (3.4), $u_1(r) > 0$ and $v_1(r) > 0$ for $0 < r < 1$.

Let

$$G_1 = \{(a, \beta)\colon\ 0 \leqslant a < 1,\ 0 \leqslant \beta \leqslant \beta_1(a)\},$$
$$G_2 = \{(a, \beta)\colon\ 0 \leqslant a < 1,\ \beta_1(a) < \beta < 1\},$$

where

$$\beta_1(a) = \frac{1}{1+2a} \quad \text{for } 0 \leqslant a < 1.$$

If $(a, \beta)\in G_2$, then $u(r) > 0$ for $0 < r < 1$. If $(a, \beta)\in G_1$, then $u(r)$ has exactly one root r_1, given by (3.3), in the interval $(0, 1)$ and $u(r) > 0$ for $0 < r < r_1$.

The polynomial $v(r)$ has one or three roots in the interval $(0, 1)$ for every $a\in[0, 1)$ and $\beta\in(0, 1)$. Let: 1° $(a, \beta)\in G_2$ or 2° $(a, \beta)\in G_1$ and $r_1 \geqslant r^*$. Therefore $u(r^*) > 0$. Hence, because of $u_1(r) > 0$, $v_1(r) > 0$ for $0 < r < 1$ and in view of (3.6) $u(r^*)$ and $v(r^*)$ must have the same sign. Thus $v(r^*) > 0$ and consequently $v(r)$ has at least one root in the interval $(r^*, 1)$. Denote by r_2 the smallest root of $v(r)$ in this interval.

Hence

$$\text{r. s. } S(a, \beta) = \begin{cases} r_1 & \text{if } (a, \beta)\in G_1 \text{ and } r_1 \leqslant r^*, \\ r_2 & \text{if } (a, \beta)\in G_1 \text{ and } r_1 > r^* \text{ or } (a, \beta)\in G_2. \end{cases}$$

b. We shall prove that $r_1 \leqslant r^*$ if and only if $0 \leqslant \beta \leqslant \beta(a)$ for every $0 \leqslant a < 1$, where $\beta(a)$ is given by (3.1)–(3.2).

To this aim we observe first that the function

$$g(r) = g(r;\ 1-2\beta,\ -1) = (1+r)[(1-2\beta)r^3 - 3(1-2\beta)r^2 + 3r - 1] \tag{3.7}$$

increases for $0 < r < 1$, thus $r_1 \leqslant r^*$ if and only if $g(r_1) \leqslant 0$.

If $a = \frac{1}{2}$ and $0 \leqslant \beta < \frac{1}{2}$, then the polynomial $u(r)$ has the root r_1 in the interval $(0, 1)$ and $r_1 = (3-4\beta)^{-1}$. Thus,

$$g(r_1) = (1+r_1)r_1^3 \cdot (\beta-1)(8\beta^2 - 7\beta + 1) < 0 \qquad \text{for } \beta < \beta(\tfrac{1}{2})$$

and in this case $r_1 < r^*$. Conversely, if $a = \frac{1}{2}$ and $r_1 < r^*$, then $\beta < \beta(\frac{1}{2})$.

Let $(a, \beta) \in G_1$ and $a \neq \frac{1}{2}$. Since $u(r_1) = 0$,

$$(1-2a)\frac{g(r_1)}{r_1} = (1-2a)\frac{g(r_1)}{r_1} + (1-2a-r_1)\frac{u(r_1)}{r_1} = \mathfrak{a}r_1 - \mathfrak{b}, \tag{3.8}$$

where

$$\mathfrak{a} = 2[1+2a^2 - 2a\beta(1+2a)], \qquad \mathfrak{b} = 2[1-2a+2a^2 - 2(1-2a)\beta]. \tag{3.9}$$

Since $(a, \beta) \in G_1$, then $\mathfrak{a} > 0$.

Let

$$G_3 = \{(a, \beta):\ 0 \leqslant a < a_0,\ \beta_2(a) \leqslant \beta < \beta_1(a)\},$$
$$G_4 = \{(a, \beta):\ \tfrac{1}{2} \leqslant a < 1,\ \ \beta_3(a) \leqslant \beta < \beta_1(a)\},$$

where a_0 is the unique root of the polynomial $4a^3 - 2a^2 + 4a - 1$ in the interval $(0, \frac{1}{2})$ and

$$\beta_2(a) = \frac{2a^2 - 2a + 1}{2(1-2a)}, \qquad \beta_3(a) = \frac{a}{2a^2 + 3a - 1}.$$

It is easy to verify that $\mathfrak{b} \leqslant 0$ for $(a, \beta) \in G_3$, $0 < \mathfrak{b} < \mathfrak{a}$ for $(a, \beta) \in G_1 - (G_3 \cup G_4)$ and $\mathfrak{b} \geqslant \mathfrak{a}$ for $(a, \beta) \in G_4$. Hence, because of (3.8) and in view of $\mathfrak{a} > 0$, we obtain that $g(r_1) > 0$ for $(a, \beta) \in G_3 \cup G_4$. If $(a, \beta) \in G_1 - (G_3 \cup G_4)$, then

$$(1-2a)\frac{g(r_1)}{r_1} = \mathfrak{a}(r_1 - r_0), \tag{3.10}$$

where $r_0 = \mathfrak{b}/\mathfrak{a}$ and $0 < r_0 < 1$. From (3.10) we obtain that $g(r_1) \leqslant 0$ if $r_1 \leqslant r_0$ and $a < \frac{1}{2}$, or if $r_1 \geqslant r_0$ and $a > \frac{1}{2}$. Since $u(r)$ decreases for $0 < r < 1$ we conclude that the condition $g(r_1) \leqslant 0$ implies

$$(2a-1)u(r_0) \geqslant 0$$

for $0 \leqslant a < 1$. We have

$$\mathfrak{a}^2 u(r_0) = (1-2a)(1-2\beta)\mathfrak{b}^2 - 2(2-a-2\beta)\mathfrak{a}\mathfrak{b} + \mathfrak{a}^2,$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are given by (3.9).

After some calculation we get

$$a^2 u(r_0) = 8(2a-1)(\beta-1)(4M\beta^2+2L\beta-1),$$

where L and M are given by (3.2). Therefore, if $g(r_1) \leqslant 0$, then $h(\beta) \leqslant 0$, where

$$h(\beta) = 4M\beta^2+2L\beta-1.$$

If $M \geqslant 0$, then the inequality $h(\beta) \leqslant 0$ is equivalent to

$$(3.11) \qquad 0 \leqslant \beta \leqslant \beta(a).$$

If $M < 0$, then, because of $h(\frac{1}{2}) > 0$ for $0 \leqslant a < 1$, the polynomial $h(\beta)$ has likewise the real root $-\beta(a)$ and $\beta^*(a)$, where

$$\beta^*(a) = \frac{1}{L-\sqrt{L^2+4M}}.$$

Since $\beta^*(a) > \beta_1(a)$, then the inequality $h(\beta) \leqslant 0$ is also equivalent to (3.11) in the case $M < 0$.

Summing, we have proved that the inequality $r_1 \leqslant r^*$ implies (3.11).

c. We shall prove that every point (a, β), where $0 \leqslant a < 1$ and $0 \leqslant \beta < \beta(a)$, belongs to the domain $G_1-(G_3 \cup G_4)$, i. e. that inequality (3.11) implies $r_1 \leqslant r^*$. To this aim we observe first that $\beta(a) < \beta_1(a)$ for every $0 \leqslant a < 1$. Next, we obtain $\beta(a) < \beta_3(a)$ for $\frac{1}{2} \leqslant a < 1$. In fact, if $\beta(a) \geqslant \beta_3(a)$ for some a, then

$$(3.12) \qquad t(a) > a\sqrt{L^2+4M},$$

where $t(a) = -2a^4+3a^3-4a^2+2a-1$.

Since the derivative $t''(a) < 0$ for every a, then $t'(a)$ decreases and because of $t'(\frac{1}{2}) < 0$ we obtain that $t(a)$ decreases in the interval $(0, 1)$. Therefore, in view of $t(\frac{1}{2}) < 0$, we obtain that $t(a) < 0$ for $\frac{1}{2} \leqslant a < 1$ which is impossible because of (3.12).

Let

$$(3.13) \qquad \Delta(a) = L+\sqrt{L^2+4M},$$

where $L = L(a)$ and $M = M(a)$ are given by (3.2). Differentiating (3.13) w. r. t. a we obtain

$$\Delta'(a) = \frac{LL'+2M'+L'\sqrt{L^2+4M}}{\sqrt{L^2+4M}}.$$

Since $L' > 0$, then $\Delta(a)$ increases if $LL'+2M' \geqslant 0$. Let $LL'+2M' < 0$. In this case $\Delta'(a) > 0$ if $\delta(a) > 0$, where

$$\delta(a) = 4a^4-4a^3+5a^2-2a+1.$$

Since the derivative $\delta''(a) > 0$ for every a, then $\delta'(a)$ increases and because of $\delta'(0) < 0$ and $\delta'(1) > 0$ $\delta(a)$ decreases for $0 \leqslant a < a_1$ and increases for $a_1 \leqslant a < 1$, where a_1 is the unique root of $\delta'(a)$ in the interval $(0, 1)$. Since $\delta(a_1)$ may be written in form

$$\delta(a_1) = \delta(a_1) + \frac{1-4a_1}{16}\delta'(a_1) = 2(14a_1^2 - 7a_1 + 7),$$

then $\delta(a) > 0$ if $LL' + 2M' < 0$. Therefore $\Delta(a)$ increases likewise in this case.

Hence, the function $\beta(a)$ decreases for $0 \leqslant a < 1$. Since $\beta_2(a)$ increases for $0 \leqslant a < a_0$ and $\beta(0) < \beta_2(0)$, then $\beta(a) < \beta_2(a)$ for $0 \leqslant a < a_0$ and this completes the proof.

COROLLARY. *For every* $0 \leqslant a < 1$

$$\text{r. s. } S(a, \beta) = \begin{cases} r_1 & \text{if } 0 \leqslant \beta \leqslant \beta_1, \\ r_2 & \text{if } \beta_2 \leqslant \beta < 1, \end{cases}$$

where

$$\beta_1 = \beta(0) = 0,1, \quad \beta_2 = \beta(1) = (1+\sqrt{5})^{-1}. \tag{3.14}$$

4. Let

$$\lambda(r) = \frac{1}{r}\sqrt{\frac{r^2+2r-1}{-r^2+2r+1}} \tag{4.1}$$

and

$$\psi(r) = \frac{\sqrt{-r^4+6r^2-1}-(1-r)^2}{2r(1-r)} \tag{4.2}$$

for $r \in I$, where $I = \{r: \sqrt{2}-1 \leqslant r \leqslant 1\}$. The functions $\lambda(r)$ and $\psi(r)$ are increasing in I and $0 \leqslant \lambda(r) \leqslant 1$, $-2^{-1/2} \leqslant \psi(r) < \infty$ for $r \in I$. If $\psi(r)$ increases from 0 to 1, then the inverse function, w. r. t. $\psi(r)$, increases from the value r' to r'', where

$$r' = \tfrac{1}{2}(1+\sqrt{3}-\sqrt[4]{12}), \quad r'' = \tfrac{1}{2}(\sqrt{6}-\sqrt{2}).$$

Therefore the function

$$\tilde{\beta}(a) = \lambda[\psi^{-1}(a)], \quad 0 \leqslant a < 1 \tag{4.3}$$

increases from $\tilde{\beta}(0) = \beta'$ to $\tilde{\beta}(1) = \beta''$, where

$$\beta' = \lambda(r') = \sqrt{(1+\sqrt{3})(1-\sqrt[4]{0,75})} \tag{4.4}$$

and

$$\beta'' = \lambda(r'') = \tfrac{1}{2}(1+\sqrt{3})(2-\sqrt{2}). \tag{4.5}$$

THEOREM 2. *Let*

$$\tilde{D}_1 = \{(\alpha, \beta):\ 0 \leqslant \alpha < 1,\ \tilde{\beta}(\alpha) \leqslant \beta < 1\},$$
$$\tilde{D}_2 = \{(\alpha, \beta):\ 0 \leqslant \alpha < 1,\ 0 < \beta < \tilde{\beta}(\alpha)\},$$

where $\tilde{\beta}(\alpha)$ *is given by* (4.3).

Then, the radius of starlikeness for the family $S(\alpha, (\beta))$,

$$\text{r. s. } S(\alpha, (\beta)) = \begin{cases} \tilde{r}_1 & \text{if } (\alpha, \beta) \in \tilde{D}_1, \\ \tilde{r}_2 & \text{if } (\alpha, \beta) \in \tilde{D}_2, \end{cases}$$

where $\tilde{r}_1$ *is the unique root of the polynomial*

$$\tilde{u}(r) = \beta^2(1-2\alpha)r^3 - \beta(\beta+2)r^2 - (1-2\alpha+2\beta)r + 1$$

in the interval $(0, 1)$, $\tilde{r}_2$ *is the smallest root of the polynomial*

$$\tilde{v}(r) = -\beta(\beta+2\alpha-1)^2 r^4 - 4\beta(1-\alpha)(\beta+2\alpha-1)r^3 + \\ + [\beta^3+2\beta^2-(3-2\alpha)^2\beta+2(1-2\alpha)]r^2 - 4(1-\alpha)(1-\beta)r + 2(1-\beta)$$

in the interval $(\tilde{r}^*, 1)$ *and* $\tilde{r}^* = \tilde{r}^*(\beta, -\beta)$ (cf. (2.10)–(2.12)).

The equality r. s. $\{f\}$ = r. s. $S(\alpha, (\beta))$ *holds for function* (2.2), *where* $F(z)$ *is given by* (3.5),

$$P(z) = \begin{cases} P^*(z;\ \beta, -\beta) & \text{for } (\alpha, \beta) \in \tilde{D}_1, \\ P^{**}(z;\ \beta, -\beta) & \text{for } (\alpha, \beta) \in \tilde{D}_2, \end{cases}$$

and $d = d(\tilde{r}_2;\ \beta, -\beta)$, (cf. (2.12)–(2.15), (2.17) and (2.19)–(2.20));

$$\operatorname{re}\frac{zf'(z)}{f(z)} = 0 \quad \text{for } z = \bar{\varepsilon} r_0,$$

where $r_0 = \tilde{r}_1$ *or* $r_0 = \tilde{r}_2$, *respectively.*

Proof. In view of (2.1)–(2.3), (2.5)–(2.9) and (2.19) we obtain

$$w(r) = \begin{cases} \tilde{u}(r)/\tilde{u}_1(r) & \text{for } 0 < r \leqslant \tilde{r}^*, \\ \tilde{v}(r)/\tilde{v}_1(r) & \text{for } \tilde{r}^* < r < 1, \end{cases}$$

where $\tilde{u}_1(r) > 0$ and $\tilde{v}_1(r) > 0$ for $0 < r < 1$.

Let

$$\tilde{G}_1 = \{(\alpha, \beta):\ 0 \leqslant \alpha < 1,\ \tilde{\beta}_1(\alpha) \leqslant \beta \leqslant 1\},$$
$$\tilde{G}_2 = \{(\alpha, \beta):\ 0 \leqslant \alpha < 1,\ 0 < \beta < \tilde{\beta}_1(\alpha)\},$$

where

$$\tilde{\beta}_1(\alpha) = \frac{\alpha}{1+\sqrt{1+\alpha^2}} \quad \text{for } 0 \leqslant \alpha < 1.$$

We observe that

$$\tilde{\beta}(\alpha) > \tilde{\beta}_1(\alpha) \quad \text{for } 0 \leqslant \alpha < 1. \tag{4.6}$$

If $(\alpha, \beta) \in \tilde{G}_2$, then $\tilde{u}(r) > 0$ for $0 < r < 1$. If $(\alpha, \beta) \in \tilde{G}_1$, then $\tilde{u}(r)$ has exactly one root in the interval $(0, 1)$ and $u(r) > 0$ for $0 < r < \tilde{r}_1$. The polynomial $v(r)$ hat at least one root in the interval $(\tilde{r}^*, 1)$.

Thus

$$\text{r. s. } S\big(\alpha, (\beta)\big) = \begin{cases} \tilde{r}_1 & \text{if } (\alpha, \beta) \in \tilde{G}_1 \text{ and } \tilde{r}_1 \leqslant \tilde{r}^*, \\ \tilde{r}_2 & \text{if } (\alpha, \beta) \in \tilde{G}_1 \text{ and } \tilde{r}_1 > \tilde{r}^* \\ & \text{or } (\alpha, \beta) \in \tilde{G}_2. \end{cases}$$

Let $(\alpha, \beta) \in \tilde{G}_1$. Since $\tilde{u}(r)$ decreases for $0 < r < 1$; then $\tilde{r}_1 \leqslant \tilde{r}^*$ if and only if $\tilde{u}(\tilde{r}^*) \leqslant 0$.

We have

$$\tilde{g}(r) = g(r;\ \beta, -\beta) = (r^2 + 2r^3 - r^4)[\lambda^2(r) - \beta^2]$$

and

$$\tilde{u}(r) = 2r(1 - \beta^2 r^2)\left[\alpha - \frac{(1+r)\beta}{1 - \beta^2 r^3} + \frac{1-r}{2r}\right]. \tag{4.7}$$

Since $\lambda(\tilde{r}^*) = \beta$, then from (4.7), in view of (4.1) and (4.2), we get

$$\tilde{u}(\tilde{r}^*) = 4\,\frac{(1 - \tilde{r}^{*2})\tilde{r}^*}{1 + 2\tilde{r}^* - \tilde{r}^{*2}}\,[\alpha - \psi(\tilde{r}^*)].$$

If $\beta < \beta'$, then $\psi(\tilde{r}^*) < 0$, for $\beta' \leqslant \beta < \beta''$ we obtain $0 \leqslant \psi(\tilde{r}^*) < 1$ and finally, if $\beta \geqslant \beta''$, then $\psi(\tilde{r}^*) \geqslant 1$. Thus, $\tilde{u}(\tilde{r}^*) \leqslant 0$ if $0 \leqslant \alpha < 1$ and $\beta'' \leqslant \beta \leqslant 1$ or

$$\alpha \leqslant \psi(\tilde{r}^*) \tag{4.8}$$

and $\beta' \leqslant \beta < \beta''$. If $\tilde{r}^*$ satisfy (4.8), then $\tilde{r}^* \geqslant \psi^{-1}(\alpha)$ and consequently $\lambda(\tilde{r}^*) \geqslant \lambda[\psi^{-1}(\alpha)]$, i. e. $\beta \geqslant \tilde{\beta}(\alpha)$.

Therefore, in view of (4.6) we conclude that $\tilde{u}(\tilde{r}^*) \leqslant 0$ if and only if $(\alpha, \beta) \in \tilde{D}_1$, which ends the proof.

Corollary. *For every* $0 \leqslant \alpha < 1$

$$\text{r. s. } S\big(\alpha, (\beta)\big) = \begin{cases} \tilde{r}_1 & \text{if } \beta'' \leqslant \beta \leqslant 1, \\ \tilde{r}_2 & \text{if } 0 < \beta \leqslant \beta'. \end{cases}$$

5. Let

$$\hat{\lambda}(r) = \frac{(1-r)^3}{2r^2(3-r)}$$

and

$$\hat{\psi}(r) = \frac{1-2r}{r^2(2-r)}$$

for $r \in \hat{I}$, where $\hat{I} = \{r\colon 2-\sqrt{3} \leqslant r \leqslant 1\}$. The functions $\hat{\lambda}(r)$ and $\hat{\psi}(r)$ are decreasing in $\hat{I}$ and $0 \leqslant \hat{\lambda}(r) \leqslant 1$, $-1 \leqslant \hat{\psi}(r) \leqslant 2+\sqrt{3}$ for $r \in \hat{I}$. If $w = \hat{\psi}(r)$ decreases from 1 to 0, then $r = \hat{\psi}^{-1}(w)$ increases from $\hat{r}'$ to $\hat{r}''$, where

$$\hat{r}' = \tfrac{1}{2}(3-\sqrt{5}), \qquad \hat{r}'' = \tfrac{1}{2}.$$

Therefore the function

$$\hat{\beta}(a) = \hat{\lambda}[\hat{\psi}^{-1}(a)], \qquad 0 < a \leqslant 1, \tag{5.1}$$

increases from $\hat{\beta}(0) = \beta_1$ to $\hat{\beta}(1) = \beta_2$, where β_1 and β_2 are given by (3.14).

THEOREM 3. *Let*

$$\hat{D}_1 = \{(a, \beta)\colon 0 < a \leqslant 1, 0 \leqslant \beta \leqslant \hat{\beta}(a)\},$$

$$\hat{D}_2 = \{(a, \beta)\colon 0 < a \leqslant 1, \hat{\beta}(a) < \beta < 1\},$$

where $\hat{\beta}(a)$ *is given by* (5.1).

Then

$$\text{r. s. } S\big((a), \beta\big) = \begin{cases} \hat{r}_1 & \textit{if } (a, \beta) \in \hat{D}_1, \\ \hat{r}_2 & \textit{if } (a, \beta) \in \hat{D}_2, \end{cases}$$

where $\hat{r}_1$ *is the unique root of the polynomial*

$$\hat{u}(r) = a(1-2\beta)r^3-(2a+1-2\beta)r^2-(a+2-4\beta)r+1$$

in the interval $(0, 1)$, $\hat{r}_2$ *is the smallest root of*

$$\hat{v}(r) = -a^2\beta r^6+[2(a^2+1)\beta-1]r^4-2a(1-\beta)r^3-(2\beta+a^2)r^2+\beta$$

in the interval $(r^*, 1)$ (cf. Theorem 1).

Equality r. s. $\{f\}$ = r. s. $S\big((a), \beta\big)$ *holds for function* (2.2), *where*

$$F(z) = F^*(z;\ a, -a), \tag{5.2}$$

$$P(z) = \begin{cases} P^*(z;\ 1-2\beta, -1) & \textit{for } (a, \beta) \in \hat{D}_1, \\ P^{**}(z;\ 1-2\beta, -1) & \textit{for } (a, \beta) \in \hat{D}_2, \end{cases}$$

and $d = d(\hat{r}_2, 1-2\beta, -1)$; $\operatorname{re}\dfrac{zf'(z)}{f(z)} = 0$ *for* $z = \bar{\varepsilon}r_0$, *where* $r_0 = \hat{r}_1$ *or* $r_0 = \hat{r}_2$, *respectively.*

Proof. Similary as in part 4 we prove that

$$\text{r. s. } S\big((a), \beta\big) = \begin{cases} \hat{r}_1 & \text{if } (a, \beta) \in \hat{G}_1 \text{ and } \hat{r}_1 \leqslant r^*, \\ \hat{r}_2 & \text{if } (a, \beta) \in \hat{G}_1 \text{ and } \hat{r} > r^* \text{ or } (a, \beta) \in \hat{G}_2, \end{cases}$$

where

$$\hat{G}_1 = \{(a, \beta): 0 < a \leqslant 1, 0 \leqslant \beta \leqslant \hat{\beta}_1(a)\},$$

$$\hat{G}_2 = \{(a, \beta): 0 < a \leqslant 1, \hat{\beta}_1(a) < \beta < 1\}$$

and

$$\hat{\beta}_1(a) = \frac{1+a}{3-a}.$$

Next, we obtain that $\hat{r}_1 \leqslant r^*$ if and only if $\hat{u}(r^*) \leqslant 0$. Since

$$g(r) = 2r^2(3-r)[\beta - \hat{\lambda}(r)]$$

(cf. (3.10)) and

$$\hat{u}(r^*) = \frac{2r^*(1+r^*)(2-r^*)}{3-r^*}[\hat{\psi}(r^*) - a]$$

then, because of $\hat{G}_1 \supset \hat{D}_1$, we conclude finally that the assertion is true.

COROLLARY. *For every* $0 < a \leqslant 1$

$$\text{r. s. } S\big((a), \beta\big) = \begin{cases} \hat{r}_1 & \textit{if } 0 \leqslant \beta \leqslant \beta_1, \\ \hat{r}_2 & \textit{if } \beta_2 \leqslant \beta < 1. \end{cases}$$

6. Similary as in parts 4 and 5 we obtain the following

THEOREM 4. *Let*

$$\hat{\psi}(r) = \frac{1}{r}\frac{1-r^2-\sqrt{-r^4+6r^2-1}}{1-r^2+\sqrt{-r^4+6r^2-1}}, \qquad \sqrt{2}-1 \leqslant r \leqslant 1,$$

$$\hat{\beta}(a) = \lambda[\hat{\psi}^{-1}(a)], \qquad 0 < a \leqslant 1,$$

where $\lambda(r)$ *is given by* (4.1),

$$\hat{D}_1 = \{(a, \beta): 0 < a \leqslant 1, \hat{\beta}(a) \leqslant \beta \leqslant 1\}$$

and

$$\hat{D}_2 = \{(a, \beta): 0 < a \leqslant 1, 0 < \beta < \hat{\beta}(a)\}.$$

Then

$$\text{r. s. } S\big((a), (\beta)\big) = \begin{cases} \hat{r}_1 & \textit{if } (a, \beta) \in \hat{D}_1, \\ \hat{r}_2 & \textit{if } (a, \beta) \in \hat{D}_2, \end{cases}$$

where $\hat{r}_1$ *is the unique root of the polynomial*

$$\hat{u}(r) = a\beta^2 r^3 - \beta(\beta+2a)r^2 - (2\beta+a)r + 1$$

in the interval $(0, 1)$, $\hat{r}_2$ *is the smallest root of the polynomial*

$$\begin{aligned}\hat{v}(r) = & -\alpha^2\beta(1-\beta)^2 r^6 + 2\alpha\beta(1-\beta^2)r^5 + \\ & + (1+\beta)[(\alpha^2-1)\beta^2 - (3\alpha^2+1)\beta + 2\alpha^2]r^4 + 2\alpha\beta(\beta^2-5)r^3 + \\ & + (1+\beta)[\beta^2+\beta-2(1+\alpha^2)]r^2 + 2(1-\beta)\end{aligned}$$

in the interval $(\tilde{r}^*, 1)$ (cf. Theorem 2).

Equality $\text{r. s.}\{f\} = \text{r. s.}\, S\big((\alpha), (\beta)\big)$ *holds for function* (2.2), *where* $F(z)$ *is given by* (5.2) *and*

$$P(z) = \begin{cases} P^*(z;\ \beta, -\beta) & \text{for } (\alpha, \beta) \in \hat{D}_1, \\ P^{**}(z;\ \beta, -\beta) & \text{for } (\alpha, \beta) \in \hat{D}_2, \end{cases}$$

and $d = d(\hat{r}_2, \beta, -\beta)$; $\operatorname{re} \dfrac{zf'(z)}{f(z)} = 0$ *for* $z = \bar{\varepsilon} r_0$, *where* $r_0 = \hat{r}_1$ *or* $r_0 = \hat{r}_2$, *respectively.*

COROLLARY. *For every* $0 < \alpha \leqslant 1$

$$\text{r. s.}\, S\big((\alpha), (\beta)\big) = \begin{cases} \hat{r}_1 & \text{if } \beta'' \leqslant \beta \leqslant 1, \\ \hat{r}_2 & \text{if } 0 < \beta \leqslant \beta', \end{cases}$$

where β' *and* β'' *are given by* (4.4) *and* (4.5), *respectively.*

References

[1] W. Janowski, *Some extremal problems for certain families of analytic functions I*, Ann. Polon. Math. 28 (1973), p. 297–326.

[2] M. Read, *On close-to-convex univalent function*, Michigan Math. J. 3 (1955), p. 59–62.

$f_{-1}(y)$; $f(x) \not\equiv 0$ signifiera que la fonction $f(x)$ n'est pas identiquement égale à zéro dans aucun intervalle de son domaine de définition.

Considérons à présent une équation différentielle linéaire avec une perturbation en forme (II) dans laquelle les fonctions $a(p), b(p), c(p)$ et $\psi(u)$ sont données et continues dans des intervalles convenables. L'équation (II) comprend comme cas particuliers beaucoup d'équations différentielles qu'on trouve dans la monographie de Kamke [2] p. e.: les équations linéaires, les équations à variables séparées, les équations de Riccati, l'équation généralisée de Bernoulli, une certaine sous-classe d'équations non-linéaires.

Considérons maintenant le problème suivant:

PROBLÈME I.1. *Soit donnée une famille de fonctions $u(p, C)$ satisfaisant à l'équation différentielle* (II) *et à la condition*

$$u'_p(p, C) \neq 0 \tag{1.1}$$

pour chaque $p \in P$ et C fixé. Soient les fonctions $a(p), b(p), c(p)$ de la classe C dans l'intervalle P et la fonction $\psi(u)$ de la classe C^1 et telle que $\psi'_u(u) \neq 0$ le long de chaque solution $u(p, C)$ de l'équation (II). *Considérons la famille de fonctions*

$$x(p, C) = \psi\big(u(p, C)\big), \tag{1.2}$$

de variable p avec paramètre C. Soient les fonctions de la famille $y(p, C)$ définies à l'aide de l'identité

$$y'_p(p, C) = x'_p(p, C)p, \tag{1.3}$$

remplie pour chaque $p \in P$ et C fixé. Nous avons donc définie une famille en forme paramétrique

$$x = x(p, C), \quad y = y(p, C). \tag{1.4}$$

Le problème consiste à trouver la forme de la famille (1.4) *et son équation différentielle.*

Nous formulons le théorème qui donne une solution partielle du problème I.1. Une certaine restriction résulte du fait que nous imposons à la fonction $a(p)$ une certaine condition qui rétrécit la classe d'équations (II) à certaines sous-classes y compris celle des équations linéaires.

THÉORÈME I.1. *Si les hypothèses suivantes sont remplies*:

1° *les fonctions $b(p)$ et $c(p)$ sont de la classe C pour $p \in P$, la fonction $a(p)$ est de la forme*

$$a(p) = -\exp\int b(p)dp, \tag{1.5}$$

2° *$u(p, C)$ est la famille de solutions de l'équation différentielle* (II),