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Some remarks on spaces provided with mixed norm

Let \((E_i, \mu_i)\) denote for \(i = 1, 2\) a measure space, let \((E_1 \times E_2, \mu_1 \times \mu_2)\) be a product measure space and let \(X_i\) denote some space of \(\mu_i\)-measurable functions in which a homogeneous \(B\)-norm \(\left\| \cdot \right\|_i\) is defined.

If \(f\) is a \(\mu_1 \times \mu_2\)-measurable function in \(E_1 \times E_2\), \(f(x, y) \in X_1\) with exception of a set \(y\) of the \(\mu_1\)-measure 0, and \(\left\| f(\cdot, y) \right\|_1 \in X_2\), then \(\left\| f(\cdot, y) \right\|_2\) is called the mixed norm of \(f\).

In this paper we consider spaces of functions of two variables equipped with mixed norms. When \(X_i\) are Orlicz spaces we give necessary and sufficient conditions that the mixed norm space is an Orlicz space of functions of two variables, \(\mu_1 \times \mu_2\)-measurable.

1. In this note \(\varphi\)-function is called a function \(\varphi\): \((0, \infty) \rightarrow R_+\), continuous, non-decreasing and equal zero only in 0, and tending to \(\infty\), as \(u \rightarrow \infty\).

If \(\varphi\) and \(\psi\) are \(\varphi\)-functions, and for some positive constants \(k_1, k_2, a, b\) the inequality

\[
(+) \quad k_1 \varphi(au) \leq k_2 \psi(bu) \quad \text{holds for } u \geq u_0,
\]

we say that \(\psi\) is non-weaker than \(\varphi\) (in \(l\)-sense, or for large \(u\)) and we write \(\varphi \prec \psi\). When \(\varphi, \psi\) are convex \(\varphi\)-functions \((+)\) is equivalent to \(\varphi(u) \leq \psi(\lambda u)\) for some constant \(\lambda > 0\) and for \(u \geq u_0\).

If \(\varphi \prec \psi\) and \(\psi \prec \varphi\), then we call \(\varphi\) and \(\psi\) \(l\)-equivalent, in symbols \(\varphi \sim \psi\); \(\varphi\) satisfies condition \(A_2\) if \(\varphi(2u) \leq k \varphi(u)\) for some constant \(k > 0\) and for \(u \geq u_0\).

Let \(\varphi\) be a convex \(\varphi\)-function satisfying the two following conditions:

\[
(0_1) \quad \frac{\varphi(u)}{u} \rightarrow 0 \quad \text{as } u \rightarrow 0,
\]

\[
(\infty_1) \quad \frac{\varphi(u)}{u} \rightarrow \infty \quad \text{as } u \rightarrow \infty,
\]

we call the function \(\varphi^*(v) = \sup_{u \geq 0} (uv - \varphi(u)), v \geq 0\) the complementary (in the Young sense) function for \(\varphi\); it is also a convex \(\varphi\)-function satisfying \((0_1)\) and \((\infty_1)\).
1.1. In what follows $E_1$ and $E_2$ will always denote non-empty sets, $\mathcal{C}_i$ ($i = 1, 2$) $\sigma$-algebra of subsets of $E_i$ on which a finite, non-atomic and $\sigma$-additive measure $\mu_i$ is defined, $\mu = \mu_1 \times \mu_2$ denotes a product measure on $\sigma$-algebra $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$.

Measure $\mu_i$ is separable on $\mathcal{C}_i$ ($i = 1, 2$) if $\mathcal{C}_i$ is a separable space in Radon–Nikodym metric $\mu_i(e' - e'')$, $e', e'' \in \mathcal{C}_i$. In a similar sense we understand separability of $\mu$ on $\mathcal{C}$. The space of real $\mu$-measurable functions defined on $Q = E_1 \times E_2$ will be denoted by $\mathcal{S}$.

1.2. Let $\varphi$ be a convex $\varphi$-function and let

$$L^\varphi(Q) = \left\{ f \in \mathcal{S} : \int_Q \varphi(\lambda |f(x, y)|) \, d\mu(x, y) < \infty \text{ for some } \lambda > 0 \right\}.$$ 

$L^\varphi(Q)$ is an Orlicz space which is, as is well known, a Banach space with the norm

$$\|f\|_\varphi^\varphi = \inf \left\{ \varepsilon > 0 : \int_Q \varphi(|f(x, y)|/\varepsilon) \, d\mu(x, y) \leq 1 \right\}$$

(Luxemburg's norm).

If $\varphi$ satisfies (0) and ($\infty$), then $L^\varphi(Q)$ can also be equipped with another homogeneous norm

$$\|f\|_\varphi^\varphi = \sup \left| \int_Q f(x, y) \varphi(x, y) \, d\mu(x, y) \right| \quad \text{ (Orlicz's norm),}$$

where supremum is taken with respect to such $g$'s that

$$\int_Q \varphi^*(|g(x, y)|) \, d\mu(x, y) \leq 1.$$

The inequalities $\|f\|_\varphi^\varphi \leq \|f\|_\varphi^\varphi \leq 2 \|f\|_\varphi^\varphi$ hold.

In this note, in addition to the Orlicz spaces of functions of two variables we consider also Orlicz spaces of functions of one variable $L^\varphi(E, \lambda)$ on which the norms $\|\cdot\|_\varphi^E$, $\|\cdot\|_\varphi^E$ (when $E = E_i$, $i = 1, 2$, we write $L^\varphi(E_i)$, $\|\cdot\|_\varphi$, $\|\cdot\|_\varphi$) are defined in a similar fashion, i.e. similarly as before for $L^\varphi(Q)$. Here $E$ denotes a non-empty set, for which $\sigma$-algebra of subsets is defined along with a finite $\sigma$-measure $\lambda$.

Whenever we talk of the measurable function $f : E_1 \rightarrow \mathbb{R}$, $f : E_2 \rightarrow \mathbb{R}$, $f : Q \rightarrow \mathbb{R}$ we understand respectively $\mu_1$-measurable, $\mu_2$- or $\mu$-measurable, function. The phrase: $f$ has property $W$ almost everywhere is to be understood that $W$ holds with exception of a set of, respectively, $\mu_1$, $\mu_2$ or $\mu$-measure 0.

2. If $f \in \mathcal{S}$ and for almost every $y, f(\cdot, y) \in L^\varphi(E_1)$, then the norms $\|f(\cdot, y)\|_\varphi$, $\|f(\cdot, y)\|_\varphi$ are measurable functions with respect of parameter.

It is well known the following formula is true for $g \in L^\varphi(E_1)$

$$\|g\|_\varphi = \inf_{t > 0} \left\{ t^{-1} + t^{-1} \int_{E_1} \varphi(tg(x)) \, d\mu_1 \right\}.$$
Spaces provided with mixed norm

Let $T$ denote the set of positive rational numbers. Defining $\|g\|_\varphi$ we can limit ourselves to $t \in T$. Since the functions $h_t(y) = t^{-1} \int_{E_1} \varphi_1(tf(x, y)) \, dx$ are $\mu_1$-measurable it follows that $\|f(\cdot, y)\|_\varphi$ is measurable. Measurability of $\|f(\cdot, y)\|_\varphi$ we prove similarly.

2.1. Let $\varphi = (\varphi_1, \varphi_2)$ denote in the sequel the pairs of convex $\varphi$-functions $\varphi_1$ and $\varphi_2$. With this notation $L^{\varphi}(Q)$ is a set of all functions having the properties: for almost all $y, f(\cdot, y) \in L^{\varphi_1}(E_1)$, the function $h(y) = \|f(\cdot, y)\|_{\varphi_1}$ belongs to $L^{\varphi_2}(E_2)$.

It is easy to see that $L^{\varphi}(Q)$ as a subspace of $S$, is a vector space. In $L^{\varphi}(Q)$ the homogeneous norm can be defined by $|f|_\varphi = \|h(\cdot)\|_{\varphi_2}$. This norm we call the mixed norm of type $\varphi = (\varphi_1, \varphi_2)$. It is convenient sometimes to write this norm in various ways:

$$|f|_\varphi = |f(x, y)|_\varphi = |f(\cdot, \cdot)|_\varphi = \|f(\cdot, y)\|_{\varphi_1}^\varphi_{\varphi_2}.$$

Instead of using Orlicz's norms in defining the norm in $L^{\varphi}(Q)$ are could use Luxemburg's norm, that is, the norm could be defined as $|f|_\varphi = \|f(\cdot, y)\|_{\varphi_1}^\varphi_{\varphi_2}$. Alternatively, in the first stage of defining the mixed norm Orlicz's norm could be used and then Luxemburg's norm, or conversely. All such norms are equivalent. The particular case of these spaces $L^{\varphi}(Q)$ when $\varphi_2 = \varphi_1^*$ was first investigated by Zaanen [6] in connection with some problems arising in the theory of integral operators, see also [4], [5]. The case when $\varphi_1$ and $\varphi_2$ are powers is well known and was considered in a number of papers. Initial and fundamental is here the paper of Benedek and Panzone [1].

These authors introduced in their paper the spaces of mixed norm for functions of $n$ variables and for finite systems $(L^{p_1}, L^{p_2}, \ldots, L^{p_n})$ of the spaces of functions integrable with th power $(i = 1, 2, \ldots, n)$.

This could be generalized as follows. For measurable functions of $n$ variables one could form the space with mixed norm from the systems $(L^{p_1}(E_1), L^{p_2}(E_2), \ldots, L^{p_n}(E_n))$. The successive application of the procedure described above for the pair of $\varphi$-functions would then generalize the concept of a mixed norm space for such case. For the area of problems which is of interest for us in this note the case of functions on $n$-variables is of the same importance as that of functions of two variables save of some computational difficulties however we shall not consider it here.

2.2. Let $f_n \in L^{\varphi}(E_1)$, where $\varphi$ is a convex $\varphi$-function satisfying (0$_1$) and (0$\infty$) and let $f_1 \leq f_2 \leq \ldots$

Let $\lim f_n(x) = f(x)$ for almost all $x \in E_1$. If $\sup_n \|f_n\|_\varphi < \infty$, then $f \in L^{\varphi_1}(E_1)$ and $\lim\|f_n\|_\varphi = \|f\|_\varphi$.

Analogous theorem holds for the norm $\|\cdot\|_{\varphi_n}$ (without assuming (0$_1$) and (0$\infty$)).
2.2.1. With the same assumptions on \( \varphi \) as in 2.2, if \( f_n \geq 0, \sum_1^{\infty} ||f_n||_{(\varphi)} < \infty \), then the series \( \sum_1^{\infty} f_n(x) = f(x) \) converges for almost every \( x, f \in L^{*\varphi}(E_2) \) and

\[
||f||_{(\varphi)} \leq \sum_1^{\infty} ||f_n||_{(\varphi)}.
\]

Analogous theorem holds for the norm \( ||\cdot||_{\varphi} \).

Easy proofs of these lemmas are omitted.

2.3. Let \( \varphi_i \) be a convex \( \varphi \)-function, \( i = 1,2 \). If \( f_n \in L^{*\varphi_i}(Q) \) for \( n = 1,2, \ldots, ||f_n||_{(\varphi_i)} \to 0 \) as \( n \to \infty \), then \( \{f_n\} \) converges to 0 in \( \mu \)-measure.

To prove it, let us define the sets

\[
e_{nk}(y) = \{(x, y) \in Q, \text{ for fixed } y \in E_2, k|f_n(x, y)| \geq 1\}, n, k = 1, 2, \ldots,
\]

\[
e_{nk} = \{(x, y) \in Q: k|f_n(x, y)| \geq 1\}.
\]

Clearly

(i)
\[
|\chi_{e_{nk}}|_{(\varphi)} \leq |f_n|_{(\varphi)} k,
\]

where \( \chi_{e_{nk}}(y) \) is the characteristic function of \( e_{nk}(y) \). For almost every \( y \) the set \( e_{nk}(y) \) is \( \mu_1 \)-measurable, let us write \( \mu_1(e_{nk}(y)) = \lambda_{nk}(y). \) Let \( h(y) = ||\chi_{e_{nk}}(y)||_{(\varphi_1)} \). For every \( \varepsilon > 0 \) there exists a \( \eta > 0 \) such that \( ||h(\cdot)||_{(\varphi_2)} < \eta \) implies \( h(y) < \varepsilon \) with exception of a set of measure \( < \varepsilon \). In view of (i) and since \( h(y) = [\varphi^{-1}(1/\lambda_{nk}(y))]^{-1} \), we get

(ii)
\[
\frac{1}{\varphi^{-1}(1/(\lambda_{nk}(y)))} \leq \varepsilon,
\]

except of the set of \( y \)'s \( a_{nk} \) whose measure is \( < \varepsilon \) for \( n \geq n_0(\varepsilon, k) \). On account of Fubini theorem we obtain

\[
\mu(e_{nk}) = \int \lambda_{nk}(y) d\mu_2 = \int_{a_{nk}}^{E_2} + \int_{E_2 \setminus a_{nk}} \cdots < \frac{1}{\varphi_1(1/\varepsilon)} \mu_2(E_2) + \varepsilon \mu_2(E_2),
\]

when \( n \geq n_0(\varepsilon, k) \), or else \( \mu(e_{nk}) \to 0 \) as \( n \to \infty \).

2.4. Let \( f_n \in L^{*\varphi}(Q) \) for \( n = 1, 2, \ldots \). If \( \sum_1^{\infty} ||f_n||_{(\varphi)} < \infty \), then the series \( \sum_1^{\infty} f_n(x, y) \) is convergent almost everywhere in \( Q \) and its sum belongs to \( L^{*\varphi}(Q) \), and

\[
|\sum_1^{\infty} f_n(x, y)|_{(\varphi)} \leq \sum_1^{\infty} ||f_n(x, y)||_{(\varphi)}.
\]
By virtue of 2.2.1, for \( \varphi = \varphi_2 \), \( h_n(y) = \| f_n(\cdot, y) \|_{(\varphi_1)} \) we obtain convergence of the series \( \sum_{n=1}^{\infty} h_n(y) \) for almost every \( y \) and

\[
\| \sum_{n=1}^{\infty} h_n(y) \|_{(\varphi_2)} \leq \sum_{n=1}^{\infty} \| h_n(y) \|_{(\varphi_2)}.
\]

Applying 2.2.1 once again, this time for \( \varphi = \varphi_1 \) we get that for almost every \( y \) the series \( \sum_{n=1}^{\infty} f_n(x, y) \) is convergent for almost every \( x \) and

\[
\| \sum_{n=1}^{\infty} f_n(\cdot, y) \|_{(\varphi_1)} \leq \sum_{n=1}^{\infty} h_n(y). \]

Now, computing the norms in \( L^{\varphi_2}(E_2) \) on both sides of this inequality we see that (*) is true.

2.5. The space \( L^{\varphi}(Q) \) is complete. Let \( f_n \in L^{\varphi}(Q) \) and let \( |f_n - f_m|_{(\varphi)} \to 0 \) as \( n, m \to \infty \). We take an increasing sequence of indices \( n_i \) in such a way that \( \sum_{n=1}^{\infty} |f_{n_i} - f_{n_{i+1}}|_{(\varphi)} < \infty \). In view of 2.4 \( \sum_{n=1}^{\infty} (f_{n_i}(x, y) - f_{n_{i+1}}(x, y)) = f(x, y) \) is convergent almost everywhere in \( Q \). This, together with (*) in 2.4 yields

\[
|f - f_{n_k}|_{(\varphi)} = \left| \sum_{i=k}^{\infty} (f_{n_i} - f_{n_{i+1}}) \right|_{(\varphi)} \leq \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}|_{(\varphi)} \to 0 \quad \text{as} \quad k \to \infty.
\]

Since \( f_{n_k} \in L^{\varphi}(Q) \) it follows that \( f \in L^{\varphi}(Q) \).

Since \( |f - f_{n_k}|_{(\varphi)} < |f - f_{n_k}|_{(\varphi)} + |f_{n_k} - f_n|_{(\varphi)} \) selecting for \( n \to \infty \) a suitable \( n_k \) we get \( |f - f_{n_k}|_{(\varphi)} \to 0 \) as \( n \to \infty \).

2.6. Let \( \varphi \) be a convex \( \varphi \)-function satisfying condition \( A_2 \), \( g \in L^{\varphi}(E, \lambda) \), \( g_n \) be \( \lambda \)-measurable functions such that \( |g_n(x)| \leq |g(x)| \) for almost every \( x \in E \). Under these assumptions

\[
\| g_n - g \|_{E^\varphi} \to 0 \quad \text{as} \quad n \to \infty.
\]

2.7. Let \( \varphi_1, \varphi_2 \) be convex \( \varphi \)-functions satisfying \( A_2 \), let \( g \in L^{\varphi}(Q) \), \( f_n \in S \), let \( \{f_n\} \) converge in \( \mu \)-measure to \( f \) in \( Q \) and let

\[
(*) \quad |f_n(x, y)| \leq |g(x, y)| \quad \text{for almost every} \quad (x, y) \in Q,
\]

\( n = 1, 2, \ldots \). Under these assumptions \( |f_n - f|_{\varphi} \to 0 \) as \( n \to \infty \).

We can assume that

\[
(i) \quad f_n(x, y) \to f(x, y) \quad \text{almost everywhere in} \quad Q.
\]

There exists in \( E_2 \) a set \( e \) of \( y \)'s whose measure is \( \mu_2(E_2) \) and such that for \( y \in e \) \((*)\) and \((i)\) holds for almost every \( x \). Thus, by 2.6

\[
(ii) \quad h_n(y) = \| f_n(\cdot, y) - f(\cdot, y) \|_{\varphi_1} \to 0 \quad \text{for} \quad n \to \infty, \quad y \in e.
\]
Since \( \|f(x, y)\| \leq |g(x, y)| \) almost everywhere in \( Q \) it follows then \( \|f(\cdot, y)\|_{\varphi_1} \leq \|g(\cdot, y)\|_{\varphi_1} \) and since also \( \|f_n(\cdot, y)\|_{\varphi_1} \leq \|g(\cdot, y)\|_{\varphi_1} \), we get \( h_n(y) \leq 2 \|g(\cdot, y)\|_{\varphi_1} \). We have \( 2\|g(\cdot, y)\|_{\varphi_1} \in L^{*\varphi_2}(E_2) \) and so, application of 2.6 again yields \( |f_n - f|_\varphi \to 0 \) for \( n \to \infty \).

2.8. Let \( \varphi_1, \varphi_2 \) denote convex \( \varphi \)-functions satisfying \( \Delta_2 \), let the measures \( \mu_1 \) and \( \mu_2 \) be separable (in the sense of Radon–Nikodym metric). Then the space \( L^{*\varphi}(Q) \) is separable.

Separability of measures \( \mu_1, \mu_2 \) implies separability of \( \mu \). Denote by \( \tilde{\mathcal{C}} \) a countable class of subsets of \( \mathcal{C} \) such that for arbitrary \( \varepsilon > 0 \) and \( e \in \mathcal{C} \) there is a set \( \tilde{e} \in \tilde{\mathcal{C}} \) for which \( \mu(\tilde{e} - e) < \varepsilon \). If \( \mu(\tilde{e}_n - e) \to 0 \) as \( n \to \infty \), where \( \tilde{e}_n \in \tilde{\mathcal{C}} \), then \( \chi_{\tilde{e}_n} \) tends to \( \chi_e \) in \( \mu \)-measure and in view of 2.7 \( |\chi_{\tilde{e}_n} - \chi_e|_\varphi \to 0 \) as \( n \to \infty \). It follows then that if \( s \) is a \( \mu \)-simple function, i.e. is of the form \( \sum_{i=1}^{n} a_i \chi_{\tilde{e}_i}, \tilde{e}_i \in \tilde{\mathcal{C}} \), then there is a sequence of simple functions \( \tilde{s}_k = \sum_{i=1}^{n} a_i \chi_{\tilde{e}_i} \), \( \tilde{e}_i \in \tilde{\mathcal{C}} \) such that \( |\tilde{s}_k - s|_\varphi \to 0 \) as \( k \to \infty \). But for \( f \in L^{*\varphi}(Q) \) there is a sequence of \( \mu \)-simple functions \( s_n' \) such that \( |s_n'(x, y)| \leq |f(x, y)| \), \( s_n' \) tends to \( f \) in \( \mu \)-measure. By virtue of 2.7 \( |s_n' - f|_\varphi \to 0 \) as \( n \to \infty \). From what we said above it follows that the countable set of simple functions of the form \( \tilde{s} = \sum_{i=1}^{n} a_i \chi_{\tilde{e}_i}, \tilde{e}_i \in \tilde{\mathcal{C}}, a_i \) — a rational number, is dense in \( L^{*\varphi}(Q) \).

2.8.1. Let \( \varphi_1, \varphi_2 \) be convex \( \varphi \)-functions satisfying \( \Delta_2 \), \( E_1 = E_2 = \langle a, b \rangle \), \( \mathcal{C}_1 = \mathcal{C}_2 \) be the algebra of Lebesgue-measurable sets, \( \mu_1 = \mu_2 \) the Lebesgue measure. Under these assumptions, continuous functions in \( Q \) are dense in \( L^{*\varphi}(Q) \), and polynomials with rational coefficients form dense and countable subset in this space.

To prove it, let us notice, that if \( s \) is a \( \mu \)-simple function then, in view of Luzin and Tietze's theorems, we can construct a sequence of continuous functions such that \( |g_n(x, y)| \leq \text{ess sup} |s| \), and \( \{g_n\} \) converges in \( \mu \)-measure to \( s \). By 2.7 \( |g_n - s|_\varphi \to 0 \) as \( n \to \infty \) and since the set of \( \mu \)-simple functions is, as it follows from Theorem 2.8, dense in \( L^{*\varphi}(Q) \), hence the first part of the theorem is proved. The second part follows easily from the fact that if a sequence from \( L^{*\varphi}(Q) \) converges uniformly in \( Q \) it also converges in norm \( |\cdot|_\varphi \).

2.9. Let \( \varphi_1, \varphi_2 \) be convex \( \varphi \)-functions satisfying \( (\infty_1) \) and \( (\infty_1) \), \( \varphi^*, \varphi^*_2 \) its complementary functions. For any functions \( f \in L^{*\varphi}(Q), g \in L^{*\varphi^*}(Q), \) where \( \varphi = (\varphi_1, \varphi_2), \varphi^* = (\varphi^*_1, \varphi^*_2) \) is satisfied the Hölder's inequality

\[
\int_Q f(x, y)g(x, y)\,d\mu(x, y) \leq |f|_{\varphi} |g|_{\varphi^*}.
\]
Applying twice Hölder’s inequality for one variable we get
\[ \left| \int_Q \int f(x, y) g(x, y) \, d\mu(x, y) \right| \leq \int_{E_2} \left| \int_{E_1} f(x, y) g(x, y) \, d\mu_1 \right| \, d\mu_2 \]
\[ \leq \int_{E_2} \|f(\cdot, y)\|_{\varphi_1} \|g(\cdot, y)\|_{(\varphi_1)^*} \, d\mu_2 \leq \left| f(\cdot, \cdot)\right|_{\varphi} \|g(\cdot, \cdot)\|_{(\varphi_1)^*}. \]

3. For \( L^{\ast \varphi}(Q) \subset L^{\tilde{\varphi}}(Q) \), where \( \varphi = (\varphi_1, \varphi_2), \tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2), \varphi_i, \tilde{\varphi}_i \)
\((i = 1, 2)\) are convex \( \varphi \)-functions, it is necessary and sufficient that
\[ (+) \quad \tilde{\varphi}_1 \preceq \varphi_1, \quad \tilde{\varphi}_2 \preceq \varphi_2. \]

From the theory of Orlicz spaces it is known that \( \tilde{\varphi} \to \varphi \) implies \( L^{\ast \varphi}(E, \lambda) \subset L^{\ast \tilde{\varphi}}(E, \lambda) \) and conversely and, moreover, \( \|f\|_{\varphi} \leq k \|f\|_{\tilde{\varphi}} \)
for some \( k > 0 \).

Let relations \((+)\) be satisfied and let \( f \in L^{\ast \varphi}(Q) \). Since \( f(\cdot, y) \in L^{\ast \varphi_1}(E_1) \)
for almost every \( y \) it follows that \( \|f(\cdot, y)\|_{\varphi_1} \leq k \|f(\cdot, y)\|_{\tilde{\varphi}_1} \). Now since
\( \|f(\cdot, y)\|_{\varphi_1} \in L^{\ast \varphi_2}(E_2) \), we get that \( \|f(\cdot, y)\|_{\tilde{\varphi}_1} \in L^{\ast \varphi_2}(E_2) \), and finally
\( L^{\ast \varphi}(Q) \subset L^{\ast \tilde{\varphi}}(Q) \). To prove necessity we observe that if \( g \in L^{\ast \varphi_1}(E_1) \),
\( h \in L^{\ast \varphi_2}(E_2) \), then \( f(x, y) = g(x)h(y) \) belongs to \( L^{\ast \varphi}(Q) \). Let \( L^{\ast \varphi}(Q) \)
\( \subset L^{\ast \tilde{\varphi}}(Q) \). When \( h(y) = 1 \) for \( y \in E_2 \), \( g \in L^{\ast \varphi_1}(E_1) \) then \( g \in L^{\ast \tilde{\varphi}_1}(E_1) \) and
it results that \( \tilde{\varphi}_1 \preceq \varphi_1 \). Similarly we demonstrate that \( \tilde{\varphi}_2 \preceq \varphi_2 \).

3.1. Let two convex \( \varphi \)-functions \( \varphi_1, \varphi_2 \) be given and let \( \varphi = (\varphi_1, \varphi_2) \).
The inclusion
\[ (*) \quad L^{\ast \varphi}(Q) \subset L^{\ast \tilde{\varphi}}(Q) \]
is true if and only if, for some positive constants \( k_2, r \) holds the inequality
\[ (+) \quad \varphi_1(u)\varphi_2(v) \leq \varphi(k_2 uv) \quad \text{for} \quad u, v \geq r. \]

By virtue of 2.3 the identity mapping of \( L^{\ast \varphi}(Q) \) into \( L^{\ast \tilde{\varphi}}(Q) \) is of
a closed graph, and therefore, for some constant \( l > 0 \) is true the inequality
\[ (i) \quad \|f\|_{(\varphi)} \leq l \|f\|_{(\tilde{\varphi})}. \]
Let \( f(x, y) = \chi_{e_1}(x)\chi_{e_2}(y) \), where \( e_1 \in C_1, e_2 \in C_2 \). As is known, when \( \mu_1(e_1) \)
> 0, \( \mu_2(e_2) > 0 \), then
\[ \|\chi_{e_1}(\cdot)\|_{(\varphi_1)} = \frac{1}{\varphi_1^{-1}(1/\mu_1(e_1))}, \quad \|\chi_{e_2}(\cdot)\|_{(\varphi_2)} = \frac{1}{\varphi_2^{-1}(1/\mu_2(e_2))}. \]
Thus it follows from \((i)\) that
\[ \frac{1}{\varphi_1^{-1}(1/\mu_1(e_1)) \varphi_2^{-1}(1/\mu_2(e_2))} \leq \frac{l}{\varphi^{-1}(1/\mu_1(e_1)) \mu_2(e_2)}. \]
and hence, taking into account the fact that \( u = (\mu_1(e_1))^{-1}, v = (\mu_2(e_2))^{-1} \) takes all the values from \((\mu_1(E_1))^{-1}\) or from \((\mu_2(E_2))^{-1}\) to \(\infty\), \(\varphi_1^{-1}(u)\varphi_2^{-1}(v) \geq l\varphi^{-1}(uv)\), for \( u, v \geq \sup[(\mu_1(E_1))^{-1}, (\mu_2(E_2))^{-1}] = \tilde{r} \). Writing it now for inverse function, we obtain

\[
\varphi(l^{-1}uv) \geq \varphi_1(u)\varphi_2(v)
\]

for \( u, v \geq r = \sup(\varphi_1^{-1}(\tilde{r}), \varphi_2^{-1}(\tilde{r})) \). Hence we get (\( \ast \)), where \( k_2 = l^{-1} \).

To prove sufficiency let \( f \) be a measurable bounded function and let \( e \in C_1 \), be such that

\[
\mu_1(e)\varphi_1(r) < \frac{1}{2}, \quad \tilde{f}(x, y) = f(x, y)\chi_e(x)
\]

and we denote

\[
\epsilon(y) = \{(x, y) : \tilde{f}(x, y)/\tilde{h}(y) \geq r, \tilde{h}(y) = \|\tilde{f}(\cdot, y)\|_{\varphi_1}\}
\]

(boundedness of \( f \) implies \( \tilde{h}(y) < \infty \)).

Let \( \tilde{h}(y) \geq r \). In view of (\( \ast \)) for \( (x, y) \in \epsilon(y) \)

(ii)

\[
\varphi_1(|\tilde{f}(x, y)|/\tilde{h}(y))\varphi_2(\tilde{h}(y)) \leq \varphi(k_2|\tilde{f}(x, y)|).
\]

For \( x \in e - e(y) \) the following inequality is satisfied:

(iii)

\[
\varphi_1(|\tilde{f}(x, y)|/\tilde{h}(y))\varphi_2(\tilde{h}(y)) \leq \varphi_1(r)\varphi_2(\tilde{h}(y)).
\]

Integrating (ii) with respect to \( x \) on \( e(y) \) and (iii) on \( e - e(y) \) and adding we get

(iv)

\[
\int_{e} \varphi_1\left(\frac{|\tilde{f}(x, y)|}{\tilde{h}(y)}\right) d\mu_1 \varphi_2(\tilde{h}(y)) \leq \int_{e} \varphi(k_2|f(x, y)|) d\mu_1 + \varphi_1(r)\mu_1(e)\varphi_2(\tilde{h}(y)).
\]

Let us notice, that for bounded functions

\[
\int_{e} \varphi_1\left(\frac{|f(x, y)|}{\tilde{h}(y)}\right) d\mu_1 = 1.
\]

Hence, we get from (iv)

\[
\epsilon \varphi_2(\tilde{h}(y)) \leq \int_{e} \varphi(k_2|f(x, y)|) d\mu_1, \quad \text{where } \frac{1}{2} < \epsilon < 1, \tilde{h}(y) \geq r.
\]

Integrating with respect to \( y \) on \( E_2 \) we get

(v)

\[
\epsilon \int_{E_2} \varphi_2(\tilde{h}(y)) d\mu_2 \leq \int_{E_2} \int \varphi(k_2|f(x, y)|) d\mu_1 d\mu_2 + \varphi_2(r)\mu_2(E_2).
\]

Let \( s \) be a positive integer such that \( 2\mu_1(E_1)\varphi_1(r) < s \). Divide \( E_1 \) on \( s \) subsets \( e_i \) of equal measures \( \mu_1(E_1)/s \), and denote \( h(y) = \|f(\cdot, y)\|_{\varphi_1} \), \( \tilde{f}_i(x, y) = f(x, y)\chi_{e_i}(x), \tilde{h}_i(y) = \|\tilde{f}_i(\cdot, y)\|_{\varphi_1} \). Since \( \mu_1(e_i) \leq 1/2\varphi_1(r) \) we
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may apply (v) to $e = e_i$ and adding respective inequalities we get

$$(vi) \sum_{i=1}^{s} \int_{E_2} \varphi_2(h_i(y)) d\mu_2 \leq 2 \int_{E_1} \int_{E_2} \varphi_2(k_2 |f(x, y)|) d\mu_1 d\mu_2 + 2s \varphi_2(r) \mu_2(E_2).$$

But $f(x, y) = \tilde{f}_1(x, y) + \tilde{f}_2(x, y) + \ldots + \tilde{f}_s(x, y)$,

$$h(y) \leq \tilde{h}_1(y) + \tilde{h}_2(y) + \ldots + \tilde{h}_s(y),$$

hence and (vi) imply that

$$\int_{E_2} \varphi_2 \left( \frac{h(y)}{s} \right) d\mu_2 \leq \frac{2}{s} \int_{E_1} \int_{E_2} \varphi_2(k_2 |f(x, y)|) d\mu_1 d\mu_2 + 2\varphi_2(r) \mu_2(E_2).$$

This last inequality we proved for arbitrary function $f$ measurable and bounded. Approximation by truncated function and the limit passage shows us that it is true for arbitrary function $f \in L^{*p}(Q)$, for which the integral on the right-hand side is finite, and when this integral $= \infty$ then the inequality in question is also true.

Known lemma yields then $f \in L^{*p}(Q)$ implies $h \in L^{*p}(E_2)$, and so $f \in L^{*p}(Q)$, and, moreover, $\|f(\cdot, y)\|_{p_1} \leq C \|f\|_{p_2}$, where $C = (1 + 2\varphi_2(r) \mu_2(E_2)/s)$, and as a result we get inclusion (*).

3.2. Let convex $\varphi$-functions $\varphi$, $\varphi_1$, $\varphi_2$ be given, and let $\varphi = (\varphi_1, \varphi_2)$. The inclusion

$$(**) \quad L^{*p}(Q) = L^{*p}(Q),$$

is true if and only if, for some positive constants $k_1, r$ holds the inequality

$$(++) \quad \varphi(k_1 uv) \leq \varphi_1(u) \varphi_2(v) \quad \text{for} \quad u, v \geq r.$$

The proof of necessity is analogous to that in the preceding theorem. We may assume that $k_1 \leq 1$. Let $f \in L^{*p}(Q)$ and let $h(y) = \|f(\cdot, y)\|_{p_1}$. For $h(y) \leq 1$ is true the inequality

$$(i) \quad \int_{E_1} \varphi(k_1 |f(x, y)|) d\mu_1 \leq \int_{E_1} \varphi_1(\|f(x, y)\|) d\mu_1 \leq h(y).$$

Let $h(y) > 1$. We may assume that $r > 1$. For fixed $y$ let $(x, y) \in \varepsilon(y) = \{(x, y): |f(x, y)|/rh(y) \geq r\}$. By virtue of $(++)$ satisfied are the inequalities

$$\varphi(k_1 |f(x, y)|) \leq \varphi_1 \left( \frac{|f(x, y)|}{rh(y)} \right) \varphi_2(rh(y)) \leq \frac{1}{r} \varphi_1 \left( \frac{|f(x, y)|}{h(y)} \right) \varphi_2(r^2 h(y)),$$

$$\int_{\varepsilon(y)} \varphi(k_1 |f(x, y)|) d\mu_1 \leq \frac{1}{r} \varphi_2(r^2 h(y)).$$

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For \((x, y) \in E_1 - e(y)\) there hold the inequalities
\[
\varphi(k_1 |f(x, y)|) \leq \varphi(k_1 r^2 h(y)) \leq \varphi_1(r) \varphi_2(r h(y)),
\]
\[
\int_{E_1 - e(y)} \varphi(k_1 |f(x, y)|) \, d\mu_1 \leq \varphi_2(r^2 h(y)) \varphi_1(r) \mu_1(E_1).
\]

As a consequence, for \(h(y) > 1\)

(ii) \[
\int_{E_1} \varphi(k_1 |f(x, y)|) \, d\mu_1 \leq \frac{1}{r} \varphi_2(r^2 h(y)) + \varphi_2(r^2 h(y)) \varphi_1(r) \mu_1(E_1).
\]

From (i) and (ii) we get
\[
\int_{E_2} \int_{E_1} \varphi(k_1 |f(x, y)|) \, d\mu_1 \, d\mu_2 \leq l_1 \int_{E_2} \varphi_2(r^2 h(y)) \, d\mu_2 + l_2,
\]
where \(l_1 = 1/r + \varphi_1(r) \mu_1(E_1), l_2 = \mu_2(E_2)\).

Since this last inequality holds for any function \(f \in L^p(Q)\) hence if \(f \in L^p(Q)\), then \(f \in L^p(Q)\) and \(\|f\|_p < C\|f\|_p\) for some constant \(C > 0\).

3.3. For some convex \(\varphi\)-functions \(\varphi, \varphi_1, \varphi_2\) holds the equation
\[
L^{\ast\varphi}(Q) = L^{\ast\varphi}(Q), \quad \text{where} \ \varphi = (\varphi_1, \varphi_2)
\]
if and only if

(a) \(\varphi \sim \varphi_1, \ \varphi \sim \varphi_2\),

(b) for some positive constants \(m_1, m_2, r\) hold the inequalities
\[
m_1 \varphi(u) \varphi(v) \leq \varphi(uv) \leq m_2 \varphi(u) \varphi(v) \quad \text{for} \ u, v > r.
\]

To prove sufficiency let us observe that if \(m_1 < 1, 1 < m_2,\) then
\[
\varphi(m_1 u) \leq m_1 \varphi(u), \ \varphi(m_2 u) \geq m_2 \varphi(u) \quad \text{and that 3 implies that} \ L^{\ast\varphi}(Q) = L^{\ast\varphi}(Q), \quad \text{where} \ \tilde{\varphi} = (\varphi, \varphi). \quad \text{From (a) and 3.1 and 3.2 it follows that} \ L^{\ast\varphi}(Q) = L^{\ast\varphi}(Q).
\]

To prove necessity let us notice that from (***) one can deduce inequalities (+), (+++) for \(u, v \geq r\). It means that \(\varphi \sim \varphi_1 \varphi \sim \varphi_2\). Indeed,
\[
\varphi_1(u) \varphi_2(r) \leq \varphi(k_1 wr), \ \varphi(k_1 wr) \leq \varphi_1(u) \varphi_2(r) \quad \text{for} \ u \geq r. \quad \text{To demonstrate that} \ \varphi \sim \varphi_2 \quad \text{we proceed analogously. Simple computation shows that} \ l\text{-equivalence of} \ \varphi \ \text{with} \ \varphi_1, \varphi_2 \quad \text{implies that the inequalities analogous to (+) and (+++) (with, of course, other constants) hold when} \ \varphi_1, \varphi_2 \quad \text{is replaced by} \ \varphi. \quad \text{But if} \ \varphi(k_1 wr) \leq \varphi(u) \varphi(v) \quad \text{for} \ u, v \geq r, \quad \text{then} \ \varphi \quad \text{satisfies} \ \Delta_2, \quad \text{more precisely: for every} \ C > 1 \quad \text{there is a constant} \ l(C) > 0 \quad \text{such that} \ \varphi(Cu) \leq l(C) \varphi(u) \quad \text{for} \ u \geq u_0(C). \quad \text{This shows us that} \ (+), (+++) \quad \text{impl} \ (b).
\]

We shall make the following remarks on Theorems 3.1–3.3. Let \(L^p(Q)\) be a mixed norm space \(L^{\ast\varphi}(Q)\), where \(\varphi = (u^p, u^p^2),\ p_1, p_2 > 1.\) If \(p_1 = p_2 = p,\) then we check trivially that \(L^p(Q) = L^p(Q).\) In this case condition (b) of 3.3 is satisfied in the simple form with \(m_1 = m_2 = 1, \ r = 0.\) However, even more general form of inequality in (b) is still a serious
limitation of \( \varphi \) and it is evident that results for a wide class of spaces with mixed norm \( \varphi = (\varphi_1, \varphi_2) \) does not automatically contain results for the spaces of \( \varphi \)-integrable functions of two variables.

3.4. For any \( \varphi \)-function we denote
\[
I_{\varphi}(a) = \liminf_{u \to \infty} \frac{\varphi(au)}{\varphi(u)}, \quad \bar{I}_{\varphi}(a) = \limsup_{u \to \infty} \frac{\varphi(au)}{\varphi(u)}.
\]
There exist the following limits
\[
s_{\varphi} = \lim_{a \to 0^+} \frac{\lg I_{\varphi}(a)}{\lg a}, \quad \sigma_{\varphi} = \lim_{a \to 0^+} \frac{\lg \bar{I}_{\varphi}(a)}{\lg a}.
\]
These numbers are called, respectively, the lower and upper index of \( \varphi \) \([2], [3]\).

Let us assume that for \( \varphi \) holds 3.3(b). Then
\[
(+) \quad s_{\varphi} = \sigma_{\varphi} = r, \quad \text{where} \ 1 \leq r < \infty,
\]
\[
(++) \quad r = \lim_{u \to \infty} \frac{\lg \varphi(u)}{\lg u}.
\]
Using nomenclature as in [2], it means that \( \varphi \) is of quasi-regular growth.

To prove it let us notice that for \( a \geq r, u \geq r \), \( m_1 \varphi(a) \leq \varphi(au)/\varphi(u) \leq m_2 \varphi(a) \leq m_2 \varphi(a)/\lg a \leq \frac{\lg m_1}{\lg a} + \frac{\lg \varphi(a)}{\lg a}, \)
and hence
\[
(+) \quad (++) \quad r \leq \frac{\lg m_1}{\lg a} + \frac{\lg \varphi(a)}{\lg a}, \quad \text{and hence} \quad (+) \quad (++) \quad \text{follow}.
\]
Convexity of \( \varphi \) imply that \( s_{\varphi} = r \geq 1 \). To show that \( r < \infty \) we observe that
\[
m_1^n \varphi(a)^n \leq \varphi(a^{n+1}) \leq m_2^n \varphi(a)^n \leq \frac{\lg m_1}{\lg a} \leq \frac{\lg \varphi(a)}{\lg a} + \frac{\lg \varphi(a)}{\lg a},
\]
and then \( r \leq \frac{\lg m_1}{\lg a} + \frac{\lg \varphi(a)}{\lg a}\).

References