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Solutions of class C^r with respect to the parameter of a linear functional equation

1. In the present paper we are concerned with the functional equation

$$\varphi[f(x), t] = g(x, t)\varphi(x, t) + F(x, t), \tag{1}$$

where $\varphi(x, t)$ is an unknown function and $f(x)$, $g(x, t)$, $F(x, t)$ are known real functions of a real variables and t is a real parameter.

We shall prove that under some assumptions concerning the given functions, the solution $\varphi(x, t)$ of equation (1) is of class C^r, $1 \leqslant r < \infty$ with respect to the parameter t.

The analogous problem (in the other case) has been investigated in [1] and for the more general equation

$$\varphi(x) = H\big(x, \varphi[f(x)], t\big) \tag{2}$$

in [2] (under different assumptions).

2. Let us introduce the notation: $f^0(x) = x$, $f^{n+1}(x) = f[f^n(x)]$, $n = 0, \pm 1, \ldots$

$$\Delta_0 = (a, b) \times T, \qquad \Delta = \langle a, b) \times T, \qquad T - \text{interval}. \tag{3}$$

Let $x_0 \in (a, b)$ be arbitrarily fixed; then

$$\delta_0 = \{(x, t)\colon\ x \in \langle f(x_0), x_0\rangle,\ t \in T\}. \tag{4}$$

Let $\langle \alpha, \beta\rangle \subset T$ and $y \in (a, b)$, we write

$$\delta(y) = \{(x, t)\colon\ x \in \langle f(y), y\rangle,\ t \in \langle \alpha, \beta\rangle\}, \tag{5}$$

$$G_n(x, t) = \prod_{v=0}^{n-1} g[f^v(x), t], \tag{6}$$

$$G(x, t) = \lim_{n\to\infty} G_n(x, t). \tag{7}$$

We assume the following hypotheses:

I (I_0). *The functions $g(x, t)$, $F(x, t)$ are continuous in $\Delta(\Delta_0)$*(1).

II. *The function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b)$ and $a < f(x) < x$ in (a, b).*

III^r (III_0^r). *There exist the derivatives $\frac{\partial^v g}{\partial t^v}(x, t)$, $\frac{\partial^v F}{\partial t^v}(x, t)$, $v = 1, \ldots, r$, continuous in $\Delta(\Delta_0)$.*

IV. *The function g fulfils the condition $g(x, t) \neq 0$ in Δ_0.*

V^r. *For every $v = 1, \ldots, r$ and for every closed interval $\langle \alpha, \beta\rangle \subset T$ there exist: interval $\langle a, a+\eta_v\rangle \subset \langle a, b)$, $\eta_v > 0$, function $B_v(x, t)$ bounded in $\langle a, a+\eta_v\rangle \times \langle \alpha, \beta\rangle$ and a constant $0 < \theta < 1$, such that the inequalities*

$$\left|\frac{\partial^v g}{\partial t^v}(x, t)\right| \leqslant B_v(x, t), \qquad B_v[f(x), t] \leqslant \theta B_v(x, t) \tag{8}$$

hold in $\langle a, a+\eta_v\rangle \times \langle \alpha, \beta\rangle$(2).

VI^r. *For every $v = 1, \ldots, r$ and for every closed interval $\langle \alpha, \beta\rangle \subset T$ there exist: interval $\langle a, a+\varrho_v\rangle \subset \langle a, b)$, $\varrho_v > 0$, a function $D_v(x, t)$ bounded in $\langle a, a+\varrho_v\rangle \times \langle \alpha, \beta\rangle$ and constant $0 < \bar{\theta} < 1$, such that the inequalities*

$$\left|\frac{\partial^v F}{\partial t^v}(x, t)\right| \leqslant D_v(x, t), \qquad D_v[f(x), t] \leqslant \bar{\theta} D_v(x, t) \tag{9}$$

hold in $\langle a, a+\varrho_v\rangle \times \langle \alpha, \beta\rangle$.

VII. $$g(a, t) = 1 \quad \textit{for } t \in T.$$

Remark. For a certain $\langle \alpha, \beta\rangle$ and v, the constant θ in (8) and $\bar{\theta}$ in (9) can always be chosen common.

We put

$$\bar{F}(x, t) \overset{\mathrm{df}}{=} F(x, t) + c(t)[g(x, t) - 1], \tag{10}$$

$$H_n(x, t) \overset{\mathrm{df}}{=} \sum_{i=0}^{n-2} \prod_{v=i+1}^{n-1} g[f^v(x), t] \cdot F[f^i(x), t]. \tag{11}$$

$$\begin{aligned} a_v(x, t) &= \frac{\partial^v \varphi}{\partial t^v}(x, t), \\ \psi_v(x, t) &= \sum_{n=1}^{v} \binom{v}{n} \frac{\partial^n g}{\partial t^n}(x, t) \frac{\partial^{v-n} \varphi}{\partial t^{v-n}}(x, t) + \frac{\partial^v F}{\partial t^v}(x, t). \end{aligned} \tag{12}$$

(1) In I and I_0 we take Δ and Δ_0 respectively.

(2) For $t = \alpha$ and $t = \beta$, $\frac{\partial^v g}{\partial t^v}$ denotes the right and left derivatives resp.

3. We note the following theorems.

THEOREM 1. *Suppose that hypotheses* I_0, II, III_0^r, IV *are fulfilled. Then, for every* $x_0 \in (a, b)$ *and every function* $\psi(x, t)$ *of class* C^r *with respect to the parameter* t *in* δ_0 *and fulfilling the conditions*

$$\psi[f(x_0), t] = g(x_0, t)\psi(x_0, t) + F(x_0, t), \qquad t \in T, \tag{13}$$

$$a_v[f(x_0), t] = g(x_0, t)a_v(x_0, t) + F_v(x_0, t), \qquad t \in T, v = 1, \ldots, r, \tag{14}$$

there exists exactly one function $\varphi(x, t)$ *of class* C^r *with respect to the parameter* t *in* Δ_0, *satisfying equation* (1) *in* Δ_0 *and such that*

$$\varphi(x, t) = \psi(x, t) \quad \text{in } \delta_0. \tag{15}$$

The proof is analogous to the proof of Theorem 1 in [5] and Theorem 1 in [3] and is therefore omitted.

THEOREM 2. *Suppose that hypotheses* II *and* IV *are fulfilled, and let* $c(t)$ *be a function continuous in* T *such that* $\overline{F}(a, t) = 0$. *If, moreover, for every interval* $\langle \alpha, \beta \rangle \subset T$ *there exists an* $y \in (a, b)$ *such that*

$$\lim_{n\to\infty} G_n(x, t) = \lim_{n\to\infty} H_n(x, t) = 0 \tag{16}$$

uniformly in $\delta(y)$, *then equation* (1) *has in* Δ *continuous solution depending on an arbitrary function. All these solutions fulfil the condition* $\varphi(a, t) = c(t)$ (3).

The proof analogous to the proof of Theorem 7 in [4] we omitted.

Now we shall prove

THEOREM 3. *Suppose that hypotheses* I, II, III^1, IV, V^1, VI^1, VII *are fulfilled and let* $c(t) = c$ (c — *constant*) *be a function such that* $\overline{F}(a, t) = 0$. *If, moreover, for every interval* $\langle \alpha, \beta \rangle \subset T$ *there exists an* $y \in (a, b)$ *such that condition* (16) *is fulfilled uniformly in* $\delta(y)$, *then for every function* $\psi(x, t)$ *of class* C^1 *with respect to the parameter* t *in* δ_0 *and fulfilling conditions* (13) *and* (14), $v = 1$, *there exists exactly one function* $\varphi(x, t)$ *of class* C^1 *with respect to the parameter* t *in* Δ, *satisfying equation* (1) *in* Δ *and fulfilling condition* (15).

Proof. On account of Theorem 1 for every function $\psi(x, t)$ fulfilling conditions of the above theorem there exists a unique function $\varphi(x, t)$ of class C^1 with respect to the parameter t in Δ_0, satisfying equation (1) in Δ_0 and fulfilling condition (15). Put

$$\varphi(a, t) = c. \tag{17}$$

Then the function $\varphi(x, t)$ is continuous solution of equation (1) in Δ (on account of the proof of Theorem 2, cf. also the proof of Theorem 7 in [4]).

(3) It implies that we must suppose continuous the function $c(t)$ in T.

It is easily seen that for every $x \in \langle a, b)$ there exists $\partial\varphi/\partial t$ for $t \in T$ and satisfying equation

$$\frac{\partial\varphi}{\partial t}[f(x), t] = g(x, t)\frac{\partial\varphi}{\partial t}(x, t) + \frac{\partial g}{\partial t}(x, t)\varphi(x, t) + \frac{\partial\psi}{\partial t}(x, t). \tag{18}$$

Now we shall prove that $\partial\varphi/\partial t$ is continuous in Δ. Let us fix an interval $\langle a, \beta\rangle \subset T$. In view of V^1, VI^1 and VII there exists constant q, $\theta < q < 1$ and interval $\langle a, d\rangle \subset \langle a, b)$, $d \leqslant \min(\eta_1, \varrho_1)$ such that

$$|g(x, t)| > q \quad \text{in } \langle a, d\rangle \times \langle a, \beta\rangle. \tag{19}$$

We can find an N such that $J = \langle f^{N+1}(y), f^N(y)\rangle \subset \langle a, d\rangle$. Then there exists a constant $K > 0$ such that

$$|\varphi(x, t)| < K \quad \text{in } J \times \langle a, \beta\rangle. \tag{20}$$

If $x \in J$, then $x = f^N(\bar{x})$, where $\bar{x} \in \langle f(y), y\rangle$ and we have

$$G_n(x, t) = \prod_{i=0}^{n-1} g\big[f^i\big(f^N(\bar{x})\big), t\big] = \frac{1}{G_N(\bar{x}, t)} \cdot G_{n+N}(\bar{x}, t),$$

whence we obtain $G_n(x, t) \rightrightarrows 0$ for $n \to \infty$ uniformly in $J \times \langle a, \beta\rangle$.

Now we shall consider equation (18). By (17) we have

$$\frac{\partial g}{\partial t}(a, t)\varphi(a, t) + \frac{\partial F}{\partial t}(a, t) = 0. \tag{21}$$

For equation (18)

$$\bar{F}(x, t) = \frac{\partial g}{\partial t}(x, t)\varphi(x, t) + \frac{\partial F}{\partial t}(x, t) + c(t)[g(x, t) - 1].$$

We shall prove that $c(t)$ can be chosen as $c(t) = 0$. Actually, by (21) we have $\bar{F}(a, t) = 0$. Moreover, we have in view of V^1, VI^1, (19) and (20)

$$\left|\sum_{v=0}^{n-2} \frac{F[f^v(x), t]}{G_{v+1}(x, t)}\right| \leqslant \sum_{v=0}^{n-2} \frac{(K+1)\theta^v\big(B_1(x, t) + D_1(x, t)\big)}{q^{v+1}} \leqslant \bar{K} \sum_{v=0}^{n-2} \left(\frac{\theta}{q}\right)^v,$$

where

$$\bar{K} = \frac{K+1}{q}\Big[\sup_{J\times\langle a,\beta\rangle} B_1(x, t) + \sup_{J\times\langle a,\beta\rangle} D_1(x, t)\Big].$$

Since $q > \theta$, the series $\sum_{v=0}^{\infty} (\theta/q)^v$ converges and obvious is bounded. Since (cf. [4])

$$H_n(x, t) = G_n(x, t) \sum_{v=0}^{n-2} \frac{F[f^v(x), t]}{G_{v+1}(x, t)}, \tag{22}$$

we have that $H_n(x, t) \rightrightarrows 0$ uniformly in $J \times \langle \alpha, \beta \rangle$. It follows from Theorem 2 that for the function $\psi_0(x, t) = \frac{\partial \varphi}{\partial t}(x, t)$ in $J \times T$ equation (18) has in Δ continuous solution $\psi(x, t)$ satisfying condition $\psi(a, t) = 0$. Evidently $\psi(x, t) = \frac{\partial \varphi}{\partial t}(x, t)$ in Δ. This completes the proof.

THEOREM 4. *Suppose that hypotheses* I, II, III^r, IV, $1 \leqslant r < \infty$, V^r, VI^r, VII *are fulfilled and let $c(t)$ be a polynomial of a degree $s \leqslant r-1$ such that $\bar{F}(a, t) = 0$. If, moreover, for every interval $\langle \alpha, \beta \rangle \subset T$ there exists an $x_0 \in (a, b)$ such that condition* (16) *is fulfilled uniformly in $\delta(x_0)$, then for every function $\psi(x, t)$ of class C^r with respect to the parameter t in δ_0 and fulfilling conditions* (13) *and* (14), *there exists exactly one function $\varphi(x, t)$ of class C^r with respect to the parameter t in Δ, satisfying equation* (1) *in Δ and fulfilling condition* (15).

The proof is analogous to the proof of Theorem 3 (by induction with respect to r) and is therefore omitted.

References

[1] S. Czerwik, *On the differentiability of solutions of a linear functional equation with respect to the parameter*, Ann. Polon. Math. 24 (1971), p. 217–225.

[2] — *On the differentiability with respect to the parameter of solutions of a functional equation* (to appear).

[3] B. Choczewski, *On differentiable solutions of a functional equation*, Ann. Polon. Math. 13 (1963), p. 133–138.

[4] — and M. Kuczma, *On the "indeterminate case" in the theory of a linear functional equation*, Fund. Math. 58 (1966), p. 163–175.

[5] J. Kordylewski and M. Kuczma, *On some linear functional equation I*, Ann. Polon. Math. 9 (1960), p. 119–136.