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Solutions of class $C^r$ with respect to the parameter of a linear functional equation

1. In the present paper we are concerned with the functional equation

$$\varphi [f(x), t] = g(x, t)\varphi (x, t) + F(x, t),$$

where $\varphi (x, t)$ is an unknown function and $f(x), g(x, t), F(x, t)$ are known real functions of a real variables and $t$ is a real parameter.

We shall prove that under some assumptions concerning the given functions, the solution $\varphi (x, t)$ of equation (1) is of class $C^r$, $1 \leq r < \infty$ with respect to the parameter $t$.

The analogous problem (in the other case) has been investigated in [1] and for the more general equation

$$\varphi (x) = H(x, \varphi [f(x)], t)$$

in [2] (under different assumptions).

2. Let us introduce the notation:

$$f^0(x) = x, \quad f^{n+1}(x) = f[f^n(x)], \quad n = 0, \pm 1, \ldots$$

$$\Delta_0 = (a, b) \times T, \quad \Delta = (a, b) \times T, \quad T \text{ - interval.}$$

Let $x_0 \in (a, b)$ be arbitrarily fixed; then

$$\delta_0 = \{(x, t): x \in \langle f(x_0), x_0 \rangle, t \in T\}.$$ 

Let $\langle a, \beta \rangle \subset T$ and $y \in (a, b)$, we write

$$\delta(y) = \{(x, t): x \in \langle f(y), y \rangle, t \in \langle a, \beta \rangle\},$$

$$G_n(x, t) = \prod_{v=0}^{n-1} g[f^v(x), t],$$

$$G(x, t) = \lim_{n \to \infty} G_n(x, t).$$
We assume the following hypotheses:

I (I₀).

The functions \( g(x, t), F(x, t) \) are continuous in \( \Lambda(A₀) \).

II.

The function \( f(x) \) is continuous and strictly increasing in an interval \( (a, b) \) and \( a < f(x) < b \) in \( (a, b) \).

III' (III₀).

There exist the derivatives \( \frac{\partial^v g}{\partial t^v}(x, t), \frac{\partial^v F}{\partial t^v}(x, t), \ v = 1, \ldots, r \), continuous in \( \Lambda(A₀) \).

IV.

The function \( g \) fulfills the condition \( g(x, t) \neq 0 \) in \( A₀ \).

Vr.

For every \( v = 1, \ldots, r \) and for every closed interval \( (a, \beta) \subset T \) there exist: interval \( (a, a+\eta_v) \subset (a, b) \), \( \eta_v > 0 \), function \( B_v(x, t) \) bounded in \( (a, a+\eta_v) \times (a, \beta) \) and a constant \( 0 < \theta < 1 \), such that the inequalities

\[
\left| \frac{\partial^v g}{\partial t^v}(x, t) \right| \leq B_v(x, t), \quad B_v[f(x), t] \leq \theta B_v(x, t)
\]

hold in \( (a, a+\eta_v) \times (a, \beta) \).

VIr.

For every \( v = 1, \ldots, r \) and for every closed interval \( (a, \beta) \subset T \) there exist: interval \( (a, a+\xi_v) \subset (a, b) \), \( \xi_v > 0 \), a function \( D_v(x, t) \) bounded in \( (a, a+\xi_v) \times (a, \beta) \) and constant \( 0 < \tilde{\theta} < 1 \), such that the inequalities

\[
\left| \frac{\partial^v F}{\partial t^v}(x, t) \right| \leq D_v(x, t), \quad D_v[f(x), t] \leq \tilde{\theta} D_v(x, t)
\]

hold in \( (a, a+\xi_v) \times (a, \beta) \).

VII.

\( g(a, t) = 1 \) for \( t \in T \).

Remark. For a certain \( (a, \beta) \) and \( v \), the constant \( \theta \) in (8) and \( \tilde{\theta} \) in (9) can always be chosen common.

We put

\[
F(x, t) = \frac{dt}{dt} F(x, t) + c(t)[g(x, t) - 1],
\]

\[
H_n(x, t) = \frac{dt}{dt} \sum_{i=0}^{n-2} \prod_{i=0}^{n-1} g[f^v(x), t] \cdot F[f^i(x), t].
\]

\[
\alpha_v(x, t) = \frac{\partial^v \varphi}{\partial t^v}(x, t),
\]

\[
\psi_v(x, t) = \sum_{n=1}^{v} \binom{v}{n} \frac{\partial^n g}{\partial t^n}(x, t) \frac{\partial^{v-n} \varphi}{\partial t^{v-n}}(x, t) + \frac{\partial^v F}{\partial t^v}(x, t).
\]

(1) In I and I₀ we take \( \Lambda \) and \( \Lambda₀ \) respectively.

(2) For \( t = a \) and \( t = \beta \), \( \frac{\partial^v g}{\partial t^v} \) denotes the right and left derivatives resp.
3. We note the following theorems.

**Theorem 1.** Suppose that hypotheses $I_0$, $II$, $III'$, $IV$ are fulfilled. Then, for every $x_0 \in (a, b)$ and every function $\psi(x, t)$ of class $C^r$ with respect to the parameter $t$ in $\delta_0$ and fulfilling the conditions

\begin{align}
\psi[f(x_0), t] &= g(x_0, t)\psi(x_0, t) + F(x_0, t), \quad t \in T, \\
\alpha_v[f(x_0), t] &= g(x_0, t)\alpha_v(x_0, t) + F_v(x_0, t), \quad t \in T, \quad v = 1, \ldots, r,
\end{align}

there exists exactly one function $\varphi(x, t)$ of class $C^r$ with respect to the parameter $t$ in $\Delta_0$, satisfying equation (1) in $\Delta_0$ and such that

\begin{equation}
\varphi(x, t) = \psi(x, t) \quad \text{in } \delta_0.
\end{equation}

The proof is analogous to the proof of Theorem 1 in [5] and Theorem 1 in [3] and is therefore omitted.

**Theorem 2.** Suppose that hypotheses $II$ and $IV$ are fulfilled, and let $c(t)$ be a function continuous in $T$ such that $F(a, t) = 0$. If, moreover, for every interval $(a, \beta) \subseteq T$ there exists an $y \in (a, b)$ such that

\begin{equation}
\lim_{n \to \infty} G_n(x, t) = \lim_{n \to \infty} H_n(x, t) = 0
\end{equation}

uniformly in $\delta(y)$, then equation (1) has in $\Delta$ continuous solution depending on an arbitrary function. All these solutions fulfil the condition $\varphi(a, t) = c(t)$.(3)

The proof analogous to the proof of Theorem 7 in [4] we omitted.

Now we shall prove

**Theorem 3.** Suppose that hypotheses $I$, $II$, $III'$, $IV$, $V'$, $VI'$, $VII$ are fulfilled and let $c(t) = c$ ($c$ — constant) be a function such that $F(a, t) = 0$. If, moreover, for every interval $(a, \beta) \subseteq T$ there exists an $y \in (a, b)$ such that condition (16) is fulfilled uniformly in $\delta(y)$, then for every function $\varphi(x, t)$ of class $C^1$ with respect to the parameter $t$ in $\delta_0$ and fulfilling conditions (13) and (14), $v = 1$, there exists exactly one function $\varphi(x, t)$ of class $C^1$ with respect to the parameter $t$ in $\Delta$, satisfying equation (1) in $\Delta$ and fulfilling condition (15).

**Proof.** On account of Theorem 1 for every function $\psi(x, t)$ fulfilling conditions of the above theorem there exists a unique function $\varphi(x, t)$ of class $C^1$ with respect to the parameter $t$ in $\Delta_0$, satisfying equation (1) in $\Delta_0$ and fulfilling condition (15). Put

\begin{equation}
\varphi(a, t) = c.
\end{equation}

Then the function $\varphi(x, t)$ is continuous solution of equation (1) in $\Delta$ (on account of the proof of Theorem 2, cf. also the proof of Theorem 7 in [4]).

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(3) It implies that we must suppose continuous the function $c(t)$ in $T$. 
It is easily seen that for every $x \in (a, b)$ there exists $\frac{\partial \varphi}{\partial t}$ for $t \in T$ and satisfying equation

\begin{equation}
\frac{\partial \varphi}{\partial t}[f(x), t] = g(x, t) \frac{\partial \varphi}{\partial t}(x, t) + \frac{\partial g}{\partial t}(x, t) \varphi(x, t) + \frac{\partial \varphi}{\partial t}(x, t).
\end{equation}

Now we shall prove that $\frac{\partial \varphi}{\partial t}$ is continuous in $A$. Let us fix an interval $\langle a, \beta \rangle \subset T$. In view of $V^1$, $V^1$ and $V^1$ there exists constant $q$, $\theta < q < 1$ and interval $\langle a, d \rangle \subset \langle a, b \rangle$, $d \leq \min(\eta_1, \theta_1)$ such that

\begin{equation}
|g(x, t)| > q \quad \text{in} \quad \langle a, d \rangle \times \langle a, \beta \rangle.
\end{equation}

We can find an $N$ such that $J = \langle f^{N+1}(y), f^N(y) \rangle \subset \langle a, d \rangle$. Then there exists a constant $K > 0$ such that

\begin{equation}
|\varphi(x, t)| < K \quad \text{in} \quad J \times \langle a, \beta \rangle.
\end{equation}

If $x \in J$, then $x = f^N(\bar{x})$, where $\bar{x} \in \langle f(y), y \rangle$ and we have

\[
G_n(x, t) = \prod_{i=0}^{n-1} g[f^i(f^N(\bar{x})), t] = \frac{1}{G^N(\bar{x}, t)} \cdot G_{n+N}(\bar{x}, t),
\]

whence we obtain $G_n(x, t) = 0$ for $n \to \infty$ uniformly in $J \times \langle a, \beta \rangle$.

Now we shall consider equation (18). By (17) we have

\begin{equation}
\frac{\partial g}{\partial t}(a, t) \varphi(a, t) + \frac{\partial F}{\partial t}(a, t) = 0.
\end{equation}

For equation (18)

\[
\bar{F}(x, t) = \frac{\partial g}{\partial t}(x, t) \varphi(x, t) + \frac{\partial F}{\partial t}(x, t) + c(t)[g(x, t) - 1].
\]

We shall prove that $c(t)$ can be chosen as $c(t) = 0$. Actually, by (21) we have $\bar{F}(a, t) = 0$. Moreover, we have in view of $V^1$, $V^1$, (19) and (20)

\[
\left| \sum_{v=0}^{n-2} \frac{F[f^v(x), t]}{G_{v+1}(x, t)} \right| \leq \sum_{v=0}^{n-2} \frac{(K + 1) \theta v B_1(x, t) + D_1(x, t)}{q^{v+1}} \leq \bar{K} \sum_{v=0}^{n-2} \left( \frac{\theta}{q} \right)^v,
\]

where

\[
\bar{K} = \frac{K + 1}{q} [\sup_{J \times \langle a, \beta \rangle} B_1(x, t) + \sup_{J \times \langle a, \beta \rangle} D_1(x, t)].
\]

Since $q > \theta$, the series $\sum_{v=0}^{\infty} (\theta/q)^v$ converges and obvious is bounded. Since (cf. [4])

\begin{equation}
H_n(x, t) = G_n(x, t) \sum_{v=0}^{n-2} \frac{F[f^v(x), t]}{G_{v+1}(x, t)},
\end{equation}

Then

\[
\lim_{n \to \infty} H_n(x, t) = F(x, t).
\]

...
we have that $H_n(x, t) \Rightarrow 0$ uniformly in $J \times \langle a, \beta \rangle$. It follows from Theorem 2 that for the function $\varphi_0(x, t) = \frac{\partial \varphi}{\partial t}(x, t)$ in $J \times T$ equation (18) has in $A$ continuous solution $\varphi(x, t)$ satisfying condition $\varphi(a, t) = 0$. Evidently $\varphi(x, t) = \frac{\partial \varphi}{\partial t}(x, t)$ in $A$. This completes the proof.

**Theorem 4.** Suppose that hypotheses I, II, III', IV, $1 \leq r < \infty$, V', VI', VII are fulfilled and let $c(t)$ be a polynomial of a degree $s < r - 1$ such that $F(a, t) = 0$. If, moreover, for every interval $\langle a, \beta \rangle \subset T$ there exists an $x_0 \in (a, b)$ such that condition (16) is fulfilled uniformly in $\delta(x_0)$, then for every function $\varphi(x, t)$ of class $C^r$ with respect to the parameter $t$ in $\delta_0$ and fulfilling conditions (13) and (14), there exists exactly one function $\varphi(x, t)$ of class $C^r$ with respect to the parameter $t$ in $A$, satisfying equation (1) in $A$ and fulfilling condition (15).

The proof is analogous to the proof of Theorem 3 (by induction with respect to $r$) and is therefore omitted.

**References**


