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Integration generated by a volume which is an infinite sum of volumes

Introduction. A non-empty family of sets V of a space X is called a *prering* if for every two sets $A, B \in V$ the sets $A \cap B, A \setminus B$ can be represented as a finite union of pair-wise disjoint sets from the family V.

A function v is called a *volume* if it is non-negative, real-valued and countably additive on a prering V. The triple (X, V, v) is called a *volume space* if v is a volume on the prering V.

If $(Y, | \ |)$ is a Banach space, denote by $S(V, Y)$ the family of all functions of the form $s = c_{A_1}y_1 + \ldots + c_{A_n}y_n$, where A_i is a finite family of pair-wise disjoint sets from V, the points $y_1, \ldots, y_n$ are from the Banach space Y, and where c_A denotes the characteristic function of the set A. Members of $S(V, Y)$ are called *simple functions*.

Let R denote the set of all real numbers. Denote by $S(V)$ the family of all sets $A \subset X$ such that $c_A \in S(V, R)$. It follows immediately that the family $S(V)$ consists of all sets of the form $A = A_1 \cup \ldots \cup A_n$, where the sets $A_i \in V$ $(i = 1, \ldots, n)$ are pair-wise disjoint. It is also easy to see that $S(V)$ is a ring of sets.

The theory of the integral and of the spaces $L_p(v, Y)$ $(1 \leqslant p < \infty)$ generated by the volume space (X, V, v) is developed in [1] and [2]. For a development of the theory of the space $M(v, Y)$ of Lebesgue–Bochner measurable functions and the family $M(v)$ of measurable sets generated by the volume v see [3].

If v is any volume on the prering V we denote by $N(v)$ the family of all subsets $A \subset X$ satisfying the condition that for each $\varepsilon > 0$ there exists a sequence A_n of sets in V such that $A \subset \bigcup_n A_n$ and $\sum_n v(A_n) < \varepsilon$. Members of $N(v)$ are called *null-sets*. A condition $C(x)$ depending on a parameter $x \in X$ is said to be satisfied *v-almost everywhere* (v-a. e.) if there exists a set $C \in N(v)$ such that the condition $C(x)$ is satisfied at every point $x \notin C$.

By $L_\infty(v, Y)$ we denote the subfamily of all functions $f \in M(v, Y)$ satisfying the condition that $|f(x)| \leqslant M$ v-a. e. for some constant M. For $f \in L_\infty(v, Y)$ we set

$$\|f\|_\infty = \inf\{M: |f(x)| \leqslant M \ v\text{-a. e.}\}.$$

If v_i $(i = 1, \ldots, n)$ are volumes on a prering V, then the set function $v = v_1 + \ldots + v_n$ is a volume on V. In [8] we established the relations that exist between the spaces $L_p(v, Y)$, $M(v, Y)$, $M(v)$ and $L_p(v_i, Y)$, $M(v_i, Y)$, $M(v_i)$ $(i = 1, \ldots, n)$ respectively.

If $v_t (t \in T)$ are volumes on a prering, where T is an infinite parameter set such that $\sum_T v_t(A) < \infty$ for all $A \in V$, then the set function v, defined by $v(A) = \sum_T v_t(A)$ for all sets $A \in V$, is a volume on V, and all the relations established in [8] for finite sums of volumes are valid for this case.

In Section 2 we point out the connections between the approach to integration based on volumes and that based on measures. Using these connections we formulate the results of Section 1 for measures.

1. Integration with respect to an infinite sum of volumes.

LEMMA 1. *Let $v_t (t \in T)$ be a family of volumes on V such that $\sum_T v_t(A) < \infty$ for all $A \in V$; then the set function v, defined by $v(A) = \sum_T v_t(A)$ for all $A \in V$, is a volume on V.*

Proof. Let us notice that if S_1 and S_2 are any abstract sets and if a is a mapping from $S_1 \times S_2$ into the extended real numbers, then we have the equality

$$\sup_{S_1 \times S_2} a(s_1, s_2) = \sup_{S_1} \sup_{S_2} a(s_1, s_2) = \sup_{S_2} \sup_{S_1} a(s_1, s_2).$$

From the previous we get

$$\sum_n v(A_n) = \sum_n \Big(\sum_T v_t(A_n)\Big) = \sum_T \Big(\sum_n v_t(A_n)\Big).$$

Since each v_t is a volume on V we have $\sum_n v_t(A_n) = v_t(A)$ yielding $\sum_n v(A_n) = \sum_T v_t(A) = v(A)$.

THEOREM 1. *Let v_t $(t \in T)$ be volumes on the prering V such that $\sum_T v_t(A) < \infty$ for all $A \in V$. Then for the volume $v = \sum_T v_t$ on the prering V we have:*

(a) *The family $N(v)$ of v-null sets is the intersection of the families $N(v_t)$.*

(b) *For every Banach space Y the family $M(v, Y)$ of v-measurable functions is the intersection of the families $M(v_t, Y)$.*

(c) *The family $M(v)$ of v-measurable sets is the intersection of the families $M(v_t)$.*

Proof. Since $v_t(A) \leqslant v(A)$ for all $t \in T$ and $A \in V$ we have from [8], Section 1, Theorem 1, that $N(v)$, $M(v, Y)$, $M(v)$ are contained in $\bigcap_T N(v_t)$, $\bigcap_T M(v_t, Y)$ and $\bigcap_T M(v_t)$ respectively.

LEMMA 2. *If $A \in \bigcap_T N(v_t)$ and if there exists a set $B \in V$ such that $A \subset B$, then $A \in N(v)$.*

Proof. Take an $\varepsilon > 0$, since $\sum_T v_t(B) < \infty$ there exists a finite set $S \subset T$ such that $\sum_{T \setminus S} v_t(B) < \varepsilon/2$. Define the volume w on the prering V by $w = \sum_S v_t$. Since $A \in N(v_t)$ for all $t \in T$ we have from [8], Section 2, Theorem 1, that $A \in N(w)$. Thus there exists a sequence of sets $C_n \in V$ such that $A \subset \bigcup_n C_n$ and $\sum_n w(C_n) < \varepsilon/2$. From the properties of the prering V we may assume that the sets C_n are pair-wise disjoint and that $C_n \subset B$ for all n. We have $\sum_n v(C_n) = \sum_n \big(\sum_S v_t(C_n)\big) + \sum_n \big(\sum_{T \setminus S} v_t(C_n)\big)$. But $\sum_n \big(\sum_{T \setminus S} v_t(C_n)\big) = \sum_{T \setminus S} \big(\sum_n v_t(C_n)\big) \leqslant \sum_{T \setminus S} v_t(B) < \varepsilon/2$ and $\sum_n \big(\sum_S v_t(C_n)\big) = \sum_n w(C_n) < \varepsilon/2$. Therefore we have $\sum_n v(C_n) < \varepsilon$.

Now take a set $A \in \bigcap_T N(v_t)$. In particular we have $A \in N(v_{t_0})$ for any fixed $t_0 \in T$. Therefore from the definition of $N(v_{t_0})$ we have that there exists a sequence $A_n \in V$ such that $A \subset \bigcup_n A_n$. From the properties of the prering V we can assume that the sets A_n are pair-wise disjoint. Let us set $B_n = A \cap A_n$. We have $B_n \in \bigcap_T N(v_t)$ and $B_n \subset A_n$. Thus by Lemma 2 the set B_n is a v-null set. Since $A = \bigcup_n B_n$ we have $A \in N(v)$.

Proof of part (b). Denote by V_σ the family of sets which are unions of countable families of sets from the family V. If $f \in \bigcap_T M(v_t, Y)$, then from the definition of the space $M(v_t, Y)$ we have that there exists a set $B \in V_\sigma$ such that $f(x) = 0$ if $x \notin B$. Since $B = \bigcup_n A_n$ for some sequence of sets $A_n \in V$ and $v(A_n) < \infty$ for all n it is easy to show that there exists a subset $T_0 \subset T$ such that T_0 is at most countable and $v_t(A_n) = 0$ for all $t \notin T_0$ and all n. Let us assume that $T_0 = \{t_1, t_2, \ldots\}$.

Denote by $S_1(Y)$ the first Baire class generated by the family $S(V, Y)$ of simple functions, that is, the set of all functions f from X into the Banach space Y for which there exists a sequence of simple functions $s_n \in S(V, Y)$ such that the sequence of values $s_n(x)$ converges to the value $f(x)$ for all $x \in X$. $S_{i+1}(Y)$, the $(i+1)$-Baire class generated by $S(V, Y)$ is the family of functions which consists of pointwise limits of sequences from the set $S_i(Y)$ $(i = 1, 2, \ldots)$. Notice that each of the Baire classes is linear and if $g \in S_k(R)$, $f \in S_k(Y)$, then $gf \in S_k(Y)$. From [3], Theorem 7, and the construction given in that theorem, it follows that a function f mapping X into Y belongs to the space $M(v, Y)$ if and only if there exists a set $D \in V_{\sigma\delta} \cap N(v)$ such that the function $c_{X \setminus D} f \in S_3(Y)$, where $V_{\sigma\delta}$ denotes the family of sets which are intersections of countable families of sets from V_σ. It is not difficult to see that if $f \in M(v, Y)$ and $f(x) = 0$ for $x \notin B \in V_\sigma$, then we may assume that $D \subset B$. Therefore for each $t_n \in T_0$ there exists a set $A_n \in V_{\sigma\delta} \cap N(v_{t_n})$ such that $A_n \subset B$ and $c_{X \setminus A_n} f \in S_3(Y)$ $(n = 1, 2, \ldots)$. Set $A = \bigcap_n A_n$. Since $A \subset A_n$ for all n we have $A \in N(v_{t_n})$ for all $t_n \in T_0$, but $A \subset B$ and therefore $A \in N(v_t)$ for all $t \notin T_0$. From part (a) of this theorem we get $A \in N(v)$. Define the functions $g_n = c_{X \setminus E_n} f$, where $E_n = A_1 \cap \ldots \cap A_n \in V_{\sigma\delta}$. Notice the identity

$$g_n = c_{X \setminus E_{n-1}} f + c_{E_{n-1}} c_{X \setminus A_n} f$$

for $n > 1$ from which using induction we get $g_n \in S_3(Y)$ $(n = 1, 2, \ldots)$. For each $x \in X$ we have $\lim_n g_n(x) = c_{X \setminus A}(x) f(x)$. This yields $c_{X \setminus A} f \in S_4(Y) \subset M(v, Y)$. But $c_{X \setminus A}(x) f(x) = f(x)$ if $x \notin A$. Therefore $f \in M(v, Y)$ according to [3], Theorem 1.

From [3] we have that the family $M(v)$ of v-measurable sets consists of all sets $A \in X$ satisfying $c_A \in M(v, R)$, where R is the space of real numbers. Thus we have that part (c) of the theorem follows immediately from part (b) and the definition of the space $M(v)$ of v-measurable sets.

If v is any volume on the prering V, then in [3], Section 3, the set function m_v is defined on the family $M(v)$ of v-measurable sets by the formula $m_v(A) = \int c_A \, dv$ for all $A \in M(v)$. In [3] it is proven that m_v is the unique measure on the sigma-ring $M(v)$ being an extension of the volume v from the prering V.

THEOREM 2. *Let v, v_t $(t \in T)$ be volumes on the prering V such that $v(A) = \sum_T v_t(A)$ for all $A \in V$. Then we have*

$$m_v(A) = \sum_T m_{v_t}(A) \quad \textit{for all } A \in M(v).$$

Proof. Let us define the set function m' on $M(v)$ by the formula $m'(A) = \sum_T m_{v_t}(A)$ for all $A \in M(v)$. From the previous theorem we have $M(v) = \bigcap_T M(v_t)$ and therefore m' is well defined. If $A \in V$, then $m_{v_t}(A) = v_t(A)$ for all $t \in T$ yielding $m'(A) = v(A)$ for all $A \in V$. If m' is a measure on $M(v)$, then from [3], Section 3, Theorem 4, (7) we have that $m'(A) = m_v(A)$ for all $A \in M(v)$. Let us now prove that m' is countably additive on $M(v)$. Let $A_n \in M(v)$ be a sequence of pair-wise disjoint sets. Set $A = \bigcup_n A_n$. Proceeding as in the proof of Lemma 1 we have

$$\sum_n m'(A_n) = \sum_n \Big(\sum_T m_{v_t}(A_n) \Big) = \sum_T \Big(\sum_n m_{v_t}(A_n) \Big) = \sum_T m_{v_t}(A) = m'(A).$$

Let (X, W, w) be a volume space. Denote by W_w^c the family of all sets $A \subset X$ such that $c_A \in L_1(w, R)$. Define the set function w^c by means of the formula

$$w^c(A) = \int c_A dw \quad \text{for all } A \in W_w^c.$$

We have that W_w^c are just the measurable sets in $M(w)$ which have finite measure $m_w(A) = w^c(A)$. For properties of the set function w^c see [5]. The volume w^c is an extension of the volume w. The volume w^c is called the *completion of the volume w*.

Let $t \to y_t$ be a function from a set T into a Banach space Y. If $y \in Y$ we shall write $y = \sum_T y_t$ if y is the limit in the space Y of the net $z_S = \sum_S y_t$, where S runs through all finite ~~subsets~~ of T ordered by inclusion.

THEOREM 3. *Let v, v_t $(t \in T)$ be volumes on the prering V such that $v(A) = \sum_T v_t(A)$ for all $A \in V$. Then we have:*

(a) $V_v^c = \{A \in \bigcap_T V_{v_t}^c\colon \sum_T v_t^c(A) < \infty\}$ *and* $v^c(A) = \sum_T v_t^c(A)$ *for all* $A \in V_v^c$.

(b) *For every Banach space Y we have $S(V_v^c, Y) \subset \bigcap_T S(V_{v_t}^c, Y)$ and $\int s dv = \sum_T \int s dv_t$ for all $s \in S(V_v^c, Y)$.*

Proof. If $A \in V_v^c \subset M(v) = \bigcap_T M(v_t)$, then $m_v(A) = v^c(A) < \infty$. Therefore $\sum_T m_{v_t}(A) < \infty$ and so $v_t^c(A) = m_{v_t}(A) < \infty$ for all $t \in T$. This gives $A \in \bigcap_T V_{v_t}^c$ and $v^c(A) = \sum_T v_t^c(A)$. Now if $A \in \bigcap_T V_{v_t}^c \subset \bigcap_T M(v_t) = M(v)$ and $\sum_T v_t^c(A) < \infty$ we have $m_{v_t}(A) = v_t^c(A)$ for all $t \in T$ and $m_v(A) = \sum_T m_{v_t}(A) < \infty$. This gives us $A \in V_v^c$ completing the proof of part (a).

Proof of part (b). The first part of (b) follows from part (a) and the definition of the space $S(V, Y)$ of simple functions generated by a prering V. Take the simple function $s = c_A y$, where $y \in Y$ and $A \in V_v^c$. From the continuity of scalar multiplication in the Banach space Y we have $yv^c(A) = \sum_T yv_t^c(A)$. If we take a simple function $s = \sum_{i=1}^n c_{A_i} y_i \in S(V_v^c, Y)$ we have

$$\int s dv^c = \sum_{i=1}^n y_i v^c(A_i) = \sum_{i=1}^n \Big(\sum_T y_i v_t^c(A_i)\Big) = \sum_T \Big(\sum_{i=1}^n y_i v_t^c(A_i)\Big) = \sum_T \int s dv_t^c.$$

From [4], Section 1, Theorem 1, we have that the spaces $L_1(v, Y)$ and $L_1(v^c, Y)$ coincide and $\int f dv = \int f dv^c$ for all $f \in L_1(v, Y)$. Therefore we have $\int s dv = \sum_T \int s dv_t$ for all $s \in S(V_v^c, Y)$.

For any volume v on the prering V denote by $M(v, [0, \infty])$ the family of non-negative extended measurable functions as defined in [3],

p. 255. Denote by $S^+(V_v^c, R)$ the family of non-negative real-valued functions $s \in S(V_v^c, R)$. In [3], Section 2, is defined the integral $\int f dv$ for $f \in M(v, [0, \infty])$. It is not difficult to see that

$$\int f dv = \sup\left\{\int s dv\colon s \in S^+(V_v^c, R),\, s \leqslant f \text{ } v\text{-a. e.}\right\}.$$

From [3], Section 2, Theorem 3, we have that if $f \in M(v, Y)$, then $|f| \in M(v, [0, \infty])$ and we have the relation $f \in L_1(v, Y)$ if and only if $\int |f| dv < \infty$.

THEOREM 4. *Let v, v_t $(t \in T)$ be volumes on the prering V such that $v(A) = \sum_T v_t(A)$ for all $A \in V$. Then we have:*

(a) *The space $M(v, [0, \infty])$ of extended non-negative v-measurable functions is the intersection of the spaces $M(v_t, [0, \infty])$,*

(b) *We have the equality $\int f dv = \sum_T \int f dv_t \to$ for all functions $f \in M(v, [0, \infty])$.*

Proof of part (a). From [3], Section 4, Theorem 5, we have that a function f from the set X into the interval $[0, \infty]$ belongs to $M(v, [0, \infty])$ if and only if there exists a set $A \in V_\sigma$ such that $f(x) = 0$ if $x \notin A$ and $A \cap f^{-1}(I) \in M(v)$ for any interval of the form $I = (-\infty, a]$. From the identity $M(v) = \bigcap_T M(v_t)$ we get part (a).

Proof of part (b). For any $f \in M(v, [0, \infty])$ it is not difficult to see that there exists a sequence $s_n \in S^+(V_v^c, R)$ such that the sequence of values $s_n(x)$ increasingly converges to the value $f(x)$ for all $x \in X$. From [3], Section 2, Theorem 3, (4), we have $\lim_n \int s_n dv = \int f dv$ and $\lim_n \int s_n dv_t = \int f dv_t$ for all $t \in T$. Since $\int s_n dv = \sum_T \int s_n dv_t$ $(n = 1, 2, \ldots)$ to complete the proof of part (b) we need only show that

$$\lim_n \sum_T \int s_n dv_t = \sum_T \int f dv_t.$$

For each positive integer n there exists a set $A_n \in V_v^c$ such that $s_n(x) = 0$ if $x \notin A_n$. Since $v^c(A_n) < \infty$ there exists a subset $T_0 \subset T$ satisfying T_0 is at most countable and $v_t^c(A_n) = 0$ for all $t \notin T_0$ and all n. Let us assume that $T_0 = \{t_1, t_2, \ldots\}$. Let us now show

$$\lim_n \sum_i \int s_n dv_{t_i} = \sum_i \int f dv_{t_i}.$$

Let w be the volume on the prering $V = \{\emptyset, \{j\}\colon j \in J\}$, where J is the set of positive integers and $w\{j\} = 1$ for all $j \in J$. As we saw the sequence of numbers $s_n(x)$ increasingly converges to the value $f(x)$ for all $x \in X$, the sequence of values $\int s_n dv_t$ increasingly converges to the value $\int f dv_t$ for all $t \in T$ and the sequence $\int s_n dv$ increasingly converges to the value

$\int f dv$. Therefore for the non-negative valued functions h_n, h on J given by the formulas $h(j) = \sum_i c_{\{i\}}(j) \int f dv_{t_i}$, $h_n(j) = \sum_i c_{\{i\}}(j) \int s_n dv_{t_i}$ for all $j \in J$ and all n we have that the sequence of numbers $h_n(j)$ increasingly converges to $h(j)$ for all $j \in J$. But $h_n, h \in M(w, \bar{R})$ for all n and therefore from the Monotone Convergence theorem for extended real valued w-measurable functions (see Theorem 3, Section 2, [3]) we have $\lim_n \int h_n dw = \int h dw$. Let us notice that

$$\int h_n dw = \sum_i \int s_n dv_{t_i} \quad \text{and} \quad \int h dw = \sum_i \int f dv_{t_i},$$

thus part (b) is proven.

If v is any volume on the prering V and $(Y, |\ |)$ is any Banach space, then for each $f \in L_1(v, Y)$ we set $\|f\|_v = \int |f| dv$.

THEOREM 5. *Let v, v_t $(t \in T)$ be volumes on the prering V such that $v(A) = \sum_T v_t(A)$ for all $A \in V$. Then we have:*

(a) $L_1(v, Y) = \{f \in \bigcap_T L_1(v_t, Y): \sum_T \|f\|_{v_t} < \infty\}$ *and* $\|f\|_v = \sum_T \|f\|_{v_t}$ *for all* $f \in L_1(v, Y)$,

(b) $\int f dv = \sum_T \int f dv_t$ *for all* $f \in L_1(v, Y)$.

Proof of part (a). From [8], Section 1, Theorem 1, we get $L_1(v, Y) \subset \bigcap_T L_1(v_t, Y)$ for every Banach space Y since the volume v dominates the volume v_t for every $t \in T$. If $f \in L_1(v, Y)$ we have $|f| \in M(v, [0, \infty])$ and $\|f\|_v = \int |f| dv < \infty$. Since $M(v, [0, \infty]) = \bigcap_T M(v, [0, \infty])$ and $\int |f| dv = \sum_T \int |f| dv_t$ from Theorem 4, we get $\sum_T \|f\|_{v_t} = \|f\|_v < \infty$ and therefore $f \in \bigcap_T L_1(v_t, Y)$. Now let us take a function $f \in \bigcap_T L_1(v_t, Y)$ such that $\sum_T \|f\|_{v_t} < \infty$. Since $M(v, Y) = \bigcap_T M(v_t, Y)$ and $L_1(v_t, Y) \subset M(v_t, Y)$ we get $f \in M(v, Y)$ and therefore $|f| \in M(v, [0, \infty])$. From Theorem 4 we get that $\int |f| dv = \sum_T \int |f| dv_t < \infty$ and therefore $f \in L_1(v, Y)$. Thus part (a) is proven.

Proof of part (b). Let v be any volume on the prering V; then a sequence of functions s_n is called *v-basic* if there exists a sequence of simple functions $h_n \in S(V, Y)$ and a constant $M > 0$ such that $s_n = h_1 + \ldots + h_n$ and $\int |h_i| dv \leqslant M 4^{-i}$ for $i = 1, 2, \ldots$ From [1] we have that a function f mapping X into Y belongs to $L_1(v, Y)$ if and only if there exists a v-basic sequence s_n such that the sequence of values $s_n(x)$ converges to the value $f(x)$ v-almost everywhere (v-a.e.). If $s \in S(V, Y)$ is given by $s = c_{A_1} y_1 + \ldots + c_{A_n} y_n$, then from [1] we have $\int |s| dv = |y_1| v(A_1) + \ldots + |y_n| v(A_n)$. Now if we take v to be the volume $v = \sum_T v_t$, then since $v_t(A) \leqslant v(A)$ for all $A \in V$ we have that a v-basic sequence is also a v_t-basic sequence for all $t \in T$. Since $N(v) = \bigcap_T N(v_t)$ we have that if s_n is a v-basic sequence, for $v = \sum_T v_t$, convergent v-a.e. to f, then this same sequence s_n is v_t-basic and convergent v_t-a.e. to the function f for all $t \in T$. This gives us $\lim_n \int s_n dv = \int f dv$ and $\lim_n \int s_n dv_t = \int f dv_t$ for all $t \in T$. From Theorem 3 we have $\int s_n dv = \sum_T \int s_n dv_t$ for all n. To complete the proof of part (b) let us show

$$\lim_n \sum_T \int s_n dv_t = \sum_T \int f dv_t.$$

As in the proof of Theorem 4 we have that for each positive integer n there exists a set $A_n \in V_v^c$ such that $s_n(x) = 0$ if $x \notin A_n$ and therefore there exists a finite or countable subset $T_0 \subset T$ such that $v_t^c(A_n) = 0$ for all $t \notin T_0$ and all n. Let us assume that $T_0 = \{t_1, t_2, \ldots\}$. We will now show

$$\lim_n \sum_i \int s_n dv_{t_i} = \sum_i \int f dv_{t_i}.$$

Let w be the volume defined as in the proof of Theorem 4. Consider the functions h, h_n $(n = 1, 2, \ldots)$ on J defined by

$$h(j) = \sum_i c_{\{i\}}(j) \int f dv_{t_i}, \qquad h_n(j) = \sum_i c_{\{i\}}(j) \int s_n dv_{t_i}$$

for all $j \in J$. Using the fact that $\|s_n\|_v = \sum_T \|s_n\|_{v_t}$ $(n = 1, 2, \ldots)$ $\|f\|_v = \sum_T \|f\|_{v_t}$ and the Dominated Convergence Theorem we get $h_n, h \in L_1(w, Y)$ $(n = 1, 2, \ldots)$, $\int h_n dw = \sum_i \int s_n dv_{t_i}$ $(n = 1, 2, \ldots)$ and $\int h dw = \sum_i \int f dv_{t_i}$. To complete the proof of part (b) let us show $\lim_n \int h_n dw = \int h dw$. We have

$$\left| \int h_n dw - \int h dw \right| = \left| \sum_i \int (s_n - f) dv_{t_i} \right| \leq \sum_i \|s_n - f\|_{v_{t_i}}.$$

But from part (a) $\sum_i \|s_n - f\|_{v_{t_i}} = \|s_n - f\|_v$. Since s_n is a v-basic sequence convergent to f v-a. e. from [1], Lemma 4 we have $\lim_n \|s_n - f\|_v = 0$.

If v is any volume on a prering V, and Y is a Banach space, then for functions $f \in L_p(v, Y)$ we set $\|f\|_{vp} = \left(\int |f|^p dv\right)^{1/p}$ for all numbers p satisfying $1 < p < \infty$. Since we will be considering different volumes, for functions $f \in L_\infty(v, Y)$ let us denote the seminorm $\|f\|_\infty$ as defined in the introduction by $\|f\|_{v\infty}$.

THEOREM 6. *Let v, $v_t (t \in T)$ be volumes on the prering V such that $v(A) = \sum_T v_t(A)$ for all $A \in V$. Then for any Banach space Y we have:*

(a) $L_p(v, Y) = \{f \in \bigcap_T L_p(v_t, Y) \colon \sum_T \|f\|_{v_t p}^p < \infty\}$ *and* $\|f\|_{vp} = (\sum_T \|f\|_{v_t p}^p)^{1/p}$ *for* $1 < p < \infty$.

(b) $L_\infty(v, Y) = \{f \in \bigcap_T L_\infty(v_t, Y) \colon \sup_T \{\|f\|_{v_t \infty}\} < \infty\}$ *and* $\|f\|_{v\infty} = \sup_T \{\|f_{v_t \infty}\|\}$.

Proof. The proof of this theorem is similar to the corresponding proofs of Theorems 6 and 5 in [8], Section 2.

2. Relations to Lebesgue, Bochner and Dunford approaches. In [13] references are made to the relations that exist between the Bochner–Pettis, Dunford and Bogdanowicz's approach to the theory of integration. In the following we will discuss the principal papers, where such relations between the various approaches are treated and point out explicitly these relations. We will then be able to formulate the results of Section 1 in the classical setting.

From the results of [3] we have that the class of spaces of Lebesgue–Bochner summable functions generated by a sigma finite complete measure coincides with the class of spaces generated by a volume. Also in [3] the measurable functions corresponding to the approach to integration based on volumes are introduced and the classical relations between weak and strong measurability and summability (due to Pettis) are established. In fact, one can prove that if m_1, m_2 are two complete measures, then for every Banach space Y we have that the space of Lebesgue–Bochner summable functions with values in Y generated by the measure m_1 is equal to the space of corresponding summable functions generated by the measure m_2 and the integrals $\int f dm_1$ and $\int f dm_2$ coincide if and only if the finite part of the measures coincide. A detailed discussion of similar relations between the classical Lebesgue integration theory and the theory of integration generated by a volume have been established in [7].

The transition from the theory of integration generated by volumes to the theory of integration of Dunford generated by a countably additive measure defined on a sigma algebra of sets on the basis of the results contained in [1] and [3] is the following. If m is a positive extended real-valued measure on a sigma algebra as in Dunford and Schwartz [9], p. 144–155, and if v is the restriction of the measure m to the prering consisting of all sets of finite measure, then v is a volume in our sense and the Dunford construction of the space $L_1^0(m, Y)$ of Lebesgue–Bochner summable functions with values in the Banach space Y as defined on p. 119 of [9] yields the space $L(v, Y)$ obtained by means of the construction in [1] and the Bochner integrals coincide on them.

It is also easy to see that the space $L_1^0(w, Y)$ generated by a complex-valued measure w on a sigma algebra coincides with the space of such functions generated by its variation $|w|$ and thus one may assume that the measure generating the space of summable functions is positive. In this case the trilinear integral $\int u_0(f, dw)$ developed in [1] is equal to the integral of Dunford $\int f dw$, where the bilinear operator u_0 is defined by the formula $u_0(y, z) = zy$ for all y in the Banach space Y and all scalars z.

Conversely, if v is any volume and m is the complete measure obtained from it in [3], then denote by M_v the sigma algebra of all subsets A of X such that $A \cap B$ yields a v-summable set for every v-summable set B. For each set A in M_v let $m'(A) = m(A)$ if A is v-summable and $m'(A) = \infty$ otherwise. One can show that m' is a measure and that the space $L_1^0(m', Y)$ coincides with the space $L(v, Y)$ developed in [1] and the integrals are equal.

The approach to integration based on volumes yields a number of new results which were not noticed in constructions employing measures. For example, we have that the class of operators obtained by means of the constructions in [1] and [3] contains as a proper subset the integral operator $\int f dw$ as defined in [9], p. 112, for functions f with values in the Banach space Y, where the complex valued measure w is defined on a sigma algebra. To see this take for example any countably additive complex valued set function q defined on a prering V of a space X such that its variation $|q|$ is infinite on X. Then the function q cannot be extended to a measure on the smallest sigma algebra containing V and thus the integral $\int f dq$ constructed in [9] cannot be defined, where as by means of the construction in [1] it can.

As is well known, the result of Kakutani [12] on abstract L-spaces shows the importance of the linear-lattice structure in the theory of integration. Let us also notice that the notion of a prering is a natural one in this context for the following reason. If V is any family of subsets of a space X and $S(V, R)$ the family of real-valued simple functions $s = r_1 c_{A_1} + \ldots + r_k c_{A_k}$, where A_i is a finite family of pair-wise disjoint sets from V, then V is a prering if and only if $S(V, R)$ is a linear lattice. In fact, even the requirement that $S(V, R)$ is a group under addition is equivalent to V being a prering. (See [6], p. 206.)

If v is any volume on the prering V, then as in Section 1 let us denote by m_v the measure defined on the family $M(v)$ of v-measurable sets by the formula

$$m_v(A) = \int c_A \, dv \quad \text{for all sets } A \in M(v),$$

and by v_c the volume on the prering $V_c = \{A \in M(v) \colon m_v(A) < \infty\}$ given by the formula $v_c(A) = m_v(A)$ for all sets $A \in V_c$.

Let m be a non-negative extended real valued measure on a sigma-ring M of a space X. The triple (X, M, m) will be called a *measure space*.

Denote by $L_m(R)$ the space of finite real-valued Lebesgue summable functions without identification of functions equal m-a. e. (see [7], p. 281–289). Let Ω_m denote the Lebesgue completion of the measure m (see [10], p. 55 or [7], p. 291).

For any family W of subsets of the space X let $\sigma(W)$ be the smallest sigma-ring of subsets of X containing the family W.

LEMMA 1. *Let (X, M, m) be a measure space. If W is a prering of subsets of X such that $W \subset V \subset \sigma(W)$, where $V = \{A \in M: m(A) < \infty\}$, then for the volume w, defined on the prering W by the formula $w(A) = m(A)$ for all $A \in W$, we have $L_1(w, R) = L_{\Omega_m}(R)$ and $\int f dw = \int f d\Omega_m$ for all $f \in L_1(w, R)$.*

Proof. Since $W \subset V \subset \sigma(W)$ we have $\sigma(W) = \sigma(V)$. Define the volume v on the prering V by $v(A) = m(A)$ for all $A \in V$. By [7], p. 294, the volumes v and w have unique extensions to measures m_1 and m_2 respectively on $\sigma(V) = \sigma(W)$ given explicitly by $m_1(A) = m_v(A)$ and $m_2(A) = m_w(A)$ for all $A \in \sigma(W)$. Since the volume v is an extension of the volume w we have $m_w(A) = m_v(A)$ for all sets $A \in \sigma(W)$. If $A \in V$, then $m_w(A) = m_v(A) < \infty$ therefore $A \in W_c$ and $v(A) = w_c(A)$. Thus w_c is an extension of the volume v. Since the volume v is an extension of the volume w we also have that the volume v_c is an extension of the volume w. From [4], Section 1, Theorem 2, we get $v_c = w_c$, and from [4], Section 1, Theorems 2 and 1, we have $L_1(v, R) = L_1(w, R)$ and $\int f dw = \int f dv$ for all $f \in L_1(v, R)$. But from [7], Theorem 2, p. 291, we have $L_{\Omega_m}(R) = L_1(v, R)$ and $\int f d\Omega_m = \int f dv$ for all $f \in L_1(v, R)$. Therefore we have

$$L_{\Omega_m}(R) = L_1(w, R) \quad \text{and} \quad \int f d\Omega_m = \int f dw \quad \text{for all } f \in L_1(w, R).$$

For convenience of notation in the remainder of this section, let us adopt the convention that if (X, M, m) is a measure space and (X, M_t, m_t) a family of measure spaces for t in an index set T, then we set $\Omega = \Omega_m$ and $\Omega_t = \Omega_{m_t}$ for all $t \in T$, where Ω_m, Ω_{m_t} denote the Lebesgue completion of the measures m and m_t.

THEOREM 1. *Let (X, M, m) be a fixed measure space and (X, M_t, m_t) a family of measure spaces, where the parameter t changes in an index set T. Assume that W is a prering of subsets of the space X such that $W \subset V = \{A \in M: m(A) < \infty\}$ and $m(A) = \sum_T m_t(A)$ for all $A \in W$. If the family $\sigma(W)$ contains each of the families V, $V_t = \{A \in M_t: m_t(A) < \infty\}$ $(t \in T)$, then*

$$L_\Omega(R) = \Big\{f \in \bigcap_T L_{\Omega_t}(R): \sum_T \int |f| d\Omega_t < \infty\Big\}$$

and

$$\int f d\Omega = \sum_T \int f d\Omega_t \quad \textit{for all } f \in L_\Omega(R).$$

Proof. From Lemma 1 for the volumes $w, w_t (t \in T)$ defined on W by $w(A) = m(A)$, $w_t(A) = m_t(A)$ for all $A \in W$ we have $L_\Omega(R) = L_1(w, R)$, $\int f d\Omega = \int f dw$ for all $f \in L_1(w, R)$ and $L_{\Omega_t}(R) = L_1(w_t, R)$, $\int f d\Omega_t = \int f dw_t$ for all $f \in L_1(w_t, R)$ and all $t \in T$. The theorem now follows immediately from Theorem 5, Section 1.

THEOREM 2. *Let $X = R$ be the space of real numbers, and m_t be a family of non-negative Borel-regular measures on the family B of Borel sets of R for t in a parameter set T. For each $t \in T$ let g_t be the left continuous increasing real-valued function on R satisfying $g_t(0) = 0$ and $m_t[a, b) = g_t(b) - g_t(a)$ for all intervals $[a, b)$ with $a, b \in R$ and $a < b$. Define the function g on R by $g(x) = \sum_T g_t(x)$ for all $x \in R$. If $g(x) < \infty$ for all $x \in R$ and D is the set of discontinuities of the function g, then for the Borel-regular measure m on B satisfying $m[a, b) = g(b) - g(a)$ for all $a, b \in R \setminus D$, $a < b$, we have:*

(i) *The Lebesgue completion B_m of the family B of Borel sets relative to the measure m is equal to the intersection over all $t \in T$ of the Lebesgue completions B_{m_t} of the family B relative to the measures m_t and $\Omega(A) = \sum_T \Omega_t(A)$ for all $A \in B_m$.*

(ii) *The space $L_\Omega(R) = \{f \in \bigcap_T L_{\Omega_t}(R): \sum_T \int |f| d\Omega_t < \infty\}$ and $\int f d\Omega = \sum_T \int f d\Omega_t$ for all $f \in L_\Omega(R)$.*

Proof. From [11], p. 332, we have that the measures m_t are Borel regular if and only if $m_t[a, b) < \infty$ for all intervals $[a, b)$. Since $g_t(0) = 0$ for all $t \in T$ it follows that the measure m defined on the family B of Borel sets by the formula

$$m(A) = \sum_T m_t(A) \quad \text{for all } A \in B$$

is Borel regular if and only if $\sum_T g_t(x) < \infty$ for all $x \in R$, but by assumption $g(x) < \infty$ for all $x \in R$. Let D be the set of discontinuities of the function g and let W be the prering

$$W = \{[a, b): a, b \in R \setminus D, a < b\}.$$

Since D is countable it follows that the smallest sigma-ring of subsets of R containing W is the family of Borel sets B. It is easy to see that the function v defined on the prering W by the formula $v[a, b) = g(b) - g(a)$ for all $[a, b) \in W$ is a volume on W and $v(A) = \sum_T v_t(A)$ for all $A \in W$, where v_t is the volume on W given by $v_t(A) = m_t(A)$ for all $A \in W$. If m' denotes the unique measure defined on the Borel sets B which is an extension of the volume v, then we have $m' = m$ since $m(A) = \sum_T v_t(A) = v(A)$ for all $A \in W$. Since we noted that $\sigma(W) = B$, from Theorem 1 of this Section we get part (ii). Part (i) follows from Theorems 1 and 2 of Section 1, and from [7] Theorem 1, p. 294.

Let (X, M, m) be a measure space with M a sigma-algebra. Set $V = \{A \in M: m(A) < \infty\}$, and define the volume v on V by $v(A) = m(A)$ for all $A \in V$. As we noted in the first part of Section 2 the space $L_1^0(m, Y)$ of Lebesgue–Bochner summable functions with values in the Banach space Y obtained by means of the construction in [9] is equal to the space $L_1(v, Y)$ obtained in [1] and the integrals $\int f dm$ and $\int f dv$ coincide. We now easily get:

THEOREM 3. *Let (X, M, m) be a fixed measure space and (X, M_t, m_t) a family of measure spaces with M, M_t sigma-algebras for all t in an index set T. Assume that W is a prering of subsets of the space X such that $W \subset V = \{A \in M: m(A) < \infty\}$ and $m(A) = \sum_T m_t(A)$ for all $A \in W$. If the sigma-ring $\sigma(W)$ contains each of the corresponding families V, $V_t = \{A \in M_t: m_t(A) < \infty\}$ $(t \in T)$, then*

$$L_1^0(m, Y) = \Big\{f \in \bigcap_T L_1^0(m_t, Y): \sum_T \int |f| dm_t < \infty\Big\}$$

and

$$\int f dm = \sum_T \int f dm_t \quad \text{for all } f \in L_1^0(m, Y).$$

Proof. Define the volumes w, w_t $(t \in T)$ on the prering W by $w(A) = m(A)$, $w_t(A) = m_t(A)$ for all $A \in W$. From the proof of Lemma 1 of this section we have for the volumes v, v_t $(t \in T)$ defined on V, V_t by $v(A) = m(A)$ for all $A \in V$, and $v_t(A) = m_t(A)$ for all $A \in V_t$ respectively, that the completions agree, that is, $w_c = v_c$, and $(w_t)_c = (v_t)_c$ $(t \in T)$. From [4], Section 1, Theorems 1 and 2, we get, for every Banach space Y, $L_1(w, Y) = L_1(v, Y)$ and $\int f dw = \int f dv$ for all $f \in L_1(w, Y)$ and the same for w_t and v_t for all $t \in T$. Thus from Theorem 5 of Section 1 the theorem follows.

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