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Two estimates for continuous functions

1. Notation. Let M and N be convex N -functions complementary in the sense of Young and let M_{-1} and N_{-1} be the inverse functions to M and N respectively. Let L_M^* be the Orlicz space of all measurable, 2π -periodic functions f such that

$$\int_0^{2\pi} M(kf(t)) dt < \infty \quad \text{for a } k > 0.$$

As well known, two equivalent norms may be defined in L_M^* , namely

$$\|f\|_M = \|f(t)\|_M = \sup \left\{ \left| \int_0^{2\pi} f(t)g(t) dt \right| : \int_0^{2\pi} N(g(t)) dt \leq 1 \right\},$$

$$\|f\|_{(M)} = \|f(t)\|_{(M)} = \inf \left\{ k > 0 : \int_0^{2\pi} M\left(\frac{f(t)}{k}\right) dt \leq 1 \right\}.$$

The function N is said to satisfy the condition (Δ_2) , if there exists a number $\kappa > 0$ such that $N(2u) \leq \kappa N(u)$ for all $u > 0$. It is easily seen that if N satisfies the condition (Δ_2) , then there exists an increasing function $\kappa(u) > 0$ defined for $u \geq 0$ such that $N(uv) \leq \kappa(u)N(v)$ for all $u, v \geq 0$ (see [3], p. 16-22, 35, 85, 95).

Let $f \in L_M^*$, and let $F(t)$, $G(t)$ be two non-negative, non-decreasing functions defined in $\langle 0, 2\pi \rangle$. The following conditions will be of importance:

$$(1) \quad \|f(t+h) - f(t)\|_M \leq C_1 F(h),$$

$$(2) \quad \|[f(x+v) - f(x+t)]\chi_0^v(t)\|_M \leq C_2 G(v)$$

for $h, v \in \langle 0, 2\pi \rangle$, $x \in (-\infty, \infty)$, where C_1, C_2 are positive constants and $\chi_0^v(t)$ is the characteristic function of the interval $\langle 0, v \rangle$. We set

$$\Omega(t) = \frac{F(t)}{2tN_{-1}(1/t)} + \int_0^t \frac{G(v)}{v^2 N_{-1}(1/v)} dv \quad \text{for } t \in \langle 0, 2\pi \rangle.$$

2. Generalization of a theorem of Hardy and Littlewood. This theorem may be formulated as

THEOREM 1. *If f is a continuous, 2π -periodic function, satisfying conditions (1) and (2), then*

$$\max_x |f(x+h) - f(x)| \leq C_3 \Omega(h/2) \quad \text{for } h \in (0, 4\pi),$$

with $C_3 = \max(4C_1, 4C_2)$.

Proof. Clearly,

$$|f(x+h) - f(x)| \leq |f(x+h) - f(x+h/2)| + |f(x+h/2) - f(x)|.$$

We shall estimate the second term at the right-hand side of this inequality; the first term is estimated in the same manner. Starting with the identity

$$f(x) = \frac{1}{\lambda} \int_0^\lambda f(x+t) dt - \int_0^\lambda \frac{1}{v^2} \left\{ \int_0^v [f(x+v) - f(x+t)] dt \right\} dv$$

(see [2], p. 129), we have

$$\begin{aligned} |f(x+h/2) - f(x)| &\leq \frac{1}{\lambda} \int_0^\lambda |f(x+h/2+t) - f(x+t)| dt + \\ &+ \int_0^\lambda \left(\frac{1}{v^2} \int_0^v |f(x+h/2+v) - f(x+h/2+t)| dt \right) dv + \\ &+ \int_0^\lambda \left(\frac{1}{v^2} \int_0^v |f(x+v) - f(x+t)| dt \right) dv = J_1 + J_2 + J_3, \end{aligned}$$

whenever $\lambda \in (0, 2\pi)$, $x \in (-\infty, \infty)$. By Hölder's inequality and (1),

$$J_1 \leq \frac{1}{\lambda} \|f(x+h/2+t) - f(x+t)\|_M \| \chi_0^\lambda(t) \|_N \leq 2C_1 \frac{F(h/2)}{\lambda N_{-1}(1/\lambda)}$$

(see [3], p. 25, 89, 91).

Analogously, the Hölder inequality and (2) lead to

$$J_2 \leq 2C_2 \int_0^\lambda \frac{G(v)}{v^2 N_{-1}(1/v)} dv; \quad J_3 \leq 2C_2 \int_0^\lambda \frac{G(v)}{v^2 N_{-1}(1/v)} dv.$$

For example,

$$J_3 \leq \int_0^\lambda \frac{1}{v^2} \|[f(x+v) - f(x+t)]\chi_0^v(t)\|_M \|\chi_0^v(t)\|_N dv$$

$$\leq 2C_2 \int_0^\lambda \frac{G(v)}{v^2 N_{-1}(1/v)} dv.$$

Putting $\lambda = h/2$, we get at once the desired assertion.

In particular, if $M(u) = u^p/p$, $F(t) = G(t) = t^a$ ($p > 1$, $ap > 1$), then

$$q^{1/a} \Omega(h/2) = \frac{(h/2)^a}{h(2/h)^{1/a}} + \int_0^{h/2} \frac{v^a}{v^2(1/v)^{1/a}} dv = h^{\alpha-1/p}$$

$$\left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

Hence Theorem 1 gives

$$f(x+h) - f(x) = O(h^{\alpha-1/p}),$$

uniformly in x , and we obtain a theorem due to Hardy and Littlewood [1] (see also [2], p. 131).

3. Generalization of Žižiashvili's theorem. As in [4], p. 77 and 94, let us denote by $\sigma_n^\alpha(x)$ ($\alpha > -1$) the Cesàro means of the Fourier series of f , with the kernel

$$K_n^\alpha(t) = \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} D_\nu(t) / A_n^\alpha.$$

We shall prove the following

THEOREM 2. *Suppose that the 2π -periodic function $f(t)$ is continuous in $(-\infty, \infty)$, satisfies conditions (1)-(2),*

$$(3) \quad nF(\pi/n) \geq \varrho_1 \quad (n = 1, 2, \dots), \text{ where } \varrho_1 = \text{const} > 0,$$

and that $\Omega(t)$ is non-decreasing in $\langle 0, \eta \rangle$ for some positive $\eta \leq 2\pi$. Consider a function $N(u)$ satisfying the condition (Δ_2) and a number $\beta \in (0, 1)$. Then if

$$(4) \quad \kappa_{-1} \left(\frac{1}{3\pi} \left\{ \int_2^{n+1} N \left(\frac{u^{1-\beta}}{\pi^{1-\beta}} \right) \frac{du}{u(u-1)} \right\}^{-1} \right) \geq \varrho_2 \frac{N_{-1}(n/\pi)}{n^{1-\beta}}$$

$$(n = 2, 3, \dots),$$

$$(5) \quad \frac{N_{-1}(n/\pi)}{n^{1-\beta}} \leq \varrho_3 \quad (n = 1, 2, \dots),$$

where q_2, q_3 are positive constants, there exists a constant $C_4 > 0$ such that

$$\max_x |\sigma_n^{-\beta}(x) - f(x)| \leq C_4 \left\{ \Omega\left(\frac{3\pi}{2n}\right) + \frac{1}{n} \int_{\pi/n}^{\pi} \frac{\Omega(t/2)}{t^2} dt \right\}$$

for $n = 1, 2, \dots$

Proof. Writing

$$\varphi(x, t) = f(x+t) - 2f(x) + f(x-t),$$

we have

$$\sigma_n^{-\beta}(x) - f(x) = \frac{1}{\pi} \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right) \varphi(x, t) K_n^{-\beta}(t) dt = \frac{1}{\pi} (P_n(x) + Q_n(x)).$$

It is easily seen ([4], p. 77),

$$|K_n^{-\beta}(t)| \leq 2n/(1-\beta) \quad \text{for } n = 1, 2, \dots \text{ and all } t.$$

Hence, by Theorem 1,

$$|P_n(x)| \leq \frac{2n}{1-\beta} \int_0^{\pi/n} |\varphi(x, t)| dt \leq \frac{4C_3 n}{1-\beta} \int_0^{\pi/n} \Omega\left(\frac{t}{2}\right) dt.$$

Consequently,

$$|P_n(x)| \leq \frac{4\pi C_3}{1-\beta} \Omega\left(\frac{\pi}{2n}\right) \quad \text{if } \frac{\pi}{2n} \leq \eta, x \in (-\infty, \infty).$$

Further ([4], p. 95),

$$K_n^{-\beta}(t) = \frac{\sin\{(n + \frac{1}{2} - \frac{1}{2}\beta)t + \frac{1}{2}\pi\beta\}}{A_n^{-\beta}(2\sin\frac{1}{2}t)^{1-\beta}} + \frac{2\theta\beta}{n(2\sin\frac{1}{2}t)^2} = \Phi_n(t) + \Psi_n(t) \quad (|\theta| \leq 1).$$

Therefore

$$Q_n(x) = \int_{\pi/n}^{\pi} \varphi(x, t) \Phi_n(t) dt + \int_{\pi/n}^{\pi} \varphi(x, t) \Psi_n(t) dt = I_1 + I_2.$$

Next, it can easily be observed that

$$\begin{aligned} I_1 &= \frac{\sin\frac{1}{2}\pi\beta}{A_n^{-\beta}} \int_{\pi/n}^{\pi} \varphi(x, t) w(t) \cos nt dt + \frac{\cos\frac{1}{2}\pi\beta}{A_n^{-\beta}} \int_{\pi/n}^{\pi} \varphi(x, t) w(t) \sin nt dt + \\ &+ \frac{\cos\frac{1}{2}\pi\beta}{A_n^{-\beta}} \int_{\pi/n}^{\pi} \varphi(x, t) \bar{w}(t) \cos nt dt - \frac{\sin\frac{1}{2}\pi\beta}{A_n^{-\beta}} \int_{\pi/n}^{\pi} \varphi(x, t) \bar{w}(t) \sin nt dt, \end{aligned}$$

where

$$w(t) = \frac{\cos(\frac{1}{2} - \frac{1}{2}\beta)t}{(2\sin\frac{1}{2}t)^{1-\beta}}, \quad \bar{w}(t) = \frac{\sin(\frac{1}{2} - \frac{1}{2}\beta)t}{(2\sin\frac{1}{2}t)^{1-\beta}}.$$

Considering the integral

$$H_n(x) = \int_{\pi/n}^{\pi} \varphi(x, t) w(t) \cos nt dt,$$

we have ([5], p. 492-494)

$$\begin{aligned} 2H_n(x) &= \int_{\pi/n}^{\pi} \{\varphi(x, t) - \varphi(x, t + \pi/n)\} w(t) \cos nt dt + \\ &+ \int_{\pi/n}^{\pi} \varphi(x, t + \pi/n) \{w(t) - w(t + \pi/n)\} \cos nt dt + \\ &+ \int_{\pi - \pi/n}^{\pi} \varphi(x, t + \pi/n) w(t + \pi/n) \cos nt dt - \\ &- \int_0^{\pi/n} \varphi(x, t + \pi/n) w(t + \pi/n) \cos nt dt = \sum_{i=1}^4 R_{n,i}(x). \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} |R_{n,1}(x)| &\leq \int_{\pi/n}^{\pi} \left| \varphi(x, t) - \varphi\left(x, t + \frac{\pi}{n}\right) \right| \left(\frac{\pi}{2t}\right)^{1-\beta} dt \\ &\leq \left\| \varphi(x, t) - \varphi\left(x, t + \frac{\pi}{n}\right) \right\|_{\mathcal{M}} \left\| \left(\frac{\pi}{2t}\right)^{1-\beta} \chi_{\pi/n}^{\pi}(t) \right\|_{\mathcal{N}} \\ &\leq 4 \left(\frac{\pi}{2}\right)^{1-\beta} \left\| f\left(t + \frac{\pi}{n}\right) - f(t) \right\|_{\mathcal{M}} \left\| \frac{1}{t^{1-\beta}} \chi_{\pi/n}^{\pi}(t) \right\|_{\mathcal{N}}. \end{aligned}$$

In order to estimate the last factor, let us first remark that

$$\begin{aligned} \int_{\pi/n}^{\pi} N\left(\frac{1}{t^{1-\beta}k}\right) dt &\leq \sum_{m=2}^n \frac{\pi}{m(m-1)} N\left(\frac{m^{1-\beta}}{\pi^{1-\beta}k}\right) \\ &\leq \kappa \left(\frac{1}{k}\right) \sum_{m=2}^n \frac{\pi}{m(m-1)} N\left(\frac{m^{1-\beta}}{\pi^{1-\beta}}\right) \leq 3\pi\kappa \left(\frac{1}{k}\right) \int_2^{n+1} N\left(\frac{u^{1-\beta}}{\pi^{1-\beta}}\right) \frac{du}{u(u-1)}. \end{aligned}$$

Consequently,

$$\left\| \frac{1}{t^{1-\beta}} \chi_{\pi/n}^{\pi}(t) \right\|_{(\mathcal{N})} \leq 1/\kappa_{-1} \left(\frac{1}{3\pi} \left\{ \int_2^{n+1} N\left(\frac{u^{1-\beta}}{\pi^{1-\beta}}\right) \frac{du}{u(u-1)} \right\}^{-1} \right) \leq \frac{n^{1-\beta}}{\varrho_2 N_{-1}(n/\pi)}$$

in view of (4).

Applying condition (1) we obtain

$$n^\beta |R_{n,1}(x)| \leq 4 \left(\frac{\pi}{2}\right)^{1-\beta} \frac{C_1}{\varrho_2} \frac{nF(\pi/n)}{N_1(n/\pi)},$$

uniformly in $x \in (-\infty, \infty)$. Similarly we may evaluate $n^\beta R_{n,3}(x)$. The expression $n^\beta R_{n,4}(x)$ is estimated in the same manner as $P_n(x)$ and

$$(6) \quad n^\beta |R_{n,2}(x)| = O\left(\Omega\left(\frac{3\pi}{2n}\right) + \frac{1}{n} \int_{\pi/n}^\pi \Omega(t/2) \frac{1}{t^2} dt\right).$$

For example, we shall deduce the estimates for $R_{n,3}(x)$ and $R_{n,2}(x)$. The term $R_{n,3}(x)$ can be rewritten in the form

$$\begin{aligned} R_{n,3}(x) &= \int_{\pi-\pi/n}^\pi \{\varphi(x, t + \pi/n) - \varphi(x, t)\} w(t + \pi/n) \cos nt dt - \\ &\quad - \int_{\pi-\pi/n}^\pi \varphi(x, t) w(t + \pi/n) \cos nt dt = S_1(x) + S_2(x). \end{aligned}$$

If $n \geq 2$, we have

$$\begin{aligned} |S_1(x)| &\leq \int_{\pi-\pi/n}^\pi |\varphi(x, t + \pi/n) - \varphi(x, t)| \left\{ \frac{3\pi}{2\sqrt{2}(t + \pi/n)} \right\}^{1-\beta} dt \\ &\leq \left(\frac{3\pi}{2\sqrt{2}}\right)^{1-\beta} \int_{\pi-\pi/n}^\pi |\varphi(x, t + \pi/n) - \varphi(x, t)| \frac{1}{t^{1-\beta}} dt \\ &\leq 4 \left(\frac{3\pi}{2\sqrt{2}}\right)^{1-\beta} \|f(t + \pi/n) - f(t)\|_M \left\| \frac{1}{t^{1-\beta}} \chi_{\pi/n}^\pi(t) \right\|_{(N)} \\ &\leq 4 \left(\frac{3\pi}{2\sqrt{2}}\right)^{1-\beta} \frac{C_1 n^{1-\beta} F(\pi/n)}{\varrho_2 N_{-1}(n/\pi)}. \end{aligned}$$

Hence

$$n^\beta |S_1(x)| \leq 4 \left(\frac{3\pi}{2\sqrt{2}}\right)^{1-\beta} \frac{C_1}{\varrho_2} \frac{nF(\pi/n)}{N_{-1}(n/\pi)}.$$

Further,

$$|S_2(x)| \leq \int_{\pi-\pi/n}^\pi |\varphi(x, t)| \left\{ \frac{3\pi}{2\sqrt{2}(t + \pi/n)} \right\}^{1-\beta} dt \leq \left(\frac{3}{2\sqrt{2}}\right)^{1-\beta} \int_{\pi-\pi/n}^\pi |\varphi(x, t)| dt.$$

Since

$$\int_{\pi-\pi/n}^\pi |\varphi(x, t)| dt \leq \frac{4\pi L}{n}, \quad \text{where } L = \max_{-\pi \leq t \leq \pi} |f(t)|,$$

we get

$$n^\beta |S_2(x)| \leq 4\pi \left(\frac{3}{2\sqrt{2}} \right)^{1-\beta} \frac{L}{n^{1-\beta}}.$$

By inequalities (3) and (5),

$$\frac{1}{n^{1-\beta}} \leq \frac{\varrho_3 n F(\pi/n)}{\varrho_1 N_{-1}(n/\pi)}.$$

Hence

$$n^\beta |S_2(x)| \leq 4\pi \left(\frac{3}{2\sqrt{2}} \right)^{1-\beta} \frac{L\varrho_3}{\varrho_1} \frac{nF(\pi/n)}{N_{-1}(n/\pi)},$$

and consequently

$$n^\beta |R_{n,3}(x)| = O\left(\frac{nF(\pi/n)}{N_{-1}(n/\pi)} \right).$$

Next, we write $R_{n,2}(x)$ is the form

$$\begin{aligned} R_{n,2}(x) &= \int_{\pi/n}^{\pi} \{\varphi(x, t + \pi/n) - \varphi(x, t)\} \{w(t) - w(t + \pi/n)\} \cos nt dt + \\ &+ \int_{\pi/n}^{\pi} \varphi(x, t) \{w(t) - w(t + \pi/n)\} \cos nt dt = U_1(x) + U_2(x). \end{aligned}$$

Applying the inequality

$$|w(t) - w(t + \pi/n)| \leq \frac{M_1}{nt^{2-\beta}} \quad \text{for } \frac{\pi}{n} \leq t \leq \pi \quad (n \geq 2),$$

where M_1 is a constant depending only on β , we obtain

$$\begin{aligned} |U_1(x)| &\leq \frac{M_1}{n} \int_{\pi/n}^{\pi} |\varphi(x, t + \pi/n) - \varphi(x, t)| \frac{1}{t^{2-\beta}} dt \\ &\leq 2C_3 M_1 \Omega(\pi/2n) \frac{1}{n} \int_{\pi/n}^{\pi} \frac{1}{t^{2-\beta}} dt \leq \frac{2C_3 M_1 \pi^{\beta-1}}{(1-\beta)n^\beta} \Omega(\pi/2n) \end{aligned}$$

and

$$n^\beta |U_1(x)| \leq \frac{2C_3 M_1 \pi^{\beta-1}}{1-\beta} \Omega(\pi/2n).$$

Further

$$U_2(x) = \left(\int_{\pi/n}^{2\pi/n} + \int_{2\pi/n}^{\pi} \right) \varphi(x, t) \{w(t) - w(t + \pi/n)\} \cos nt dt = T_1(x) + T_2(x).$$

Changing in $T_2(x)$ the variable t into $t + \pi/n$ and adding two expressions for $U_2(x)$, we get

$$\begin{aligned}
 2U_2(x) &= 2T_1(x) - \int_{\pi/n}^{2\pi/n} \varphi(x, t + \pi/n) \{w(t + \pi/n) - w(t + 2\pi/n)\} \cos nt dt + \\
 &+ \int_{2\pi/n}^{\pi} \{\varphi(x, t) - \varphi(x, t + \pi/n)\} \{w(t) - w(t + \pi/n)\} \cos nt dt + \\
 &+ \int_{2\pi/n}^{\pi} \varphi(x, t + \pi/n) \{(w(t) - w(t + \pi/n)) - (w(t + \pi/n) - w(t + 2\pi/n))\} \cos nt dt + \\
 &+ \int_{\pi-\pi/n}^{\pi} \varphi(x, t + \pi/n) \{w(t + \pi/n) - w(t + 2\pi/n)\} \cos nt dt = 2T_1(x) + \sum_{i=1}^4 T_{2,i}(x).
 \end{aligned}$$

Now, if $n \geq 3$,

$$\begin{aligned}
 |T_1(x)| &\leq \frac{M_1}{n} \int_{\pi/n}^{2\pi/n} |\varphi(x, t)| \frac{1}{t^{2-\beta}} dt \leq \frac{2C_3 M_1}{(1-\beta)\pi^{1-\beta}} \frac{\Omega(\pi/n)}{n^\beta}, \\
 |T_{2,1}(x)| &\leq \frac{M_1}{n} \int_{\pi/n}^{2\pi/n} |\varphi(x, t + \pi/n)| \frac{1}{t^{2-\beta}} dt \leq \frac{2C_3 M_1}{(1-\beta)\pi^{1-\beta}} \frac{\Omega\left(\frac{3\pi}{2n}\right)}{n^\beta}, \\
 |T_{2,2}(x)| &\leq \frac{M_1}{n} \int_{2\pi/n}^{\pi} |\varphi(x, t) - \varphi(x, t + \pi/n)| \frac{1}{t^{2-\beta}} dt \leq \frac{2C_3 M_1 \Omega(\pi/2n)}{(1-\beta)\pi^{1-\beta} n^\beta}.
 \end{aligned}$$

The expression $T_{2,3}(x)$ will be written as

$$\begin{aligned}
 T_{2,3}(x) &= \int_{2\pi/n}^{\pi} \{\varphi(x, t + \pi/n) - \varphi(x, t)\} \{w(t) - w(t + \pi/n) - \\
 &\quad - (w(t + \pi/n) - w(t + 2\pi/n))\} \cos nt dt + \\
 &+ \int_{2\pi/n}^{\pi} \varphi(x, t) \{w(t) - w(t + \pi/n) - \\
 &\quad - (w(t + \pi/n) - w(t + 2\pi/n))\} \cos nt dt = W_1(x) + W_2(x).
 \end{aligned}$$

Taking into account the inequality

$$\left| (w(t) - w(t + \pi/n)) - (w(t + \pi/n) - w(t + 2\pi/n)) \right| \leq \frac{M_2}{n^2 t^{3-\beta}},$$

where M_2 is a constant depending on β , we obtain

$$\begin{aligned} |W_1(x)| &\leq \frac{M_2}{n^2} \int_{2\pi/n}^{\pi} |\varphi(x, t + \pi/n) - \varphi(x, t)| \frac{1}{t^{3-\beta}} dt \\ &\leq \frac{2C_3 M_2 \Omega(\pi/2n)}{n^2} \int_{2\pi/n}^{\pi} \frac{1}{t^{3-\beta}} dt \leq \frac{2C_3 M_2}{(2-\beta)(2\pi)^{2-\beta}} \frac{\Omega(\pi/2n)}{n^\beta}, \\ |W_2(x)| &\leq \frac{M_2}{n^2} \int_{2\pi/n}^{\pi} |\varphi(x, t)| \frac{1}{t^{3-\beta}} dt = \frac{M_2}{n^2} \int_{2\pi/n}^{\pi} |\varphi(x, t)| \frac{1}{t^{1-\beta}} \frac{1}{t^2} dt \\ &\leq \frac{M_2}{(2\pi)^{1-\beta} n^\beta} \frac{1}{n} \int_{2\pi/n}^{\pi} |\varphi(x, t)| \frac{1}{t^2} dt \leq \frac{2C_3 M_2}{(2\pi)^{1-\beta} n^\beta} \frac{1}{n} \int_{2\pi/n}^{\pi} \Omega(t/2) \frac{1}{t^2} dt. \end{aligned}$$

From the estimates for $|W_1(x)|$ and $|W_2(x)|$ it follows that

$$n^\beta |T_{2,3}(x)| = O\left(\Omega\left(\frac{\pi}{2n}\right) + \frac{1}{n} \int_{\pi/n}^{\pi} \Omega(t/2) \frac{1}{t^2} dt\right).$$

The expression $|T_{2,4}(x)|$ is estimated analogously to $|R_{n,3}(x)|$. Taking into account the estimates for $|T_1(x)|$ and $|T_{2,i}(x)|$ ($i = 1, 2, 3, 4$) we get at once relation (6).

Hence

$$n^\beta H_n(x) = O\left(\Omega\left(\frac{3\pi}{2n}\right) + \frac{1}{n} \int_{\pi/n}^{\pi} \Omega(t/2) \frac{1}{t^2} dt\right),$$

uniformly in x , and since $A_n^{-\beta} \simeq 1/\{n^\beta \Gamma(1-\beta)\}$ ([4], p. 77), the first term in I_1 is of the order indicated for $n^\beta H_n(x)$. The remaining three terms of I_1 are of the same order, too.

By Theorem 1,

$$|I_2| \leq \frac{\pi^2 \beta}{2n} \int_{\pi/n}^{\pi} \frac{|\varphi(x, t)|}{t^2} dt \leq \frac{\pi^2 C_3 \beta}{n} \int_{\pi/n}^{\pi} \frac{\Omega(t/2)}{t^2} dt.$$

Thus, the proof is completed.

In case when $M(u) = u^p/p$, $F(t) = G(t) = t^\alpha$ ($1/p < \alpha \leq 1$), we obtain Theorem 5 of [5], p. 500.

References

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