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On the characterization of some families of distributions

Introduction. It is known that if we have n independent random variables $X_1, X_2, \ldots, X_n$ with the same distribution $F(x)$, then the distributions of some functions U of these variables may be determined uniquely. The distribution of the random variable U, on the other hand, determines uniquely $F(x)$ only in a few cases. In this paper we give some functions U_i $(i = 1, 2, \ldots, n-1)$ of random variables X_k $(k = 1, 2, \ldots, n)$ with the same distribution function $F(x)$ such that under some assumptions the distribution of the n-dimensional random variable $(U_1, U_2, \ldots, U_{n-1})$ determines uniquely $F(x)$. We will also see that the assumption concerning the identity of distributions of the random variables X_k for $k = 1, 2, \ldots, n$ may be weakened. It suffices namely that the distributions of the random variables X_k belong to the same class differing (if they do) in some parameters.

1. General theorems.

THEOREM 1. *Let $X_1, X_2, \ldots, X_n$ be independent random variables nd let*

$$Z_k = a_k X_k + b_k X_n \quad \text{for } k = 1, 2, \ldots, n-1 \tag{1}$$

a_k, b_k being arbitrary real numbers different from zero. If the charakteristic function $\varphi(t_1, \ldots, t_{n-1})$ of the $n-1$ dimensional random variable $(Z_1, Z_2, \ldots, Z_{n-1})$ is not equal zero at any point, then the joined distribution of $t(Z_1, Z_2, \ldots, Z_{n-1})$ defines the distribution of X_k, $k = 1, 2, \ldots, n$, precisely ao the displacement.

Lemma 1 in paper [2] is a particular case of this theorem for $k = 1, 2, \ldots, n-1$, $n = 3$.

Proof. Denote by $\varphi_k(t)$ the characteristic function of the random variable X_k, $k = 1, 2, \ldots, n$, by the definition of the characteristic function of the random variable $(Z_1, Z_2, \ldots, Z_{n-1})$ we have:

$$(2)\quad \varphi(t_1, t_2, \ldots, t_{n-1}) = E\Big[\exp\Big(\sum_{k=1}^{n-1} it_k Z_k\Big)\Big] = E\Big[\exp \sum_{k=1}^{n-1} it_k(a_k X_k + b_k X_n)\Big]$$
$$= E\Big\{\exp\Big[i\Big(\sum_{k=1}^{n-1} a_k t_k X_k + X_n \sum_{k=1}^{n-1} t_k b_k\Big)\Big]\Big\}$$

the symbol E denoting the expected value.

From the independence of the variables X_k and from formula (2) it follows that:

$$(3)\quad \varphi(t_1, t_2, \ldots, t_{n-1}) = \varphi_1(a_1 t_1) \ldots \varphi_{n-1}(a_{n-1}t_{n-1}) \cdot \varphi_n(b_1 t_1 + \ldots + b_{n-1}t_{n-1}).$$

The fact that the function $\varphi(t_1, t_2, \ldots, t_{n-1})$ does not vanish for any system of numbers $(t_1, t_2, \ldots, t_{n-1})$ is equivalent to the non-vanishing of $\varphi_k(t)$ for any value of t for $k = 1, 2, \ldots, n$.

Let now $U_1, U_2, \ldots, U_n$ be other independent random variables whose characteristic functions are $\psi_k(t)$ for $k = 1, 2, \ldots, n$ and let:

$$V_k = a_k U_k + b_k U_n \quad \text{for } k = 1, 2, \ldots, n-1.$$

Assume that the characteristic function of the joined random variable $(V_1, V_2, \ldots, V_{n-1})$ does not vanish at any point. By the same argument as above we find that

$$(4)\quad \psi(t_1, t_2, \ldots, t_{n-1}) = \psi_1(a_1 t_1) \ldots \psi_{n-1}(a_{n-1}t_{n-1})\psi_n(b_1 t_1 + \ldots + b_{n-1}t_{n-1}).$$

Assume now that the n-dimensional random variables: $(Z_1, Z_2, \ldots, Z_{n-1})$ and $(V_1, V_2, \ldots, V_{n-1})$ have equal joined distributions and that, consequently their characteristic functions are also equal. Comparing (3) and (4) we obtain:

$$(5)\quad \psi_1(a_1 t_1) \ldots \psi_{n-1}(a_{n-1}t_{n-1})\psi_n(b_1 t_1 + \ldots + b_{n-1}t_{n-1})$$
$$= \varphi_1(a_1 t_1) \ldots \varphi_{n-1}(a_{n-1}t_{n-1}) \cdot \varphi_n(b_1 t_1 + \ldots + b_{n-1}t_{n-1})$$

for $-\infty < t_k < \infty$, $k = 1, 2, \ldots, n-1$. Put

$$\frac{\psi_k(t)}{\varphi_k(t)} = p_k(t) \quad \text{for } k = 1, 2, \ldots, n,$$

then we obtain from equality (5) an functional equation with n unknown functions:

$$(6)\qquad p_1(a_1 t_1) \ldots p_{n-1}(a_{n-1}t_{n-1})p_n(b_1 t_1 + \ldots + b_{n-1}t_{n-1}) = 1,$$

$p_k(t)$ being complex continuons functions satisfying the condition $p_k(0) = 1$ for $k = 1, \ldots, n$.

Next we shall solve the functional equation (6). For this purpose we substitute in it successively:

$$t_i = \begin{cases} t & \text{for } i = k, \\ 0 & \text{for } i \neq k \text{ with } k = 1, 2, \ldots, n-1. \end{cases}$$

Then we obtain instead of functional equation (6) a system of $n-1$ functional equations of the form:

(7) $$p_k(a_k t)\cdot p_n(b_k t) = 1 \quad \text{for } k = 1, 2, \ldots, n-1.$$

Hence we find:

$$p_k(a_k t) = \frac{1}{p_n(b_k t)}$$

and substitute into equation (6), then we obtain a functional equation with one functional unknown $p_n(t)$:

(8) $$p_n(b_1 t + \ldots + b_{n-1} t) = p_n(b_1 t) \ldots p_n(b_{n-1} t).$$

The function satisfying condition (8) and the condition $p_n(0) = 1$ can only be an exponential function, thus:

(9) $$p_n(t) = e^{rt},$$

where r may be an arbitrary complex number.

Making use of (9) and (7) we determine:

$$p_k(a_k t) = e^{-b_k rt} \quad \text{for } k = 1, 2, \ldots, n-1,$$

$$p_k(t) = e^{-\frac{b_k}{a_k} rt}.$$

Hence

$$\psi_k(t) = e^{-\frac{b_k}{a_k} rt}\cdot \varphi_k(t), \qquad \psi_n(t) = e^{rt}\varphi_n(t).$$

From the property of characteristic functions: $\varphi(-t) = \overline{\varphi(t)}$ it follows that

(10) $$\psi_k(t) = e^{\beta_k ti}\cdot \varphi_k(t), \qquad k = 1, 2, \ldots, n,$$

$$\beta_k = \begin{cases} -\dfrac{b_k}{a_k} imr & \text{for } k = 1, 2, \ldots, n-1, \\ imr & k = n \end{cases}$$

being real numbers.

Equality (10) indicates that:

$$U_k = X_k + B_k \quad \text{for } k = 1, 2, \ldots, n.$$

Hence we draw the conclusion that the joined distribution of $(Z_1, Z_2, \ldots, Z_{n-1})$ determines the distributions of the random variables X_k for $k = 1, 2, \ldots, n$ precisely to a displacement, which ends the proof.

It follows from the above theorem:

THEOREM 2. *Let* $X_1, X_2, \ldots, X_n$ *be positive independent random variables and*

$$Y_k = X_k^{a_k} \cdot X_n^{b_k} \quad \text{for } k = 1, 2, \ldots, n-1, \tag{11}$$

a_k, b_k *being arbitrary real numbers different from zero.*

If the characteristic function of the $(n-1)$*-dimensional random variable* $(\ln Y_1, \ldots, \ln Y_{n-1})$ *does not vanish at any point, then the joined distribution of* $(Y_1, Y_2, \ldots, Y_{n-1})$ *defines the distributions:* $X_1, X_2, \ldots, X_n$ *precisely to a constant real factor (the so-called parameter of the scale).*

Proof. The proof is obvious for the random variables $\ln X_k$ for $k = 1, 2, \ldots, n$ satisfy the assumptions of Theorem 1, and the random variable $(\ln Y_1, \ln Y_2, \ldots, \ln Y_{n-1})$ is a random variable of the form:

$$(a_1 \ln X_1 + b_1 \ln X_n, \ldots, a_{n-1} \ln X_{n-1} + b_{n-1} \ln X_n), \tag{12}$$

and thus by Theorem 1, if the characteristic function of random variable (12) does not vanish at any point, then its distribution defines the distributions of the variables $\ln U_k$ for $k = 1, 2, \ldots, m$, related to the random variable by:

$$\ln U_k = \ln X_k + \beta_k \quad \text{for } k = 1, 2, \ldots, n$$

the constants β_k having been determined be (10).

Hence it follows that the distribution of $(Y_1, Y_2, \ldots, Y_{n-1})$ defines the distributions of X_k for $k = 1, 2, \ldots, n$ precisely to a constant real factor of the form:

$$a_k = e^{\beta_k} \quad \text{for } k = 1, 2, \ldots, n$$

which was to be proved.

By an argument similar to that used in the proof of Theorem 1 one may obtain:

THEOREM 3. *Let* $X_1, X_2, \ldots, X_n$ *be independent random variables, and* $S_1 = X_1 - \bar{X}$, $S_2 = X_2 - \bar{X}, \ldots, S_{n-1} = X_{n-1} - \bar{X}$, *where*

$$\bar{X} = \frac{1}{n} \sum_{k=1}^{n} X_k.$$

If the characteristic function of the $(n-1)$*-dimensional random variable* $(S_1, S_2, \ldots, S_{n-1})$ *does not vanish at any point, then the joined distribution of this variable defines the distributions of the random variables* X_k *for* $k = 1, 2, \ldots, n$ *precisely to an identical displacement for each of the variables.*

Further in this paper we shall make use of the above proved theorems to characterize some distributions.

2. The characterization of the generalized gamma distribution. Now we shall deal with random variables X_k for $k = 1, 2, \ldots, n$ obeying the so-called generalized gamma distribution with the parameters p_k, a the density of that distribution is of the form:

$$f_k(x) = \frac{a}{a^{\frac{p_k}{a}}\Gamma\left(\frac{p_k}{a}\right)} X^{p_k-1}\exp\left(-\frac{x^a}{a}\right) \quad \text{for } x>0 \tag{13}$$

with

$$p_k > 0,\ a > 0,\ a > 0.$$

A number of properties of the above distribution have been proved by T. Śródka in [3].

We are interested in the distribution of an $(n-1)$-dimensional random variable $(Y_1, \ldots, Y_{n-1})$, where Y_k for $k = 1, 2, \ldots, n-1$ have been defined by formula (11) with $a_k = 1$, $b_k = -1$ and thus are some functions of the random variables X_k.

THEOREM 4. *Let $X_1, X_2, \ldots, X_n$ be positive indenpendent random variables and*

$$Y_k = \frac{X_k}{X_n} \quad \textit{for } k = 1, 2, \ldots, n-1. \tag{14}$$

The necessary and sufficient condition for X_k with $k = 1, 2, \ldots, n$ to be subject to distribution (13) *is that the joined distribution $(Y_1, Y_2, \ldots, Y_{n-1})$ be an $(n-1)$ dimensional distribution with the density:*

$$f(y_1, \ldots, y_{n-1}) = \begin{cases} \dfrac{a^{n-1}\Gamma\left(\frac{1}{a}\sum\limits_{k=1}^{n} p_k\right)}{\prod\limits_{k=1}^{n}\Gamma\left(\frac{p_k}{a}\right)} \dfrac{\prod\limits_{k=1}^{n-1} y_k^{p_k-1}}{\left(1+\sum\limits_{k=1}^{n-1} y_k^a\right)^{\frac{1}{a}\sum\limits_{k=1}^{n} p_k}} & \textit{for } y_i > 0, \\ 0 & \textit{otherwise} \end{cases} \tag{15}$$

with $i = 1, 2, \ldots, n-1$.

Proof. The characteristic function $\varphi_k(t)$ of the random variable $\ln X_k$, X_k being subject to distribution (13) is:

$$\varphi_k(t) = E(X_k^{it}) = \frac{a}{a^{\frac{p_k}{a}}\Gamma\left(\frac{p_k}{a}\right)}\int\limits_0^\infty x^{it+p_k-1}\exp\left(-\frac{x^a}{a}\right)dx = \frac{a^{\frac{it}{a}}\Gamma\left(\frac{it+p_k}{a}\right)}{\cdot\Gamma\left(\frac{p_k}{a}\right)}$$

for $k = 1, 2, \ldots, n$.

By (3) the characteristic function $\varphi(t_1, t_2, \ldots, t_{n-1})$ of the $(n-1)$ dimensional random variable $(\ln Y_1, \ln Y_2, \ldots, \ln Y_{n-1})$ with $a_i = 1$,

$b_i = -1$ for $i = 1, \ldots, n-1$ may be expressed in terms of $\varphi_k(t)$ in the following way:

$$\varphi(t_1, t_2, \ldots, t_{n-1}) = \prod_{k=1}^{n-1} \frac{\Gamma\left(\frac{it_k + p_k}{a}\right)}{\Gamma\left(\frac{p_k}{a}\right)} \cdot \frac{\Gamma\left(p_n - i\sum_{k=1}^{n-1} t_k\right)}{\Gamma\left(\frac{p_n}{a}\right)}. \tag{16}$$

On the other hand the characteristic function of the random variable $(\ln Y_1, \ldots, \ln Y_{n-1})$, if $(Y_1, Y_2, \ldots, Y_{n-1})$ being subject to distribution (15) is of the form:

$$\varphi^*(t_1, \ldots, t_{n-1}) = E(y_1^{it_1}, \ldots, y_{n-1}^{it_{n-1}})$$

$$= \frac{a^{n-1}\Gamma\left(\frac{1}{a}\sum_{k=1}^{n} p_k\right)}{\prod_{k=1}^{n}\Gamma\left(\frac{p_k}{a}\right)} \int_0^\infty \ldots \int_0^\infty \frac{\prod_{k=1}^{n-1} y_k^{it_k+p_k-1} dy_1 \ldots dy_{n-1}}{\left(1+\sum_{k=1}^{n-1} y_k^a\right)^{\frac{1}{a}\sum_{k=1}^{n} p_k}}.$$

Making use of formula 4.638 in tables [1] we obtain:

$$\varphi^*(t_1, \ldots, t_{n-1}) = \frac{a^{n-1}\Gamma\left(\frac{1}{a}\sum_{k=1}^{n} p_k\right)\prod_{k=1}^{n-1}\Gamma\left(\frac{it_k + p_k}{a}\right) \cdot \Gamma\left(\frac{p_n - i\sum_{k=1}^{n-1} t_k}{a}\right)}{a^{n-1}\Gamma\left(\frac{1}{a}\sum_{k=1}^{n} p_k\right)\prod_{k=1}^{n}\Gamma\left(\frac{p_k}{a}\right)}. \tag{17}$$

As we know, for a given distribution, the characteristic function is uniquely determined by the density and vice versa, this and formulas (16) and (17) imply the validity of our theorem for $a = 1$, $n = 3$ we obtain Theorem 1 of paper [2],

THEOREM 5. *Let $X_1, X_2, \ldots, X_n$ be positive independent random variables and*

$$U_{i-1} = \frac{\sum_{k=1}^{i-1} X_k^a}{\sum_{k=1}^{i} X_k^a} \quad \text{for } i = 2, 3, \ldots, n$$

with $a > 0$.

The necessary and sufficient condition for the random variables X_k $(k = 1, 2, \ldots, n)$ to be subject to the generalized gamma distribution with the density defined by (13) *is that $U_1, U_2, \ldots, U_{n-1}$ be independent random variables with the beta distributions whose parameters are respectively equal*

$$\left(\frac{1}{a}\sum_{k=1}^{i-1} p_k; \frac{p_i}{a}\right) \quad \text{with } i = 2, 3, \ldots, n. \tag{19}$$

The proof of necessity. It may be easily verified that if a random variable X_k is subject to distribution (13), then the random variable $Z_k = X_k^a$ is subject to a gamma distribution with the density

$$h_k(z) = \frac{1}{a^{\frac{p_k}{a}} \Gamma\left(\frac{p_k}{a}\right)} z^{\frac{p_k}{a}-1} \cdot e^{-\frac{z}{a}} \quad \text{for } z > 0. \tag{20}$$

Hence and from the theorem on addition for a gamma distribution in relation to p_k/a for the same value of the parameter a it follows that: $W_{i-1} = X_i^a + \ldots + X_{i-1}^a$ for $i = 2, 3, \ldots, n$ is a random variable subject to a gamma distribution with the density:

$$k_{i-1}(w) = \frac{1}{a^{\frac{1}{a}\sum_{k=1}^{i-1} p_k} \cdot \Gamma\left(\frac{1}{a}\sum_{k=1}^{i-1} p_k\right)} w^{\frac{1}{a}\sum_{k=1}^{i-1} p_k - 1} \cdot e^{-\frac{w}{a}} \quad \text{for } w > 0. \tag{21}$$

Then we make use of theorem 7.6.1 on page 189 of paper [4]: If s_1 and s_2 are independent random variables subject to a gamma distribution with the density

$$f_k(s) = \begin{cases} \dfrac{s^{p_k-1}e^{-s}}{\Gamma(p_k)} & \text{for } s > 0 \\ 0 & \text{for } s \leqslant 0 \end{cases} \quad \text{for } k = 1, 2.$$

Then the random variable $U = \dfrac{s_1}{s_1+s_2}$ is subject to a beta distribution with the parameters p_1, p_2.

It can easily be proved that the theorem is also true if the random variables s_k are subject to a gamma distribution with two parameters: p_k and a for $k = 1, 2$ and thus if they are subject to a distribution with the density

$$f_k(s) = \frac{1}{a^{p_k}\Gamma(p_k)} s^{p_k-1} e^{-\frac{s}{a}} \quad \text{for } s > 0, k = 1, 2.$$

Putting successively: $s_1 = W_{i-1}$, $s_2 = x_i^a$ for $i = 2, 3, \ldots, n$ and applying the above theorem we find that the random variables U_{i-1} $(i = 2, 3, \ldots, n)$ defined by formulas (18) subject to a beta distribution with the parameters: $\dfrac{1}{a}\sum_{k=1}^{i-1} p_k$; $\dfrac{p_i}{a}$.

Next we shall prove that the randm variables $U_1, U_2, \ldots, U_{n-1}$ are independent. We observe that if the random variables X_k are subject to distributions (13), then the random variables $S_k = X_k^a/a$ are subject to the gamma distributions with the densities:

$$r_k(s) = \frac{1}{\Gamma\left(\frac{p_k}{a}\right)} \cdot S^{\frac{p_k}{a}-1} \cdot e^{-s} \quad \text{for } s > 0, \quad k = 1, 2, \ldots, n,$$

thus with one parameter $p_k' = \frac{p_k}{a}$.

Accepting the above denotations we may represent the random variables u_{i-1} in the following way:

$$u_{i-1} = \frac{\sum_{k=1}^{i-1} S_k}{\sum_{k=1}^{i} S_k} \quad \text{for } i = 2, 3, \ldots, n. \tag{21a}$$

The independence of random variables of form (21a) has been proved by Aitchison in paper [5].

The proof of sufficiency. Since, by assumption, $u_1, u_2, \ldots, u_{n-1}$ are independent random variables, the density of the $(n-1)$-dimensional random variable $(u_1, u_2, \ldots, u_{n-1})$ is given by the formula:

$$g(u_1, u_2, \ldots, u_n) = \frac{1}{\prod_{i=2}^{n} B\left(\frac{1}{a}\sum_{k=1}^{i-1} p_k; \frac{p_i}{a}\right)} \prod_{i=2}^{n} u_{i-1}^{\frac{1}{a}\sum_{k=1}^{i-1} p_k - 1} \cdot \prod_{i=2}^{n} (1-u_{i-1})^{\frac{p_i}{a}-1}, \quad k = 1, 2, \ldots, n-1$$

for $0 < u_k < 1$.

Expressing the functions beta by the functions gamma we obtain after some simplifications:

$$g(u_1, u_2, \ldots, u_{n-1}) = \frac{\Gamma\left(\frac{1}{a}\sum_{k=1}^{n} p_k\right)}{\prod_{k=1}^{n} \Gamma\left(\frac{p_k}{a}\right)} \prod_{i=2}^{n} \left[U_{i-1}^{\frac{1}{a}\sum_{k=1}^{i-1} p_k - 1} \cdot (1-u_{i-1})^{\frac{p_i}{a}-1}\right]. \tag{22}$$

Observe that by (14) equality (18) may be written in the form

$$u_{i-1} = \frac{\sum_{k=1}^{i-1} Y_k^a}{\sum_{k=1}^{i} Y_k^a} \quad \text{for } i = 2, 3, \ldots, n-1 \tag{23}$$

and

$$U_{n-1} = \frac{\sum\limits_{k=1}^{n-1} Y_k^a}{1 + \sum\limits_{k=1}^{n-1} Y_k^a}.$$

To determine the density $h(y_1, \ldots, y_{n-1})$ of the $(n-1)$-dimensional random variable $(Y_1, Y_2, \ldots, Y_{n-1})$ we change in (22) the variables according to (23). The jacobian determinant of this transformation equals (A).

After extracting the factors $a/(\sum\limits_{k=1}^{i+1} Y_k^a)^2$ appearing in each term of the i-th row for $i = 1, 2, \ldots, n-2$ and from the last row $a/(1 + Y_1^a + \ldots + Y_{n-1}^a)^2$ as well as the factors Y_i for $i = 1, 2, \ldots, n-1$ from the last column we obtain a determinant from which by subtracting the terms of the second column from the terms of the first column we obtain a determinant in which all the numbers in the first column except the first are equal zero. Expending the determinant according to this column, we obtain a determinant of order $(n-2)$. Proceeding in this way $n-3$ times we reduce the evalution of the jacobian determinant to evalution of a determinant of the second order.

Ultimately

$$J = \frac{a^{n-1} \prod\limits_{i=1}^{n-1} Y_i^{a-1}}{\prod\limits_{i=2}^{n-1} (\sum\limits_{k=1}^{i} Y_k^a) \cdot (1 + Y_1^a + \ldots + Y_{n-1}^a)^2}.$$

Thus the density:

$$h(y_1, \ldots, y_{n-1}) = \frac{\Gamma\left(\frac{1}{a} \sum\limits_{k=1}^{n} p_k\right)}{\prod\limits_{k=1}^{n} \Gamma\left(\frac{p_k}{a}\right)} \frac{y_1^{p_1 - a}}{(y_1^a + y_2^a)^{\frac{p_1}{a} - 1}} \frac{(y_1^a + y_2^a)^{\frac{p_1 + p_2}{a} - 1}}{(y_1^a + y_2^a + y_3^a)^{\frac{p_1 + p_2}{a} - 1}} \ldots$$

$$\ldots \frac{(y_1^a + \ldots + y_{n-1}^a)^{\frac{1}{a} \sum\limits_{k=1}^{n-1} p_k - 1}}{(1 + y_1^a + \ldots + y_{n-1}^a)^{\frac{1}{a} \sum\limits_{k=1}^{n-1} p_k - 1}} \cdot \frac{y_2^{p_2 - a}}{(y_1^a + y_2^a)^{\frac{p_2}{a} - 1}} \ldots$$

$$\ldots \frac{y_{n-1}^{p_{n-1} - a}}{(y_1^a + \ldots + y_{n-1}^a)^{\frac{p_{n-1}}{a} - 1}} \cdot \frac{|J|}{(1 + y_1^a + \ldots + y_{n-1}^a)^{\frac{p_n}{a} - 1}}.$$

$$J = \left| \begin{array}{ccccc} \dfrac{aY_1^{a-1}Y_2^a}{(Y_1^a+Y_2^a)^2} & -\dfrac{aY_1^aY_2^a}{(Y_1^a+Y_2^a)^2} & 0 \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots 0 & \ldots\ldots\ldots\ldots \\ \dfrac{aY_1^{a-1}Y_3^a}{(Y_1^a+Y_2^a+Y_3^a)^2} & \dfrac{aY_2^{a-1}Y_3^a}{(Y_1^a+Y_2^a+Y_3^a)^2} & \dfrac{-aY_3^{a-1}(Y_1^a+Y_2^a)}{(Y_1^a+Y_2^a+Y_3^a)^2} & 0 \ldots\ldots\ldots 0 & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \dfrac{aY_1^{a-1}Y_{n-1}^a}{(Y_1^a+\ldots+Y_{n-2}^a)^2} & \dfrac{aY_2^{a-1}Y_{n-1}^a}{(Y_1^a+\ldots+Y_{n-1}^a)^2} & \ldots\ldots\ldots\ldots & \dfrac{aY_{n-2}^{a-1}Y_{n-1}^a}{(Y_1^a+\ldots+Y_{n-1}^a)^2} & \dfrac{-a(Y_1^a+\ldots+Y_{n-2}^a)^2Y_{n-1}^{a-1}}{(Y_1^a+\ldots+Y_{n-1}^a)^2} \\ \dfrac{aY_1^{a-1}}{(1+Y_1^a+\ldots+Y_{n-1}^a)^2} & \dfrac{aY_2^{a-1}}{(1+Y_1^a+\ldots+Y_{n-1}^a)^2} & \ldots\ldots\ldots\ldots & \dfrac{aY_{n-2}^{a-1}}{(1+Y_1^a+\ldots+Y_{n-1}^a)^2} & \dfrac{aY_{n-1}^{a-1}}{(1+Y_1^a+\ldots+Y_{n-1}^a)^2} \end{array} \right| . \tag{A}$$

$$J = \begin{vmatrix} \dfrac{Y_2^2}{(Y_1^2+Y_2^2)^{3/2}} & \dfrac{-Y_1Y_2}{(Y_1^2+Y_2^2)^{3/2}} & 0 \;\cdots\cdots & \cdots\cdots \\ \dfrac{Y_1Y_3^2}{(Y_1^2+Y_2^2+Y_3^2)^{3/2}(Y_1^2+Y_2^2)^{1/2}} & \dfrac{Y_2Y_3^2}{(Y_1^2+Y_2^2+Y_3^2)^{3/2}(Y_1^2+Y_2^2)^{1/2}} & \dfrac{-Y_3\sqrt{Y_1^2+Y_2^2}}{(Y_1^2+Y_2^2+Y_3^2)^{3/2}} & 0 \;\cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \dfrac{Y_1Y_{n-1}^2}{(Y_1^2+\ldots+Y_{n-1}^2)^{3/2}(Y_1^2+\ldots+Y_{n-2}^2)^{1/2}} & \dfrac{Y_2Y_{n-1}^2}{(Y_1^2+\ldots+Y_{n-1}^2)^{3/2}(Y_1^2+\ldots+Y_{n-2}^2)^{1/2}} & \cdots & \dfrac{-Y_{n-1}\sqrt{Y_1^2+\ldots+Y_{n-2}^2}}{(Y_1^2+\ldots+Y_{n-1}^2)^{3/2}} \\ \dfrac{Y_1}{(Y_1^2+\ldots+Y_{n-1}^2)^{1/2}(1+Y_1^2+\ldots+Y_{n-1}^2)^{3/2}} & \dfrac{Y_2}{(Y_1^2+\ldots+Y_{n-1}^2)^{1/2}(1+Y_1^2+\ldots+Y_{n-1}^2)^{3/2}} & \cdots & \dfrac{Y_{n-1}}{(Y_1^2+\ldots+Y_{n-1}^2)^{1/2}(1+Y_1^2+\ldots+Y_{n-1}^2)^{3/2}} \end{vmatrix}.$$

After carrying out some simplifications:

$$h(y_1, \ldots, y_{n-1}) = \frac{a^{n-1}\Gamma\left(\frac{1}{a}\sum_{k=1}^{n} p_k\right)}{\left(1+\sum_{k=1}^{n-1} y_k^a\right)^{\frac{1}{a}\sum_{k=1}^{n} p_k}} \frac{\prod_{k=1}^{n-1} y_k^{p_k-1}}{\prod_{k=1}^{n}\Gamma\left(\frac{p_k}{a}\right)}.$$

Making use of Theorem 4 we conclude that the random variables $X_1, \ldots, X_n$ are subject to distributions (13), which ends the proof.

By means of an argument similar to that employed in the proof of the necessary condition of the above theorem one may prove.

THEOREM 6. *If the random variables $X_1, X_2, \ldots, X_n$ are independent and subject to distribution* (13), *then the random variables*:

$$U_{i-1} = \frac{\sum_{k=1}^{i-1} X_k^{a_k}}{\sum_{k=1}^{i} X_k^{a_k}} \quad \textit{for } i = 2, 3, \ldots, n$$

are subject to beta distributions with the parameters

$$\sum_{k=1}^{i-1} \frac{p_k}{a_k}; \ \frac{p_i}{a_i} \quad \textit{for } i = 2, 3, \ldots, n.$$

3. Characterization of a family of distributions symmetrical in relation to $x = 0$. Now we shall be concerned with a family of distributions symmetrical in relation $x = 0$ to which to normal distribution $N(0, \sigma)$ also belongs.

THEOREM 7. *Let for $k = 1, 2, \ldots, n$, X_k be independent random variables whose distributions are symmetrical in relation to the origin and satisfy the $P(X_k = 0) = 0$. The necessary and sufficient condition for X_k to be subject to distributions with the densities*

$$(24) \qquad f_k(X) = \frac{a}{2a^{\frac{p_k-1}{a}}\Gamma\left(\frac{p_k-1}{a}\right)} \cdot x^{p_k-2}e^{-x^a/a} \quad \textit{for } -\infty < x < \infty,$$

where $a > 0$ while p_k and a are arbitrary even numbers is that the joined distribution of the $(n-1)$-dimensional random variable $(Y_1, Y_2, \ldots, Y_{n-1})$, where Y_k are given by (14) *be*:

$$(25) \qquad g(y_1, \ldots, y_{n-1}) = \frac{a^{n-1}\Gamma\left[\frac{1}{a}\left(\sum_{k=1}^{n} p_k - n\right)\right]\prod_{k=1}^{n-1} y_k^{p_k-2}}{2^{n-1}\left(1+\sum_{k=1}^{n-1} y_k^a\right)^{\frac{1}{a}\left(\sum_{k=1}^{n} p_k - n\right)} \prod_{k=1}^{n}\Gamma\left(\frac{p_k-1}{a}\right)}$$

for $-\infty < y_k < \infty$, $k = 1, 2, \ldots, n-1$.

The proof of necessity. If X_k is subject to distribution (24), then density $|X_k|$ is:

$$(26)\qquad h_k(x) = \begin{cases} \dfrac{a}{a^{\frac{p_k-1}{a}}\Gamma\left(\dfrac{p_k-1}{a}\right)} x^{p_k-2}\exp\left(-\dfrac{x^a}{a}\right) & \text{for } x>0, \\ 0 & \text{otherwise.}\end{cases}$$

Hence the characteristic function $\ln|X_k|$ is:

$$(27)\qquad \varphi_k(t) = E(X^{it}) = \frac{a}{a^{\frac{p_k-1}{a}}\Gamma\left(\dfrac{p_k-1}{a}\right)}\int_0^\infty x^{p_k+it-2}e^{-x^a/a}dx$$

$$= \frac{a^{it/a}}{\Gamma\left(\dfrac{p_k-1}{a}\right)}\Gamma\left(\frac{p_k+it-1}{a}\right).$$

Making use of formula (3) for $a_k = 1$, $b_k = -1$, $k = 1, 2, \ldots, n-1$ we obtain the characteristic function of the $(n-1)$ dimensional random variable $\ln|Y_1|\ldots\ln|Y_{n-1}|$:

$$\varphi(t_1, \ldots, t_{n-1}) = \frac{\Gamma\left(\dfrac{p_n-1-i\sum\limits_{k=1}^{m-1}t_k}{a}\right)}{\Gamma\left(\dfrac{p_n-1}{a}\right)}\prod_{k=1}^{n-1}\frac{\Gamma\left(\dfrac{p_k-1-it_k}{a}\right)}{\Gamma\left(\dfrac{p_k-1}{a}\right)}.$$

On the other hand if $(Y_1, Y_2, \ldots, Y_{n-1})$ is subject to distribution (25), then the characteristic function of the $(n-1)$ dimensional random variable $(\ln|Y_1|\ldots\ln|Y_{n-1}|)$ is:

$$\varphi^*(t_1, t_2, \ldots, t_{n-1}) = E\left\{\exp i\left(\sum_{k=1}^{n-1}t_k\ln|y_k|\right)\right\}$$

$$= \frac{a^{n-1}\Gamma\left[\dfrac{1}{a}\left(\sum\limits_{k=1}^{n}p_k-n\right)\right]}{\prod\limits_{k=1}^{n}\Gamma\left(\dfrac{p_k-1}{a}\right)}\int_0^\infty\cdots\int_0^\infty\frac{y_1^{p_1+it-2}\cdots y_{n-1}^{p_{n-1}+it_{n-1}-2}}{(1+y_1^a+\ldots+y_{n-1}^a)^{\frac{1}{a}(\sum_{k=1}^{n}p_k-n)}}dy_1\ldots dy_{n-1}.$$

After making use of formula 4.6338 in tables [1] we obtain

$$(28)\qquad \varphi^*(t_1, \ldots, t_{n-1}) = \frac{\Gamma\left(\dfrac{p_n-1-i\sum\limits_{k=1}^{n-1}t_k}{a}\right)}{\Gamma\left(\dfrac{p_n-1}{a}\right)}\prod_{k=1}^{n-1}\frac{\Gamma\left(\dfrac{p_k+it_k-1}{a}\right)}{\Gamma\left(\dfrac{p_k-1}{a}\right)}.$$

We observe that $\varphi(t_1, t_2, \ldots, t_{n-1}) = \varphi^*(t_1, \ldots, t_{n-1})$, thus in view of the fact that the characteristic function determines uniquely the density of the distribution and vice versa we have proved the validity of the necessary condition.

The proof of sufficiency. If the $(n-1)$-dimensional random variable $(Y_1, \ldots, Y_{n-1})$ is subject to distribution (25), then the characteristic function $\varphi^*(t_1, \ldots, t_{n-1})$ of the random variable: $(\ln(Y_1), \ldots, (Y_{n-1}))$ is of form (28). On the other hand, if we assume that the random variables $|X_k|$ for $k = 1, 2, \ldots, n$ are subject to distribution (26) then by means of the same argument as in the proof of necessity we can prove that the characteristic function $\varphi(t_1, t_2, \ldots, t_{n-1})$ of the $(n-1)$ dimensional random variable $(\ln|Y_1| \ldots \ln|Y_{n-1}|)$ is identical with $\varphi^*(t_1, \ldots, t_{n-1})$. Thus by Theorem 2 and the fact that a characteristic function determines the density uniquely we have proved that the random variables $|X_k|$ for $k = 1, 2, \ldots, n$ are subject to distribution (26). It follows from the symmetry in relation to the origin of the distributions of the random variables X_k for $k = 1, 2, \ldots, n$ that the random variables X_k for $k = 1, 2, \ldots, n$ are subject to distribution (24), which ends the proof.

In particular if we accept: $a = 2$, $a = 2\sigma^2$, $p_k = 2$ for $k = 1, 2, \ldots, n$, we obtain:

COROLLARY. *The necessary and sufficient condition for independent random variables X_n symmetrical in relation to the origin and satisfying the condition $P(X_k = 0) = 0$ for $k = 1, 2, \ldots, n$ be subject to the same normal distribution $N(0, \sigma)$, where σ is an arbitrary positive number is that the joined distribution of the random variable $(Y_1, Y_2, \ldots, Y_{n-1})$, where Y_k have been defined by* (14), *be an $(n-1)$ dimensional Cauchy distribution with the density*:

$$g(y_1, \ldots, y_{n-1}) = \frac{\Gamma(n/2)}{\pi^{n/2}} \frac{1}{(1+y_1^2+\ldots+y_{n-1}^2)^{n/2}}$$

for $-\infty < y_k < \infty$, $k = 1, 2, \ldots, n-1$.

A particular case of this corollary is Theorem 3 of paper [2].

In the sequel we will be concerned with some functions of the random variables X_k for $k = 1, 2, \ldots, n$ of the form:

$$(29) \qquad v_1 = \frac{x_1}{\sqrt{X_1^2+X_2^2}}, \qquad v_l = \frac{\sqrt{\sum_{i=1}^{l} X_i^2}}{\sqrt{\sum_{i=1}^{l+1} X_i^2}} \quad \text{for } l = 2, \ldots, n-1.$$

THEOREM 8. *Let for $k = 1, 2, \ldots, n$, X_k be independent random variables with distributions symmetrical in relation to the origion and satisfying the condition $P(X_k = 0) = 0$.*

The necessary and sufficient condition for the random variables X_k to be subject to the same normal distribution $N(0, \sigma)$ is that the random variables $V_1, \ldots, V_{n-1}$ defined by formulas (29) be independent random variables whose densities are respectively equal:

$$
h_1(v) = \begin{cases} \dfrac{1}{\pi}\dfrac{1}{\sqrt{1-v^2}} & \text{for } |v| < 1, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\tag{30} h_l(v) = \begin{cases} \dfrac{2\Gamma\left(\dfrac{l+1}{2}\right)}{\Gamma\left(\dfrac{l}{2}\right)\sqrt{\pi}} \dfrac{v^{l-1}}{\sqrt{1-v^2}} & \text{for } 0 < v < 1, \\ 0 & \text{otherwise}, \end{cases}
$$

with $l = 2, 3, \ldots, n-1$.

The proof of necessity. It is known that if X_k for $k = 1, 2, \ldots, n$ are independent random variables subject to the same normal distribution $N(0, \sigma)$, then the random variable:

$$
U_l = \sum_{i=1}^{l} X_i^2 \quad \text{for } l = 2, 3, \ldots, n-1
$$

is subject to the gamma distribution with the parameters $p_l = l/2$, $a = 2\sigma^2$ and thus with density of the form:

$$
f_l(u) = \frac{1}{(2\sigma^2)^{l/2}\Gamma(l/2)} u^{l/2-1} e^{-u/2\sigma^2} \quad \text{for } u > 0.
$$

Since the random variable $Z_{l+1} = X_{l+1}^2$ is also subject to a gamma distribution but with the parameters $p = \frac{1}{2}$; $a = 2\sigma^2$, making use of Theorem 7.6.1 from paper [4] quoted on page 10 the random variable:

$$
W_l = \frac{U_l}{U_l + Z_{l+1}} \quad \text{for } l = 2, 3, \ldots, n-1
$$

is subject to the beta distribution with the parameters: $l/2, 1/2$ thus with the density:

$$
r_l(w) = \begin{cases} \dfrac{\Gamma(l+1)/2}{\Gamma(l/2)\Gamma(1/2)} w^{l/2-1}(1-w)^{\frac{1}{2}-1} & \text{for } 0 < w < 1, \\ 0 & \text{otherwise}. \end{cases}
$$

New we may easily determine the density of the random variable

$$v_l = \sqrt{w_l},$$

$$h_l(v) = \begin{cases} \dfrac{2\Gamma(l+1/2)}{\Gamma(l/2)\sqrt{\pi}} \dfrac{v^{l-1}}{\sqrt{1-v^2}} & \text{for } 0 < v < 1, \\ 0 & \text{otherwise} \end{cases}$$

for $l = 2, 3, \ldots, n-1$.

Now the density of the random variable:

$$V_1 = \frac{X_1}{\sqrt{X_1^2+X_2^2}}$$

remains to be determined which unlike the random variables V_l takes for $l = 2, \ldots, n-1$ all the values from the interval $(-1, +1)$. To this end we consider two independent random variables X_1, X_2 with the distribution $N(0, \sigma)$; then it is known that the joined random variable (X_1, X_2) is:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma^2} e^{-\left(x_1^2+x_2^2\right)/2\sigma^2}.$$

Let us change the variables:

$$V = \frac{X_1}{\sqrt{X_1^2+X_2^2}}; \qquad U = X_2,$$

hence

$$x_1 = \frac{uv}{\sqrt{1-v^2}}, \qquad X_2 = U.$$

Evaluating the jacobian determinant of this transformation we find:

$$J = \begin{vmatrix} \dfrac{u\sqrt{1-v^2}+\dfrac{v^2u}{\sqrt{1-v^2}}}{1-v^2}; & \dfrac{v}{\sqrt{1-v^2}} \\ 0 & 1 \end{vmatrix} = \frac{u}{(1-v^2)^{3/2}}.$$

Thus the density of the joined random variable $(U; V)$ is of the form:

$$g(U, V) = \frac{1}{2\sigma^2\pi(1-V^2)^{3/2}} |U| e^{-U^2/2\sigma^2(1-V^2)}.$$

Integrating the above function in relation to U we obtain the density $h_1(V)$ of the random variable V:

$$h_1(v) = \frac{1}{2\sigma^2\pi(1-v^2)^{3/2}} \int\limits_{-\infty}^{\infty} |u| e^{-u^2/2\sigma^2(1-v^2)} du$$

$$= \frac{1}{\sigma^2\pi(1-v^2)^{3/2}} \int\limits_{0}^{\infty} u e^{-u^2/2\sigma^2(1-v^2)} du.$$

Hence ultimately

$$h_1(v) = \frac{1}{\pi\sqrt{1-v^2}} \quad \text{for } -1 < v < 1.$$

To prove the independence of the random variables $V_1, V_2, \ldots, V_{n-1}$ it suffices to observe that the random variables

$$T_l = V_l^2 \quad \text{for } l = 1, 2, \ldots, n-1$$

are of the form

$$T_l = \frac{\sum\limits_{i=1}^{l} Z_i}{\sum\limits_{i=1}^{l+1} Z_i} \quad \text{for } l = 1, 2, \ldots, n-1,$$

Z_i being independent random variables with a gamma distribution and with the parameters: $p = \frac{1}{2}$, $a = 2\sigma^2$.

The independence of the random variables $T_1, T_2, \ldots, T_{n-1}$ has been proved by Aitchison in paper [5] and the independence of the variables $V_1, V_2, \ldots, V_{n-1}$ follows from the independence of $V_1, V_2, \ldots, V_{n-1}$.

The proof of sufficiency. Making use of formulas (30) and the independence of V_k for $k = 1, 2, \ldots, n-1$ we may find the density of the $n-1$ dimensional random variable $(V_1, V_2, \ldots, V_{n-1})$:

$$(31) \quad g(v_1, \ldots, v_{n-1}) = \frac{1}{\pi} \frac{1}{\sqrt{1-\iota_1^2}} \frac{2\Gamma(\frac{3}{2})}{\Gamma(1)\sqrt{\pi}} \frac{v_2}{\sqrt{1-v_2^2}} \frac{2\Gamma(\frac{4}{2})}{\Gamma(\frac{3}{2})\sqrt{\pi}} \frac{v_3^2}{\sqrt{1-v_3^2}} \cdots$$

$$\cdots \frac{2\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)\sqrt{\pi}} \frac{v_{n-2}^{n-3}}{\sqrt{1-v_{n-2}^2}} \frac{2\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}} \frac{v_{n-1}^{n-2}}{\sqrt{1-v_{n-1}^2}}$$

$$\text{for } |v_1| < 1, \quad 0 < v_k < 1, \ k = 2, 3, \ldots, n-1.$$

Observe that formulas (29) may be written in the following form:

$$(32)\qquad v_1=\frac{Y_1}{\sqrt{Y_1^2+Y_2^2}},\qquad v_k=\sqrt{\frac{\sum\limits_{i=1}^{k}Y_i^2}{\sum\limits_{i=1}^{k+1}Y_i^2}},\qquad v_{n-1}=\sqrt{\frac{\sum\limits_{i=1}^{n-1}Y_i^2}{1+\sum\limits_{i=1}^{n-1}Y_1^2}}$$

for $k=2,3,\ldots,n-2$.

Applying transformation (32) to density (31) we may find the density $g(y_1,\ldots,y_{n-1})$ of the $(n-1)$-dimensional random variable $(Y_1,\ldots$ $\ldots,Y_{n-1})$, Y_k being defined for $k=1,\ldots,n-1$ by formulas (14),

$$(33)\qquad g(y_1,\ldots,y_{n-1})$$

$$=\frac{\Gamma\left(\frac{n}{2}\right)\cdot 2^{n-2}}{\pi^{\frac{n}{2}}\cdot 2^{n-2}}\,\frac{(y_1^2+y_2^2)^{\frac{1}{2}}(y_1^2+y_2^2+y_3^2)(y_1^2+\ldots+y_4^2)^{\frac{3}{2}}\ldots(y_1^2+\ldots+y_{n-2}^2)^{\frac{n-3}{2}}}{(y_1^2+y_2^2+y_3^2)^{\frac{1}{2}}(y_1^2+\ldots+y_4^2)(y_1^2+\ldots+y_5^2)^{\frac{3}{2}}\ldots(y_1^2+\ldots+y_{n-1}^2)^{\frac{n-3}{2}}}\times$$

$$\times\frac{(y_1^2+\ldots+y_{n-1}^2)^{\frac{n-2}{2}}}{(1+y_1^2+\ldots+y_{n-1}^2)^{\frac{n-2}{2}}}\cdot\frac{(y_1^2+y_2^2)^{\frac{1}{2}}}{y_2}\cdots\frac{(y_1^2+\ldots+y_{n-1}^2)^{\frac{1}{2}}}{y_{n-1}}\,\frac{(1+y_1^2+\ldots+y_{n-1}^2)^{\frac{1}{2}}}{1}|J|$$

$$=\frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}}\,\frac{(y_1^2+y_2^2)(y_1^2+y_2^2+y_3^2)\ldots(y_1^2+\ldots+y_{n-1}^2)}{y_2\cdot y_3\cdots y_{n-1}}\cdot\frac{1}{(1+y_1^2+\ldots+y_{n-1}^2)^{\frac{n-3}{2}}}\cdot|J|$$

for $-\infty<y_i<\infty$, $i=1,2,\ldots,n-1$.

We shall find the jacobian determinant (B) of transformation (32).

Extracting before the determinant sign the factors of the form $\dfrac{1}{\left(\sum\limits_{k=1}^{i+1}Y_n^2\right)^{3/2}\left(\sum\limits_{k=1}^{i}Y_k^2\right)^{1/2}}$ appearing in each term of the i-th row for $i=1,2,\ldots,n-2$ and the factor $\dfrac{1}{(Y_1^2+\ldots+Y_{n-1}^2)^{1/2}(1+y_1^2+\ldots+y_{n-1}^2)^{3/2}}$ from the last row we obtain a determinant which my easily be reduced to a determinant whose one column consists of mere zeros except the first term. Expanding this determinant according to the terms of this column we decrease its order by 1. Proceeding in this way $(n-3)$ times we reduce the evaluation of the jacobian determinant to evaluation of a determinant of the second order.

Ultimately we obtain:

$$J = \frac{Y_2 Y_3 \ldots Y_{n-1}}{(Y_1^2+Y_2^2)\ldots(Y_1^2+\ldots+Y_{n-1}^2)(1+Y_1^2+\ldots+Y_{n-1}^2)^{3/2}}.$$

Returning to formula (30) we obtain:

$$g(y_1, \ldots, y_{n-1}) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \frac{1}{(1+y_1^2+\ldots+y_{n-1}^2)^{\frac{n}{2}}}.$$

The corollary from Theorem 7 implies that the random variables X_k for $k = 1, 2, \ldots, n$ are subject to the distributions $N(0, \sigma)$ which was to be proved.

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